UNDER PEER REVIEW

Linear Estimation in the Type II Generalized Logistic Distribution Under Progressive Censoring

Abstract: Generalized distributions have become increasingly popular in applications. They are highly flexible in data analysis, especially with skewed data, which are common in many applications. The Generalized Logistic Distribution (GLD) and its special cases have recently received a lot of interest in the literature. We derived estimators of the unknown parameters of type II Generalized Logistic Distribution (Type II GLD) based on progressively type II censored data. A variety of point estimation methods are employed. We considered the best linear unbiased estimator (BLUE) and the best (affine) linear equivariant estimator (BLEE). In addition, we considered Bayesian estimation. Simulation approaches were used to study the estimators and compare them with the maximum likelihood estimator (MLE) in a range of progressive censoring schemes. The mean squared error (MSE) and bias were employed as comparison criteria. An example based on real data is presented.

Keywords: Point Estimation; Best Linear Unbiased Estimation; Best linear equivariant estimation; Type II Generalized Logistic Distribution, Progressive Censoring.

1. Introduction

Considerable attention has been paid in the literature to inference in parametric distributions based on progressively censored data. Balakrishnan and Sandhu (1995) considered progressive Type II censored sample to find the best linear unbiased estimators to estimate the parameters of the exponential distributions. In addition, they found the maximum likelihood estimators (MLE's) and found that they are equal to the BLUE's of the two-parameter exponential distribution. Also, they drew the attention to the fact that the accuracy of the estimators of the location and scale parameters (BLUE) depends on r, n and m but not the progressive censoring scheme R. The generalized exponential distribution was studied by Kundu and Pradhan (2009). They considered Bayesian inference of the parameters of based on the progressively censored data assuming independent gamma priors for the scale and shape parameters. Bayes estimates are approximated using Lindley's approximation and by the importance sampling and Markov chain Monte Carlo techniques. The authors noted that the Bayes estimates have strong advantages over the MLEs, if suitable prior information is available. The generalized Rayleigh distribution was considered by Maiti and Kayal (2019) where they considered estimation of parameters and reliability characteristics a under progressive type-II censored sample. The MLEs and Bayes estimates of the parameters were obtained under various loss functions. Salah (2020) considered estimating the unknown parameters of α-power exponential distribution under progressively Type II censored data using the MLEs. He found the approximate best linear unbiased estimators (ABLUE's) as an initial guess of the MLEs. The author discovered that ABLUEs and MLEs are so closely related of the exponential distribution with two parameters. This closeness provides good initial estimates of MLEs. Aly and Bleed (2013) considered Bayesian estimation of the generalized logistic distribution based on progressively censored data under accelerated testing.

In this paper, we shall consider the type II generalized logistic distribution whose probability density function is given by

$$f(x|\lambda,\mu,\sigma) = \frac{\lambda^{\alpha}}{\sigma\Gamma(\alpha)} \exp[-\alpha\frac{x-\mu}{\sigma}]\exp[-\lambda\exp\frac{x-\mu}{\sigma}], -\infty < x, \mu < \infty; \sigma, \alpha, \lambda > 0. \qquad (1)$$

Nassar and Elmasri (2012); Azizpour and Asgharzadeh (2018) and Aljarrah et al. (2020) studied MLEs for the Generalized Logistic Distribution and other distributions under progressive censoring. Balakrishnan and Hossain (2007) found that the approximate maximum likelihood estimators (AMLEs) and the MLEs have similar performance in terms of bias and variance. Moreover, Rimawi and Baklizi (2021) investigated the type II Generalized Logistic Distribution estimators based on type II progressive censoring data. They analyzed the MLE and the Lindley's approximation to the Bayes estimator.

In this work, we will derive approximate linear estimators of the parameters of the type II generalized logistic distribution using type II progressively censored data. Progressive censoring is a type of censoring where we have n units that are placed simultaneously on the life-testing experiment. Immediately following the first failure, r_1 surviving units are randomly chosen and removed from the experiment. Immediately after the second failure, r_2 items are withdrawn and so on. The procedure is continued until all r_m remaining units are removed after the m^{th} failure. Note that the r_i's are fixed prior to study. If $r_1= r_2= \ldots = r_m= 0$, then n = m which corresponds to the complete sample, while when $r_1= r_2= \ldots = r_{m-1}= 0$, we have $r_m = n - m$ which corresponds to the conventional Type II right-censoring scheme.

2. Approximate Best Linear Unbiased Estimator (ABLUE)

Linear statistics have an easy and accurate structure. Researchers have been interested in using linear inference for parametric distributions with ordered data in a variety of applications because of their ease and accuracy. Suppose we have $(\boldsymbol{X} = X_{1:m:n}, \ldots, X_{m:m:n})`$ be a random vector of progressively Type-II censored order statistics from a distribution with location parameter μ and scale parameter σ. Let Y = $(Y_{1:m:n}, \ldots, Y_{m:m:n})`$ be such that:

$$Y_{j:m:n} = \frac{X_{j:m:n} - \mu}{\sigma}, j = 1, \ldots, m. \qquad (2)$$

Let $W = \sigma(Y - E(Y))$, $b = E(Y)$, $\theta = (\mu, \sigma)`$ and $B = [\mathbb{1}, b]$. It follows that X can be presented as a linear equation:

$$X = \mu.\mathbb{1} + \sigma.Y = \mu.\mathbb{1} + \sigma.E(Y) + W = [\mathbb{1}, b]\binom{\mu}{\sigma} + W = B\,\theta + W`. \qquad (3)$$

Let Σ be the covariance matrix cov(Y), assuming Σ is regular, and non-singular covariance matrix, then

$$\Sigma = \Delta\,\Sigma_{U^{\mathcal{R}}}\,\Delta. \qquad (4)$$

The best linear unbiased estimator (BLUE) for the parameters under study depends on the evaluation of the variance covariance matrix of the order statistics from the progressively censored data. This matrix is very complicated and can not be obtained in closed form. An approximate best linear unbiased estimator is available. It is derived in Balakrishnan and Cramer (2014). We will apply this approximation to the location and scale parameters of our model as follows:

Suppose we have $m \geq 2$ and $n = \sum_{j=1}^{m} r_j + 1$, the BLUE estimators of μ and σ are given by

$$\hat{\mu}_{LU} = \frac{1}{\Delta}.((b`\Sigma^{-1} b)(\mathbb{I}` \Sigma^{-1}X) - (\mathbb{I}` \Sigma^{-1}b)(b`\Sigma^{-1} X)), \tag{5}$$

$$\hat{\sigma}_{LU} = \frac{1}{\Delta}.((\mathbb{I}`\Sigma^{-1} \mathbb{I})(b` \Sigma^{-1}X) - (\mathbb{I}` \Sigma^{-1}b)(\mathbb{I}`\Sigma^{-1} X)), \tag{6}$$

where $\Delta = ((\mathbb{I}`\Sigma^{-1} \mathbb{I})(b` \Sigma^{-1}b) - (\mathbb{I}`\Sigma^{-1} b)^2 > 0$.

In order to find the approximate covariance matrix, we calculate the following quantities;

$$\gamma_j = n - j + 1,\ j = 1, \dots, n \quad , \ c_r = \prod_{j=1}^{r} \gamma_j ,\ r = 1, \dots, m,\ d_r = \prod_{j=1}^{r} (\gamma_j + 1), r = 1, \dots, m,$$

$$e_r = \prod_{j=1}^{r} (\gamma_j + 2),\ r = 1, \dots, m, \quad a_r = \frac{d_r}{e_r},\ r = 1, \dots, m,\ b_r = \frac{c_r}{d_r}, r = 1, \dots, m,$$

$$EU_r = \Pi_r = 1 - b_r,\ r = 1, \dots, m, COVU_rU_s = (a_r - b_r)b_s , r = 1, \dots, m , s = 1, \dots, m.$$

The last quantity $COVU_rU_s$ gives the approximate covariance matrix $\Sigma_{U^{\mathcal{R}}}$. Now Calculate the diagonal matrix Δ with diagonal elements $\left(\frac{1}{f(F^{-1}(\Pi_1))}, \dots, \frac{1}{f(F^{-1}(\Pi_r))}\right)$ where

$f(x) = \frac{e^{-\alpha\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha+1}}$ and $F(x) = 1 - \left[\left(\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\right)^{\alpha}\right]$. We obtain the required covariance matrix, $\Sigma = \Delta\, \Sigma_{U^{\mathcal{R}}}\, \Delta$.

The best linear equivariant estimators (BLEE) are approximated in a similar manner. Using the same notation used for the BLUEs, and let $\Delta_1 = \Delta + ((\mathbb{I}`\Sigma^{-1} \mathbb{I})$ we obtain

$$\hat{\mu}_{LE} = \frac{1}{\Delta_1}.((b`\Sigma^{-1} b + 1)(\mathbb{I}` \Sigma^{-1}X) - (\mathbb{I}` \Sigma^{-1}b)(b`\Sigma^{-1} X)), \tag{7}$$

$$\hat{\sigma}_{LE} = \frac{1}{\Delta_1}.((\mathbb{I}`\Sigma^{-1} \mathbb{I})(b` \Sigma^{-1}X) - (\mathbb{I}` \Sigma^{-1}b)(\mathbb{I}`\Sigma^{-1} X)). \tag{8}$$

3. Bayesian estimator of location and scale parameters:

Bayesian statistical methods begin with established 'prior' beliefs and update them with data to generate 'posterior' beliefs that can be used to make inferences. Based on this technique, we will derive Bayes estimators for the parameters of the type II generalized logistic distribution (GLD) location and scale parameters (μ and σ) with type II progressively censored data.

To facilitate comparison with the classical estimators, we will assume non-informative prior distributions for the location and scale parameters, that is, $\pi(\mu) = 1$ and $\pi(\sigma) = 1/\sigma$. The likelihood function is given by

$$l(data|\alpha,\mu,\sigma) \quad \alpha \quad \frac{1}{\sigma^m}\prod_{i=1}^{m} f(z_{i:m:n})[1-F(z_{i:m:n})]^{r_i}\,. \tag{9}$$

Therefore, the joint posterior density of, μ and σ given the data, is given by

$$\pi(\mu,\sigma|data) \propto \frac{1}{\sigma} l(data|\mu,\sigma), -\infty < \mu < \infty, \sigma > 0. \tag{10}$$

The Bayes estimator of a function of the parameters, say $t = t(\mu,\sigma)$ under the squared error loss function is given by its posterior expectation

$$\hat{t} = \int_0^\infty \int_{-\infty}^\infty t(\mu,\sigma)\pi(\mu,\sigma|data)d\mu d\sigma. \tag{11}$$

This integral is difficult to obtain analytically and therefore we can approximate it using either importance sampling procedures or the Lindley approximation.

Importance Sampling can be explained as a weighted average of random samples taken from another distribution h_v(x) "importance sampling" density function to estimate an expectation with respect to the target density function f_x(x). The prior distribution of μ and σ are non-informative priors for the location and scale parameters (μ and σ)

$$\pi_1(\mu) = 1, -\infty < \mu < \infty, \tag{12}$$

$$\pi_2(\sigma) = \frac{1}{\sigma}, \sigma > 0. \tag{13}$$

The joint prior distribution is

$$\pi(\mu,\sigma) = \frac{1}{\sigma}, -\infty < \mu < \infty, \sigma > 0. \tag{14}$$

It follows that the posterior distribution is given by

$$\pi(\mu,\sigma|data) = k\frac{\alpha^m}{\sigma^{m+1}}\prod_{i=1}^{m}\left\{\frac{1}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)}\left(\frac{e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}{1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}}\right)^{\alpha(R_i+1)}\right\}.$$

$$\propto \left\{\frac{e^{m/\sigma}}{m^{m-1}}\left(1+e^{-\left(\frac{\mu-\bar{x}}{\sigma/m}\right)}\right)^2 \prod_{i=1}^{m}\left\{\frac{e^{-(\alpha(R_i+1)-1)\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha(R_i+1)+1}}\right\}\right\}. \tag{15}$$

We can rewrite the posterior function as:

$$\pi(\mu,\sigma|data) \propto f_1(\mu)f_2(\sigma)h(\mu,\sigma), \tag{16}$$

where $f_1(\mu) = \left\{ \frac{m}{\sigma} \frac{e^{\frac{\mu-\bar{x}}{\sigma/m}}}{\left(1+e^{\frac{\mu-\bar{x}}{\sigma/m}}\right)^2} \right\}$, this is the logistic distribution with parameters $\bar{x} = \frac{\sum_{i=1}^{m} x_i}{m}$ and σ/m. $f_2(\sigma) = \left\{ \frac{m^{m-1}}{\Gamma(m-1)} \left(\frac{1}{\sigma}\right)^m e^{-m/\sigma} \right\}$, which is the inverse gamma distribution's pdf with parameters $\mathrm{m} - 1$ and m, and

$$h(\mu, \sigma) = \left\{ \frac{e^{m/\sigma}}{m^{m-1}} \left(1 + e^{-\left(\frac{\mu-\bar{x}}{\sigma/m}\right)}\right)^2 \prod_{i=1}^{m} \left\{ \frac{e^{-(\alpha(R_i+1)-1)\left(\frac{x_i-\mu}{\sigma}\right)}}{\left(1+e^{-\left(\frac{x_i-\mu}{\sigma}\right)}\right)^{\alpha(r_i+1)+1}} \right\} \right\}. \tag{17}$$

To find the estimate of the GLD parameters we do the following steps:

<u>Algorithm</u> 1:

Step 1: Generate σ from inverse gamma distribution with parameters $m - 1$ and m.

Step 2: Generate μ from the logistic distribution with parameters $\bar{x} = \frac{\sum_{i=1}^{m} x_i}{m}$ and σ/m, where σ is generated from Step 1.

Step 3: Repeat steps 1 and 2 to obtain $((\mu_1, \sigma_1), (\mu_2, \sigma_2), \ldots, (\mu_N, \sigma_N))$.

Step 4: Calculate the Bayes estimate as $\sum_{i=1}^{N} t(\mu_i, \sigma_i) h((\mu_i, \sigma_i)) / \sum_{i=1}^{N} h((\mu_i, \sigma_i))$.

4. Simulation Study

A Monte Carlo simulation study is conducted to investigate and compare the performance of the estimators under various experimental situations. We considered various progressive censoring schemes as explained in tables 1 – 6 below, corresponding to sample sizes of 50, 70 and 90. The location and scale parameters were set to zero and one respectively. The parameter α is taken to be 0.5, 1 and 1.5 to cover the various shapes of the distribution. We used the algorithm proposed by Balakrishnan and Sandhu (1996) to generate progressive Type II censored samples from Type II GLD. The findings are presented in Tables 1 and 6. We used 5000 replications in all our simulation runs.

The results include the biases and mean squared errors for the estimators developed in this paper in addition to the Lindley's approximation of the Bayes estimators and the maximum likelihood estimators developed and studied in Balakrishnan and Hossain (2007) and Rimawi and Baklizi (2021).

Table 1. Results of Simulation for parameter μ with GLD (α =1.5, μ = 0, , σ =1)

N	m	Scheme	MLE	Lindley	I.S	BLUE	BLEE
50	30	(0*29,20)					
		Bias	-0.0316	-0.0411	-1.7436	0.0295	0.0101
		MSE	0.0010	0.0017	3.0400	0.0660	0.0648
	30	(0*10,2*10,0*10)					
		Bias	-0.0293	-0.0466	-1.3551	2.2187	2.1775
		MSE	0.0009	0.0022	1.8362	4.9878	0.0648
	30	(20,0*29)					
		Bias	-0.0092	-0.0929	-0.8390	2.6077	2.5681
		MSE	0.0001	0.0086	0.7040	6.8653	0.0648
50	40	(0*39,10)					
		Bias	-0.0160	-0.0226	-1.2661	0.0172	0.0094
		MSE	0.0003	0.0005	1.6030	0.0497	0.0493
	40	(0*15,1*10,0*15)					
		Bias	-0.0137	-0.0421	-1.0062	0.9233	0.9108
		MSE	0.0002	0.0018	1.0125	0.9019	0.0493
	40	(10,0*39)					
		Bias	-0.0067	-0.0586	-0.7654	1.1288	1.1166
		MSE	0.0000	0.0034	0.5858	1.3237	0.0493
70	40	(0*39,30)					
		Bias	-0.0246	-0.0294	-1.7559	0.0285	0.0129
		MSE	0.0006	0.0009	3.0832	0.0506	0.0495
	40	(0*10,2*15,0*15)					
		Bias	-0.0246	-0.0366	-1.2942	2.6859	2.6498
		MSE	0.0006	0.0013	1.6750	7.2640	0.0495
70	50	(0*49,20)					
		Bias	-0.0147	-0.0224	-1.4289	0.0164	0.0085
		MSE	0.0002	0.0005	2.0419	0.0389	0.0385
	50	(0*20,2*10,0*20)					
		Bias	-0.0166	-0.0557	-1.0992	1.5217	1.5064
		MSE	0.0003	0.0031	1.2083	2.3542	0.0385
	50	(20,0*49)					
		Bias	-0.0101	-0.0557	-0.7403	1.8189	1.8040
		MSE	0.0001	0.0031	0.5481	3.3470	0.0385
90	50	(0*49,40)					
		Bias	-0.0248	-0.0259	-1.7668	0.0183	0.0053
		MSE	0.0006	0.0007	3.1217	0.0406	0.0401
	50	(0*15,2*20,0*15)					
		Bias	-0.0153	-0.0312	-1.3673	2.8937	2.8620
		MSE	0.0002	0.0010	1.8696	8.4135	0.0401
90	60	(0*59,30)					
		Bias	-0.0076	-0.0180	-1.5100	0.0143	0.0067
		MSE	0.0001	0.0003	2.2800	0.0323	0.0321
	60	(0*20,2*15,0*25)					
		Bias	-0.0067	-0.0252	-1.1241	2.0089	1.9925
		MSE	0.0000	0.0006	1.2636	4.0679	0.0321
	60	(30,0*59)					
		Bias	-0.0029	-0.0420	-0.7201	2.2792	2.2635
		MSE	0.0000	0.0018	0.5185	5.2268	0.0321

Table 2. Results of Simulation for parameter μ with GLD (α =1.0, μ = 0, , σ =1)

N	m	Scheme	MLE	Lindley	I.S	BLUE	BLEE
50	30	(0*29,20)					
		Bias	-0.0145	-0.0260	-1.2894	0.0078	-0.0010
		MSE	0.0002	0.0007	1.6625	0.0649	0.0648
	30	(0*10,2*10,0*10)					
		Bias	-0.0223	-0.0400	-0.8053	1.8900	1.8698
		MSE	0.0005	0.0016	0.6485	3.6369	0.0648
	30	(20,0*29)					
		Bias	-0.0030	-0.0845	-0.2378	2.4078	2.3881
		MSE	0.0000	0.0071	0.0565	5.8622	0.0648
50	40	(0*39,10)					
		Bias	-0.0044	-0.0148	-0.7395	-0.0040	-0.0056
		MSE	0.0000	0.0002	0.5468	0.0584	0.0584
	40	(0*15,1*10,0*15)					
		Bias	-0.0108	-0.0322	-0.4200	0.6519	0.6492
		MSE	0.0001	0.0010	0.1764	0.4834	0.0584
	40	(10,0*39)					
		Bias	0.0044	-0.0779	-0.1488	0.9265	0.9239
		MSE	0.0000	0.0061	0.0221	0.9169	0.0584
70	40	(0*39,30)					
		Bias	-0.0140	-0.0206	-1.3127	0.0046	-0.0028
		MSE	0.0002	0.0004	1.7231	0.0482	0.0482
	40	(0*10,2*15,0*15)					
		Bias	-0.0094	-0.0276	-0.7473	2.3503	2.3314
		MSE	0.0001	0.0008	0.5585	5.5720	0.0482
	40	(30,0*39)					
		Bias	-0.0027	-0.0730	-0.1854	2.8241	2.8059
		MSE	0.0000	0.0053	0.0344	8.0237	0.0482
70	50	(0*49,20)					
		Bias	-0.0020	-0.0148	-0.9359	-0.0020	-0.0045
		MSE	0.0000	0.0002	0.8759	0.0432	0.0432
	50	(0*20,2*10,0*20)					
		Bias	-0.0093	-0.0213	-0.5268	1.1800	1.1749
		MSE	0.0001	0.0005	0.2775	1.4356	0.0432
	50	(20,0*49)					
		Bias	-0.0081	-0.0561	-0.1273	1.5672	1.5622
		MSE	0.0001	0.0032	0.0162	2.4993	0.0432
90	50	(0*49,40)					
		Bias	-0.0120	-0.0179	-1.3227	0.0062	-0.0002
		MSE	0.0001	0.0003	1.7496	0.0385	0.0384
	50	(0*15,2*20,0*15)					
		Bias	-0.0150	-0.0156	-0.8236	2.5062	2.4892
		MSE	0.0002	0.0002	0.6784	6.3193	0.0384
90	60	(0*59,30)					
		Bias	-0.0057	-0.0175	-1.0327	0.0018	-0.0010
		MSE	0.0000	0.0003	1.0664	0.0346	0.0346
	60	(0*20,2*15,0*25)					
		Bias	-0.0045	-0.0221	-0.5478	1.6323	1.6258
		MSE	0.0000	0.0005	0.3001	2.6990	0.0346
	60	(30,0*59)					
		Bias	0.0012	-0.0510	-0.1158	2.0324	2.0260
		MSE	0.0000	0.0026	0.0134	4.1650	0.0346

Table 3. Results of Simulation for parameter μ with GLD (α =0.5, μ = 0, , σ =1)

N	m	Scheme	MLE	Bayesian Lindley's	Importance	BLUE	BLEE
50	30	(0*29,20)					
		Bias	0.0155	-0.0507	-0.3528	-0.0283	-0.0219
		MSE	0.0002	0.0026	0.1245	0.0997	0.0989
	30	(0*10,2*10,0*10)					
		Bias	-0.0015	-0.0836	0.3704	0.8626	0.8792
		MSE	0.0000	0.0070	0.1372	0.8430	0.0989
	30	(20,0*29)					
		Bias	0.0007	-0.2832	1.1404	1.6587	1.6758
		MSE	0.0000	0.0802	1.3005	2.8502	0.0989
50	40	(0*39,10)					
		Bias	0.0140	-0.0257	0.3215	-0.0389	-0.0319
		MSE	0.0002	0.0007	0.1033	0.1003	0.0987
	40	(0*15,1*10,0*15)					
		Bias	0.0081	-0.1002	0.8464	0.0444	0.0564
		MSE	0.0001	0.0100	0.7164	0.1007	0.0987
	40	(10,0*39)					
		Bias	0.0062	-0.2277	1.2132	0.4070	0.4193
		MSE	0.0000	0.0519	1.4719	0.2644	0.0987
70	40	(0*39,30)					
		Bias	0.0072	-0.0312	-0.4076	-0.0225	-0.0183
		MSE	0.0001	0.0010	0.1661	0.0720	0.0715
	40	(0*10,2*15,0*15)					
		Bias	-0.0026	-0.0649	0.4517	1.2506	1.2631
		MSE	0.0000	0.0042	0.2040	1.6354	0.0715
	40	(30,0*39)					
		Bias	0.0013	-0.2201	1.1894	2.0300	2.0426
		MSE	0.0000	0.0484	1.4147	4.1924	0.0715
70	50	(0*49,20)					
		Bias	0.0022	-0.0221	0.0621	-0.0313	-0.0263
		MSE	0.0000	0.0005	0.0039	0.0723	0.0713
	50	(0*20,2*10,0*20)					
		Bias	0.0092	-0.0650	0.7188	0.3066	0.3177
		MSE	0.0001	0.0042	0.5167	0.1653	0.0713
	50	(20,0*49)					
		Bias	0.0082	-0.1819	1.2491	0.8419	0.8532
		MSE	0.0001	0.0331	1.5603	0.7801	0.0713
90	50	(0*49,40)					
		Bias	0.0094	-0.0294	-0.4368	-0.0169	-0.0138
		MSE	0.0001	0.0009	0.1908	0.0563	0.0560
	50	(0*15,2*20,0*15)					
		Bias	0.0023	-0.0443	0.3366	1.3371	1.3468
		MSE	0.0000	0.0020	0.1133	1.8439	0.0560
	50	(40,0*49)					
		Bias	0.0066	-0.1864	1.2254	2.2811	2.2910
		MSE	0.0000	0.0348	1.5017	5.2593	0.0560
90	60	(0*59,30)					
		Bias	0.0086	-0.0152	-0.0725	-0.0217	-0.0178
		MSE	0.0001	0.0002	0.0053	0.0563	0.0558
	60	(0*20,2*15,0*25)					
		Bias	0.0041	-0.0531	0.6870	0.5890	0.5989
		MSE	0.0000	0.0028	0.4719	0.4027	0.0558
	60	(30,0*59)					
		Bias	0.0071	-0.1501	1.2685	1.1942	1.2042
		MSE	0.0001	0.0225	1.6090	1.4820	0.0558

Table 4. Results of Simulation for parameter σ with GLD (α =1.5, μ = 0, , σ =1)

N	m	Scheme	MLE	**Bayesian**	Importance	BLUE	BLEE
50	30	(0*29,20)					
		Bias	-0.0289	-0.0009	0.3606	0.0558	0.0291
		MSE	0.0008	0.0000	0.1300	0.0290	0.0253
	30	(0*10,2*10,0*10)					
		Bias	-0.0211	-0.0069	0.0971	1.2428	1.1861
		MSE	0.0004	0.0000	0.0094	1.5704	0.0253
	30	(20,0*29)					
		Bias	-0.0154	0.0060	0.0508	1.1522	1.0979
		MSE	0.0002	0.0000	0.0026	1.3535	0.0253
50	40	(0*39,10)					
		Bias	-0.0190	0.0063	0.1550	0.0460	0.0278
		MSE	0.0004	0.0000	0.0240	0.0198	0.0174
	40	(0*15,1*10,0*15)					
		Bias	-0.0152	0.0001	0.0689	0.6908	0.6614
		MSE	0.0002	0.0000	0.0047	0.4949	0.0174
	40	(10,0*39)					
		Bias	-0.0134	0.0010	0.0526	0.6559	0.6272
		MSE	0.0002	0.0000	0.0028	0.4479	0.0174
70	40	(0*39,30)					
		Bias	-0.0189	-0.0043	0.3667	0.0448	0.0247
		MSE	0.0004	0.0000	0.1345	0.0216	0.0192
	40	(0*10,2*15,0*15)					
		Bias	-0.0154	-0.0017	0.0614	1.4195	1.3730
		MSE	0.0002	0.0000	0.0038	2.0347	0.0192
70	50	(0*49,20)					
		Bias	-0.0153	0.0000	0.2044	0.0359	0.0210
		MSE	0.0002	0.0000	0.0418	0.0159	0.0144
	50	(0*20,2*10,0*20)					
		Bias	-0.0126	0.0015	0.0639	0.9904	0.9617
		MSE	0.0002	0.0000	0.0041	0.9955	0.0144
	50	(20,0*49)					
		Bias	-0.0100	0.0015	0.0413	0.9326	0.9047
		MSE	0.0001	0.0000	0.0017	0.8843	0.0144
90	50	(0*49,40)					
		Bias	-0.0178	-0.0025	0.3658	0.0389	0.0228
		MSE	0.0003	0.0000	0.1338	0.0173	0.0228
	50	(0*15,2*20,0*15)					
		Bias	-0.0108	-0.0062	0.0843	1.5284	1.4892
		MSE	0.0001	0.0000	0.0071	2.3518	0.0155
90	60	(0*59,30)					
		Bias	-0.0115	-0.0008	0.2394	0.0315	0.0188
		MSE	0.0001	0.0000	0.0573	0.0134	0.0123
	60	(0*20,2*15,0*25)					
		Bias	-0.0092	-0.0006	0.0529	1.2133	1.1860
		MSE	0.0001	0.0000	0.0028	1.4845	0.0123
	60	(30,0*59)					
		Bias	-0.0121	0.0044	0.0405	1.1111	1.0851
		MSE	0.0001	0.0000	0.0016	1.2469	0.0123

Table 5. Results of Simulation for parameter σ with GLD (α =1.0, μ = 0, , σ =1)

N	m	Scheme	MLE	Lindley	I.S	BLUE	BLEE
50	30	(0*29,20)					
		Bias	-0.0256	0.0105	0.1913	0.0559	0.0298
		MSE	0.0007	0.0001	0.0366	0.0285	0.0247
	30	(0*10,2*10,0*10)					
		Bias	-0.0189	-0.0015	0.0560	1.4334	1.3733
		MSE	0.0004	0.0000	0.0031	2.0801	0.0247
	30	(20,0*29)					
		Bias	-0.0144	-0.0049	0.0532	1.3737	1.3151
		MSE	0.0002	0.0000	0.0028	1.9125	0.0247
50	40	(0*39,10)					
		Bias	-0.0150	0.0064	0.0746	0.0473	0.0292
		MSE	0.0002	0.0000	0.0056	0.0199	0.0173
	40	(0*15,1*10,0*15)					
		Bias	-0.0160	0.0019	0.0416	0.7485	0.7182
		MSE	0.0003	0.0000	0.0017	0.5779	0.0173
	40	(10,0*39)					
		Bias	-0.0103	-0.0013	0.0398	0.7399	0.7098
		MSE	0.0001	0.0000	0.0016	0.5651	0.0173
70	40	(0*39,30)					
		Bias	-0.0173	0.0067	0.1925	0.0424	0.0228
		MSE	0.0003	0.0000	0.0371	0.0209	0.0188
	40	(0*10,2*15,0*15)					
		Bias	-0.0161	-0.0015	0.0332	1.6443	1.5946
		MSE	0.0003	0.0000	0.0011	2.7228	0.0188
	40	(30,0*39)					
		Bias	-0.0091	-0.0003	0.0343	1.5484	1.5005
		MSE	0.0001	0.0000	0.0012	2.4167	0.0188
70	50	(0*49,20)					
		Bias	-0.0130	0.0095	0.0982	0.0349	0.0202
		MSE	0.0002	0.0001	0.0096	0.0157	0.0142
	50	(0*20,2*10,0*20)					
		Bias	-0.0115	0.0011	0.0292	1.1164	1.0863
		MSE	0.0001	0.0000	0.0009	1.2608	0.0142
	50	(20,0*49)					
		Bias	-0.0088	-0.0008	0.0325	1.0805	1.0509
		MSE	0.0001	0.0000	0.0011	1.1820	0.0142
90	50	(0*49,40)					
		Bias	-0.0149	0.0006	0.1943	0.0357	0.0200
		MSE	0.0002	0.0000	0.0378	0.0167	0.0152
	50	(0*15,2*20,0*15)					
		Bias	-0.0129	0.0017	0.0374	1.7541	1.7123
		MSE	0.0002	0.0000	0.0014	3.0922	0.0152
90	60	(0*59,30)					
		Bias	-0.0126	0.0030	0.1154	0.0308	0.0183
		MSE	0.0002	0.0000	0.0133	0.0132	0.0121
	60	(0*20,2*15,0*25)					
		Bias	-0.0100	-0.0007	0.0269	1.3707	1.3420
		MSE	0.0001	0.0000	0.0007	1.8909	0.0121
	60	(30,0*59)					
		Bias	-0.0081	0.0004	0.0262	1.3090	1.2812
		MSE	0.0001	0.0000	0.0007	1.7258	0.0121

Table 6. Results of Simulation for parameter σ with GLD (α =0.5, μ = 0, , σ =1)

N	M	Scheme	MLE	Bayesian	Importance	BLUE	BLEE
50	30	(0*29,20)					
		Bias	-0.0206	0.0537	0.0684	0.0528	1.0274
		MSE	0.0004	0.0029	0.0047	0.0275	0.0241
	30	(0*10,2*10,0*10)					
		Bias	-0.0170	-0.0005	0.0779	1.7266	1.6609
		MSE	0.0003	0.0000	0.0061	3.0060	0.0241
	30	(20,0*29)					
		Bias	-0.0151	-0.0060	0.1052	1.8265	1.7584
		MSE	0.0002	0.0000	0.0111	3.3607	0.0241
50	40	(0*39,10)					
		Bias	-0.0124	0.0022	0.0422	0.0506	-0.0319
		MSE	0.0002	0.0000	0.0018	0.0208	0.0179
	40	(0*15,1*10,0*15)					
		Bias	-0.0169	0.0018	0.0696	0.8021	0.7697
		MSE	0.0003	0.0000	0.0048	0.6616	0.0179
	40	(10,0*39)					
		Bias	-0.0132	-0.0071	0.0963	0.8504	0.8172
		MSE	0.0002	0.0001	0.0093	0.7414	0.0179
70	40	(0*39,30)					
		Bias	-0.0189	0.0416	0.0590	0.0466	0.0275
		MSE	0.0004	0.0017	0.0035	0.0207	0.0182
	40	(0*10,2*15,0*15)					
		Bias	-0.0140	-0.0017	0.0670	2.0821	2.0260
		MSE	0.0002	0.0000	0.0045	4.3539	0.0182
	40	(30,0*39)					
		Bias	-0.0121	-0.0085	0.0948	2.1116	2.0549
		MSE	0.0001	0.0001	0.0090	4.4772	0.0182
70	50	(0*49,20)					
		Bias	-0.0093	0.0114	0.0332	0.0383	0.0234
		MSE	0.0001	0.0001	0.0011	0.0160	0.0143
	50	(0*20,2*10,0*20)					
		Bias	-0.0113	0.0030	0.0657	1.2792	1.2465
		MSE	0.0001	0.0000	0.0043	1.6509	0.0143
	50	(20,0*49)					
		Bias	-0.0106	-0.0089	0.0832	1.3285	1.2951
		MSE	0.0001	0.0001	0.0069	1.7796	0.0143
90	50	(0*49,40)					
		Bias	-0.0146	0.0334	0.0548	0.0354	0.0202
		MSE	0.0002	0.0011	0.0030	0.0161	0.0147
	50	(0*15,2*20,0*15)					
		Bias	-0.0134	-0.0001	0.0459	2.2139	2.1669
		MSE	0.0002	0.0000	0.0021	4.9164	0.0147
	50	(40,0*49)					
		Bias	-0.0081	-0.0030	0.0860	2.3172	2.2686
		MSE	0.0001	0.0000	0.0074	5.3844	0.0147
90	60	(0*59,30)					
		Bias	-0.0121	0.0179	0.0277	0.0312	0.0188
		MSE	0.0001	0.0003	0.0008	0.0131	0.0120
	60	(0*20,2*15,0*25)					
		Bias	-0.0076	-0.0008	0.0602	1.6402	1.6085
		MSE	0.0001	0.0000	0.0036	2.7023	0.0120
	60	(30,0*59)					
		Bias	-0.0088	-0.0036	0.0773	1.6694	1.6373
		MSE	0.0001	0.0000	0.0060	2.7990	0.0120

5. Real Data Example: Breakdown of an Insulating Fluid

To evaluate and analyze the quality of transformers and their insulating fluids, a variety of tests have been devised. To explain this, for example, let's consider the Dielectric Breakdown Test, which assesses an insulating liquid's capacity to endure electrical stress up to the point of failure. It displays the voltage at which there will be a breakdown. Moisture, dirt, and conductive particle contamination will induce failure at levels below what is considered tolerable. Nelson (1982) provided a data for the breakdown of an insulating fluid testing experiment. This data collection was examined and evaluated by Balakrishnan and Hossain (2007) examining Type II generalized logistic distribution inference under progressive Type II censoring. Balakrishnan and Hossain evaluated and examined the data set that fits the Type II Generalized Logistic Distribution and finding out that MLE and Approximate MLE are very close in the inferencing. In this example n= 19 and m=8 with α =1. The data and the results are shown in Tables 7 and 8 below.

Table 7: Insulating Fluid Data

I	1	2	3	4	5	6	7	8
x_i	-1.6608	-0.2485	-0.0409	0.2700	1.0224	1.5789	1.8718	1.9947
r_i	0	0	3	0	3	0	0	5

Table 8: Parameter Estimates Based on Insulating Fluid Data

Estimator	σ	μ
MLE	0.9027	1.8757
Bayesian – Lindley's Approach	0.9716	1.8511
Bayesian – Importance Sampling	1.4455	-0.2370
BLUE	1.4211	2.5867
BLEE	1.2786	2.4809

The results show that the MLE and the Bayes estimator based on Lindley's approximation are close to each other and somewhat smaller than the linear estimators. Based on our simulation study, the former estimators are more precise and reliable.

6. Summary and Conclusions

In this study, based on progressively type II censored data, we considered point estimation of location and scale parameters in type II Generalized Logistic Distribution (Type II GLD). We developed three estimators (ABLUE and ABLEE and Importance Sampling Estimator) for the unknown parameters. We also included the maximum likelihood estimators (MLE) and Bayes estimators approximated by the Lindley's Approach for comparison purposes.

The results of the simulation study reveal that MLE and Lindley's approximation to the Bayes estimator perform better than the other estimators developed in this paper. They have the smallest bias and MSE values as shown during the simulation study. As for the effect of the parameter α value on the location and scale estimator's bias and MSE values, ee got better results for smaller values of α.

The final conclusion of this work is that the MLE has the overall best performance for estimating the parameters of the type II generalized logistic distribution. However, for small sample sizes, numerical problems can occur. In such situations, the approximate linear estimators like the ABLUE and ABLEE can provide a viable alternative. The Bayes estimator performs very well too, especially the approximation based on Lindley's approach.

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