Original Research Article

Unit Monsef distribution with regression model

Abstract

A new distribution with one parameter will be introduced in this paper called unit Monsef distribution, derived from a transformation on Monsef distribution. it is probability function has been generated also distribution function. reliability function and hazard function, and it is statistical measures. parameters estimation is obtained by more than one method like maximum likelihood method, method of moment and least square method. Stress strengths parameter is investigated, the simulation scheme for unit Monsef parameters. Finally, real data is used for studying the flexibility of UNMD.

Keywords: Unit distributions. Monsef distribution, stress strength model, regression model.

Introduction

There are many phenomena have the uncertainty of a bounded, which lie on the interval (0, 1) such as the scores of ability tests and different rates. The transformation of the random variable generates new distribution with domain (0,1), which be more flexible and more fitting to data which lie in this interval. Unit distributions are the best to describe this phenomenon one of the most important distributions that use in this data is the Beta distribution (or the Pearson type IV distribution) because of its flexibility there is many unit distributions which introduced as alternative to beta distribution and Kumaraswamy distribution like the unit-Gamma distribution (1981), the unit-Logistic distribution (1982), a gamma regression model (2016) obtained depend on unit gamma distribution. unit Burr XII distribution and regression model (2021) of the most important regression models are beta regression model (2004) and Kumaraswamy regression model (2013). to in this paper a new unit distribution will be obtained depend on Monsef distribution (2021), which has more flexibility, this paper has organized as follow, the first section introduced the method of transformation , pdf and cdf of the

UMD, the survival and hazard rate and the behavior of the pdf and hazard function, the moments and ,Lorenz and Bonferroni curve are obtained in section 2, stress strength parameter were obtained in section 3, more than one estimation method were obtained to estimate the parameter of the new distribution in section 4, section 5 and 6 introduced the simulation study and application.

1. the Unit Monsef distribution

Monsef distribution was introduced by Monsef (2021) with cdf:

$$F(y) = 1 - \frac{e^{-x\theta}}{(\theta+1)^2+1}[(\theta(y+1)+1)^2+1], \quad y, \theta > 0 \quad (1)$$

and the pdf of *Monsef* distribution is:

$$f(y) = \frac{\theta^3}{2+\theta(2+\theta)}(y+1)^2 e^{-y\theta}, \quad y, \theta > 0 \quad (2)$$

From (1) using the transformation $X = \frac{y}{1+y}$ a new distribution was proposed with support on the unit interval. The c.d.f and the p.d.f of the UMD are:

$$F(x) = 1 - \frac{e^{\frac{x\theta}{-1+x}}\left(2+2x^2-2x(2+\theta)+\theta(2+\theta)\right)}{(-1+x)^2\left(2+\theta(2+\theta)\right)}, \quad 0 < x < 1,\ \theta > 0 \quad (3)$$

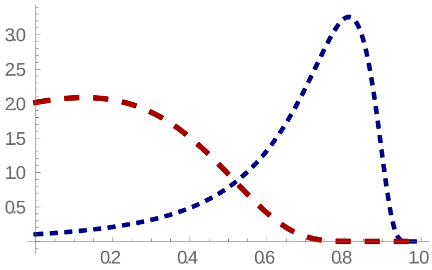
$$f(x) = \frac{e^{\frac{x\theta}{-1+x}}\theta^3}{(-1+x)^4(2+\theta(2+\theta))}, \quad 0 < x < 1,\ \theta > 0 \quad (4)$$

The survival and hazard functions can be obtained as following:

$$S(x) = \frac{e^{\frac{x\theta}{-1+x}}(2+2x^2-2x(2+\theta)+\theta(2+\theta))}{(-1+x)^2(2+\theta(2+\theta))}$$

$$H(x) = \frac{\theta^3}{(-1+x)^2(2+2x^2-2x(2+\theta)+\theta(2+\theta))}$$

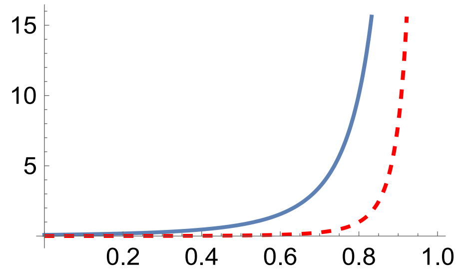
The behavior of the p.d.f and the hazard rate of the new distribution at different values of θ will be showing in the below figure.



----- at $\theta = 0.4$

----- at $\theta = 0.9$

Fig. (1). The behavior of the unit Monsef pdf



----- at $\theta = 0.2$

----- at $\theta = 0.7$

Fig. (2). The behavior of the unit Monsef hazard function

2. Moments and related measures

Some of the important statistical properties will be obtained in this section like moments, skewness and Kurtosis coefficients also Lorenz and Bonferroni curves.

2.1 Moments about the origin (raw moments).

The rth moment about the origin of the unit Monsef distribution is defined by

$$\acute{\mu}_r = \int_0^1 x^r \frac{e^{\frac{x\theta}{-1+x}}\theta^3}{(-1+x)^4(2+\theta(2+\theta))}\,dx\,,\quad r = 1,2,3,\ldots$$

The first four moments about the origin are written as:

$$\acute{\mu}_1 = \frac{2+\theta}{2+\theta(2+\theta)}\qquad , \qquad \acute{\mu}_2 = \frac{2}{2+\theta(2+\theta)}$$

$$\acute{\mu}_3 = \frac{2+(-1+\theta)\theta + e^{\theta}\theta^3(\text{CoshIntegral}[\theta] - \text{SinhIntegral}[\theta])}{2+\theta(2+\theta)}$$

$$\acute{\mu}_4 = \frac{2+\theta(-2+\theta(3+\theta)) + e^{\theta}\theta^3(4+\theta)(\text{CoshIntegral}[\theta] - \text{SinhIntegral}[\theta])}{2+\theta(2+\theta)}$$

2.2 Central moments.

The rth moment about the mean of the unit Monsef distribution is defined by

$$\mu_r = \int_0^1 (x-\mu)^r \frac{e^{\frac{x\theta}{-1+x}}\theta^3}{(-1+x)^4(2+\theta(2+\theta))}\,dx$$

Where

the variance $(\sigma^2) = \mu_2 = \frac{\theta^2}{(2+\theta(2+\theta))^2}$

$$\mu_4 = \frac{\theta^3(8+\theta(3+\theta(3+\theta))(11+\theta(3+\theta)^2)+e^{\theta}\theta(2+\theta(2+\theta))^2(6+\theta(6+\theta))(\text{CoshIntegral}[\theta]-\text{SinhIntegral}[\theta]))}{(2+\theta(2+\theta))^4}$$

The skewness and kurtosis measures can be obtained by:

$$\beta_1 = (2+\theta(2+\theta))^4(\frac{(1+\theta)(4+\theta(2+\theta))}{(2+\theta(2+\theta))^2} + e^{\theta}(\text{CoshIntegral}[\theta] - \text{SinhIntegral}[\theta]))^2$$

$$\beta_2 = \frac{8}{\theta} + (3+\theta(3+\theta))(11+\theta(3+\theta)^2) + e^{\theta}(2+\theta(2+\theta))^2(6+\theta(6+\theta))(\text{CoshIntegral}[\theta] - \text{SinhIntegral}[\theta])$$

coefficient The of variation (CV) for the new distribution can be written as :

$$CV = \frac{\sqrt{\frac{\theta^2}{(2+\theta(2+\theta))^2}(2+\theta(2+\theta))}}{2+\theta}$$

2.3 Lorenz and Bonferroni curves

The Lorenz and Bonferroni curves have importance in studying economics and in reliability applications.

The Lorenz curve for the unit Monsef distribution is defined as

$$L(F(x)) = \frac{2 + \frac{e^{\frac{x\theta}{-1+x}}(-2+x(-2+x-\theta)(-2+\theta)-\theta)}{(-1+x)^2} + \theta}{2+\theta}$$

The Bonferroni curve $B_F[F(x)]$ for the unit Monsef distribution is defined as

$$B_F[F(x)] = \frac{2+\frac{e^{\frac{x\theta}{-1+x}}(-2+x(-2+x-\theta)(-2+\theta)-\theta)}{(-1+x)^2}+\theta}{(2+\theta)(1-\frac{e^{\frac{x\theta}{-1+x}}(2+2x^2-2x(2+\theta)+\theta(2+\theta))}{(-1+x)^2(2+\theta(2+\theta))})}$$

3. stress strength reliability

Studying the survival and failure rate under conditions is an important to the system the system will perform its intended function adequately. Let X and Y are two independent unit Monsef r.vs with parameters θ_1 and θ_2 then the stress strength measures can be obtained as:

$$R = (\theta_2^3(\theta_1^6 + \theta_1^5(6 + 4\theta_2) + \theta_1^4(20 + 20\theta_2 + 6\theta_2^2) + 2\theta_2^2(2 + \theta_2(2 + \theta_2)) + 2\theta_1\theta_2(5 + \theta_2)(2 + \theta_2(2 + \theta_2)) + \theta_1^2(2 + \theta_2(2 + \theta_2))(20 + \theta_2(10 + \theta_2)) + \theta_1^3(40 + 2\theta_2(25 + 2\theta_2(6 + \theta_2)))))/((2 + \theta_1(2 + \theta_1))(\theta_1 + \theta_2)^5(2 + \theta_2(2 + \theta_2)))$$

4. The Estimation Methods

In this section some methods of parameter estimation will be introduced for the new distribution unit Monsef.

4.1 Maximum likelihood Estimation

Suppose there is a sample $x_1.\dots\dots\dots.x_n$of n iid observations. Coming from a distribution with probability density function identified in equation (4). The log–likelihood function is given by

$$ln\,\ell(x;\theta) = n\,ln\left[\frac{\theta^3}{(2+\theta(2+\theta))}\right] - \theta\sum_{i=1}^{n}\frac{x_i}{1-x_i} - 4\sum_{i=1}^{n} ln(1-x)$$

The partial derivatives of the log-likelihood for the parameter θ can be written:

$$\frac{\partial ln\ell(x;\theta)}{\partial\theta} = \frac{n(6+\theta(4+\theta))}{\theta(2+\theta(2+\theta))} - \sum_{i=1}^{n}\frac{x_i}{1-x_i}$$

The MLEs can be obtained by solving the derivatives of the $ln\,\ell(x;\theta)$ to zero. For the UMD the solution of the nonlinear equations has no closed form, also some numerical methods are needed for the solution.

4.2 the method of moment Estimator

By letting $x_1,\dots\dots\dots,x_n$be a r.s with size n from the UMD, the moment estimates (MOM), for the parameter θ is given solving equation

$$\mu = \frac{2+\theta}{2+\theta(2+\theta)}$$

By solving the above equation

$$\hat{\theta} = \frac{(2+\theta(2+\theta))(1-\frac{2(2+\theta)}{2+\theta(2+\theta)}+\sqrt{\frac{(2+\theta(4+\theta))^2}{(2+\theta(2+\theta))^2}})}{2(2+\theta)}$$

The solution of above equation has not closed form, but by using, μ we can compute the estimated values of θ.

4.3 Method of Least Square Estimation

The LSE and WLSE were proposed by Swain, Venkatraman & Wilson (1988) to estimate the parameters of Beta distributions.

The least square estimators can be obtained by minimizing

$$\sum_{j=1}^{n}\left[F(X_j)-\frac{j}{n+1}\right]^2$$

the least square estimator of θ can be introduced as:

$$\sum_{j=1}^{n}\left[1-\frac{e^{\frac{\theta x_j}{-1+x_j}}(2+\theta(2+\theta)-2(2+\theta)x_j+2x_j^2)}{(2+\theta(2+\theta))(-1+x_j)^2}-\frac{j}{n+1}\right]^2$$

The partial derivatives of the Least square method for the parameters θ can be obtained as follow:

$$\frac{\partial \mathrm{LSE}(x;\theta)}{\partial\theta}=\sum_{j=1}^{n}2\left[1-\frac{e^{\frac{\theta x_j}{-1+x_j}}(2+\theta(2+\theta)-2(2+\theta)x_j+2x_j^2)}{(2+\theta(2+\theta))(-1+x_j)^2}-\frac{j}{n+1}\right]\left(-\frac{e^{\theta\left(1+\frac{1}{-1+x_j}\right)}\theta^2x_j(6+\theta(4+\theta)+2x_j(-3-\theta+x_j))}{(2+\theta(2+\theta))^2(-1+x_j)^3}\right)$$

4.4 Method of Cramér- Von- Mises

Another method of estimation parameter is Cramér-von-Mises estimators of θ can be obtained by:

$$C(\theta,\lambda,p,c)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n}/\theta,\lambda,p,c)-\frac{2i-1}{2n}\right]^2$$

$$C(\theta,\lambda,p,c)=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\frac{e^{\frac{\theta x_j}{-1+x_j}}(2+\theta(2+\theta)-2(2+\theta)x_j+2x_j^2)}{(2+\theta(2+\theta))(-1+x_j)^2}-\frac{2i-1}{2n}\right]^2$$

5. Simulation study

A simulation Scheme for the UMD will be made by generating 5000 samples for the parameters θ at different sample sizes. The simulation nodes were chosen at different values at θ.

5.1 Simulation study for the new distribution using MLE.

Table (1) shows the MSE and the Bias of the estimates using MLE, since the MSE and the Bias values are decreases while the sample size increases.

N	$\theta = 0.1$		$\theta = 0.75$		$\theta = 1.5$		$\theta = 3.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
10	0.00041	0.0010	0.0251	0.0319	0.1213	0.0775	0.9700	0.2518
20	0.00017	0.0038	0.0107	0.0084	0.0495	0.0210	0.3713	0.0826
30	0.00011	0.0009	0.0069	0.0079	0.0318	0.0099	0.2337	0.0761
40	0.00009	0.0006	0.0059	0.0053	0.0273	0.0269	0.2023	0.0750
50	0.00006	0.0004	0.0041	0.0034	0.0187	0.0192	0.1363	0.0609
60	0.00006	0.0002	0.0034	0.0032	0.0157	0.0127	0.1143	0.0569
70	0.00005	0.0002	0.0033	0.0029	0.0151	0.0077	0.1099	0.0399
80	0.00004	0.0003	0.0025	0.0022	0.0116	0.0068	0.0845	0.0255
90	0.00003	0.0004	0.0021	0.0021	0.0098	0.0059	0.0702	0.0241
100	0.00003	0.0003	0.0019	0.0011	0.0089	0.0058	0.0643	0.0214

5.2 study for the new distribution using LSE method.

Table (2) shows the MSE and the Bias of the estimates using MLE, since the MSE and the Bias values are decreases while the sample size increases.

N	$\theta = 0.1$		$\theta = 0.75$		$\theta = 1.5$		$\theta = 3.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
10	0.0042	0.0013	0.02599	0.01116	0.12817	0.02981	1.0691	0.12038
20	0.00021	0.00081	0.01247	0.00650	0.05923	0.01677	0.46404	0.06643
30	0.00014	0.00075	0.00827	0.00595	0.03902	0.01419	0.30041	0.05410
40	0.00011	0.00065	0.00664	0.00514	0.03119	0.01202	0.23976	0.04709
50	0.000085	0.00051	0.00511	0.00395	0.02384	0.00894	0.18094	0.03659
60	0.000069	0.00034	0.00416	0.00263	0.01935	0.00575	0.14495	0.02633
70	0.000059	0.00028	0.00354	0.00211	0.01647	0.00445	0.12309	0.02239
80	0.000053	0.00027	0.00317	0.00306	0.01466	0.00636	0.10929	0.02134
90	0.000046	0.00024	0.00274	0.00199	0.01267	0.00387	0.09454	0.02037
100	0.000042	0.00022	0.00248	0.00164	0.01146	0.00299	0.08563	0.01779

5.3 Simulation study for the new distribution using Cramér- Von- Mises method.

Table (3) shows the MSE and the Bias of the estimates using Cramér- Von- Mises method, since the MSE and the Bias values are decreases while the sample size increases.

N	$\theta = 0.1$		$\theta = 0.75$		$\theta = 1.5$		$\theta = 3.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
10	0.00042	0.0018	0.0261	0.0155	0.1296	0.0406	1.091	0.1570
20	0.00021	0.0011	0.0125	0.0087	0.0596	0.0224	0.4699	0.0854
30	0.00013	0.0009	0.0083	0.0075	0.0392	0.0180	0.3035	0.0671
40	0.00011	0.0008	0.0066	0.0063	0.0313	0.0149	0.2414	0.0568
50	0.00009	0.0006	0.0051	0.0048	0.0239	0.0112	0.1821	0.0444
60	0.00007	0.0004	0.0042	0.0034	0.0194	0.0076	0.1458	0.0329
70	0.00006	0.0003	0.0035	0.0027	0.0165	0.0061	0.1237	0.0281
80	0.00005	0.0004	0.0032	0.0036	0.0147	0.0078	0.1098	0.0323
90	0.00004	0.0003	0.0027	0.0025	0.0127	0.0052	0.0949	0.0248
100	0.00004	0.0002	0.0024	0.0021	0.0114	0.0041	0.0859	0.0218

6.1 Application

In this section, application was applied on real data to shows the flexibility of UMD. The MLE is used to estimate the parameters. The statistics criteria (WT, AIC, BIC, AICC, HQIC, AD and CVM), K-S and p-value (K-S) were calculated for the fitted distributions.

The Education attainment

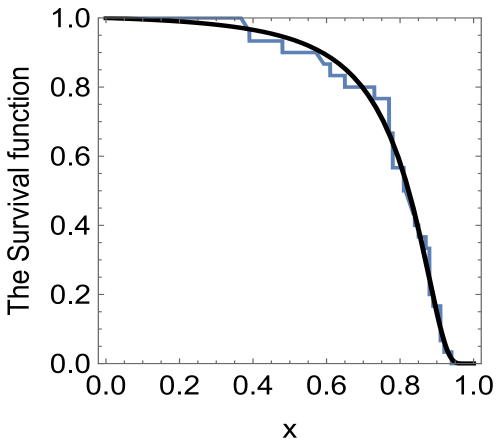
The used data set comes from the BLI of OECD countries. The data source is https://stats.oecd.org/index.aspx?DataSetCode=BLI. The educational attainment values of the OECD countries

Table (4), show the MLEs for the parameters, -log and K-S for the Unit Monsef distribution comparing with unit Lindley, Unit Weibull, Beta and Kumaraswamy Distribution.

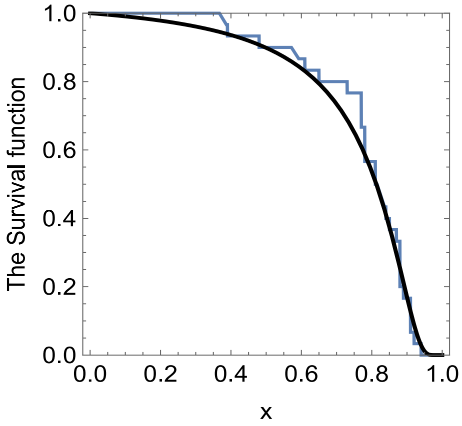
Distribution	**MLEs**	**-log**	**K-S**	**P-value**
Unit Monsef	$\hat{\theta} = 0.477$	-24.298	0.1013	0.9181
Unit Lindley	$\hat{\theta} = 0.3269$	-23.623	0.1559	0.4595
Kumaraswamy	$\hat{\alpha} = 6.154$ $\hat{\beta} = 2.466$	-21.096	0.7532	0.000
Beta	$\hat{\alpha} = 6.899$ $\hat{\beta} = 2.037$	-20.519	0.2027	0.1698
Unit Weibull	$\hat{\alpha} = 4.991$ $\hat{\beta} = 1.352$	-19.142	0.2101	0.1416

Table (5), show the statistics criteria for the Unit Monsef distribution comparing with unit Lindley, Unit Weibull, Beta and Kumaraswamy Distribution.

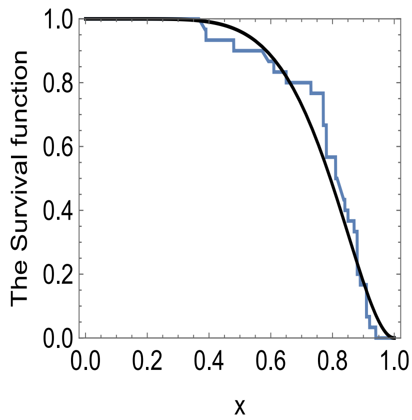
Distribution	**AIC**	**BIC**	**AICC**	**HQIC**	**AD**	**CVM**	**WT**
Unit Monsef	-46.596	-45.195	-45.596	-46.158	0.2537	0.03849	0.0384
Unit Lindley	-45.247	-43.846	-45.104	-44.799	0.42934	0.0724	0.05647
Beta	-37.519	-34.237	-36.595	-36.143	1.0531	0.1755	0.1383
Kumaraswamy	-38.192	-35.389	-37.747	37.295	67.523	6.895	1.443
Unit Weibull	-34.283	-33.839	-33.834	-33.387	1.2077	0.1975	0.1800



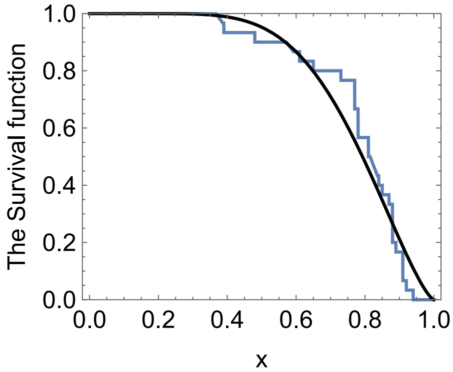
Unit Monsef Distribution



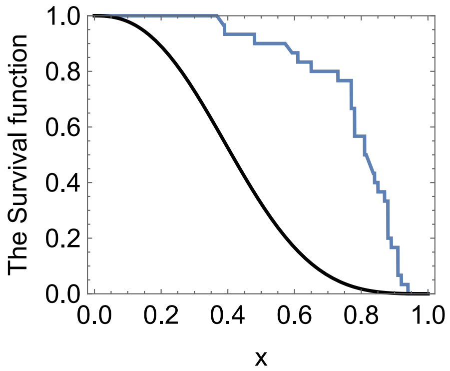
Unit Lindley Distribution



Beta Distribution



Unit Weibull Distribution



Kumaraswamy Distribution

7. Regression model

In this section, a regression structure for the unit-Monsef distribution will be obtained. In regression analysis it is very common to model the mean of the response. Since the unit-Monsef distribution has closed form expression for mean it can be used in this context. It is noteworthy that the re-parametrized p.d.f of unit Monsef can be written as follow:

$$f(x,\mu) = \frac{e^{\frac{x(1-2\mu+\sqrt{1-4(-1+\mu)\mu})}{2(-1+x)\mu}}\left(1-6\mu+4\mu^2+\sqrt{1-4(-1+\mu)\mu}\right)}{2(-1+x)^4\mu} \qquad (5)$$

Let $Y_1,\ldots,Y_n$be n independent random variables, where $Yi \sim UM(\mu_i), i = 1,\ldots,n$. The regression model is defined assuming that the mean of Yi satisfies the following functional relation

$$g(\mu_i) = x_i{}^{\tau}\beta$$

where $\beta = (\beta_1,\ldots,\beta_p)$ is a p-dimensional vector of regression coefficients ($p < n$) and $x_i = (x_{i1},\ldots,x_{ip})$ denotes the observations on p known covariates.

The proposed regression model is

$$logit(\mu_i) = \delta_0 + \delta_1 x_{i1} + \delta_2 x_{i2} + \delta_3 x_{i3}, \quad i = 1, \ldots, 30$$

The Education attainment

The used data set comes from the BLI of OECD countries. The educational attainment values of the OECD countries (y) is considered as response (dependent) variable The goal is to explore the effects of following covariates on the conditional mean of the response variable: homicide rate (HR), dwellings without basic facilities (DWBF), and labor market insecurity (LMI). The logit link function which ensures that the estimated mean lies between 0 and 1, is used for all fitted regression models. The fitted regression model is

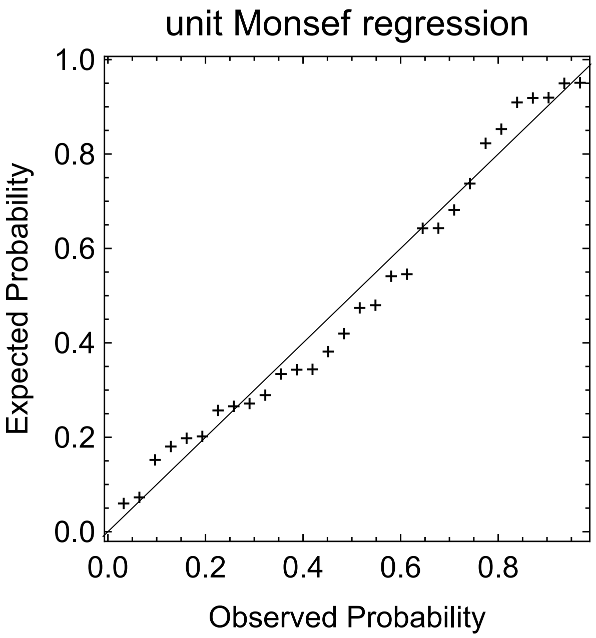
$$Logit(\mu_i) = \beta_0 + \beta_1\, HR + \beta_2\, DWBF + \beta_3 LMI$$

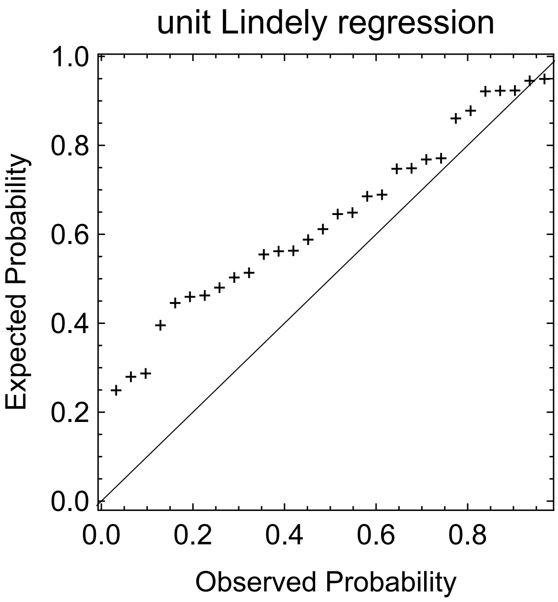
Table (6) show the estimated value of the covariates for the Unit Monsef and comparing regression models.

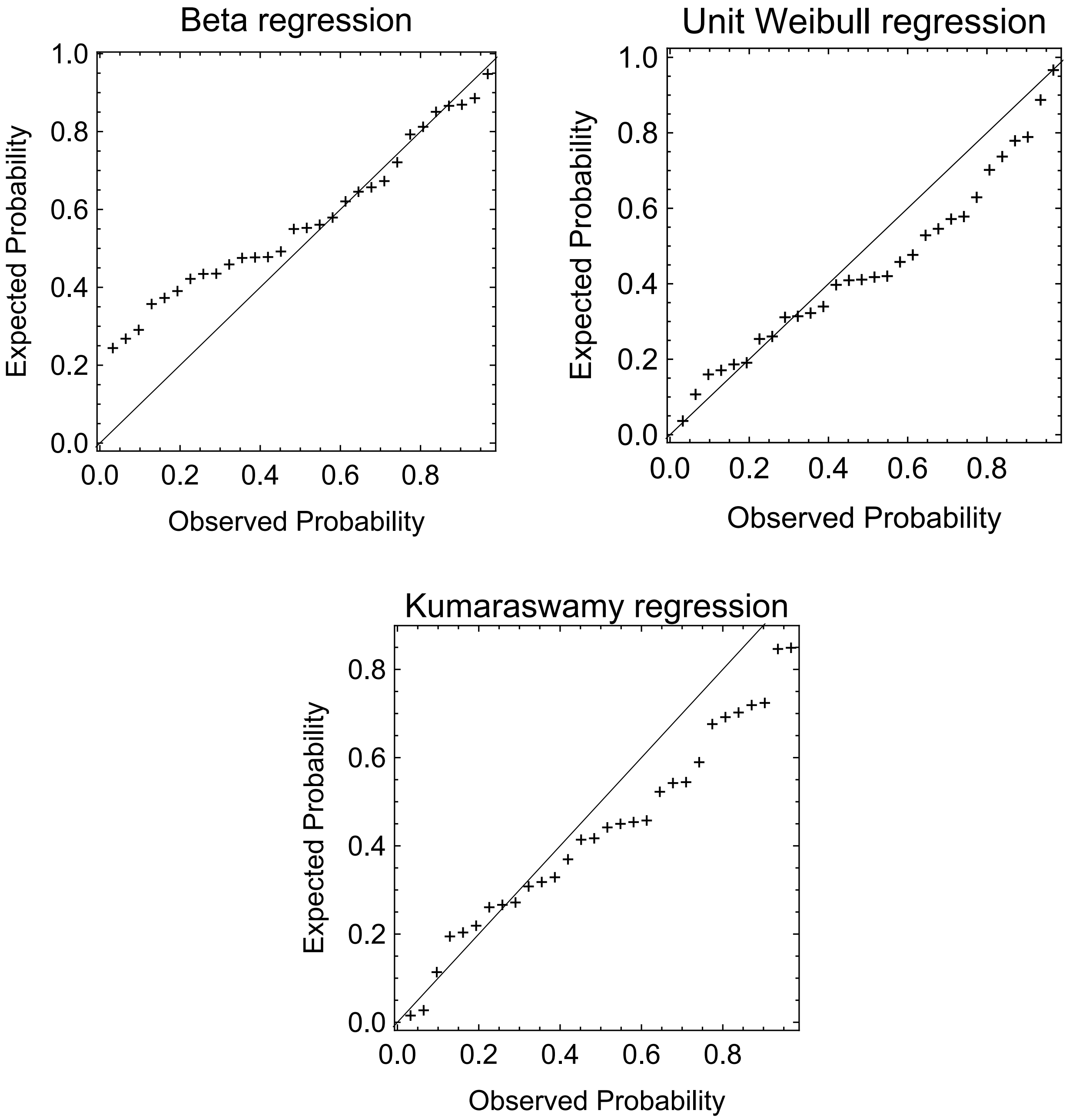
MLE	Unit Monsef	Unit Lindley	Unit Weibull	Kumaraswamy	Beta
$\boldsymbol{\beta}$	-------	-------	1.8894	3.2183	13.1724
$\boldsymbol{\delta_0}$	1.668	1.502	1.9856	1.8935	1.7749
$\boldsymbol{\delta_1}$	0.0213	-0.0241	0.05202	0.00756	-0.0216
$\boldsymbol{\delta_2}$	-0.0320	-0.0288	-0.09362	-0.06828	-0.0618
$\boldsymbol{\delta_3}$	-0.0446	-0.0411	-0.0667	0.04769	-0.0451

Table (7) show the estimated criteria for the Unit Monsef and comparing regression models.

Criteria	Unit Monsef	Unit Lindley	Unit Weibull	Beta	Kumaraswamy
AIC	-47.1035	-44.045	-45.352	-42.2007	-39.4771
BIC	-41.4987	-38.440	-38.346	-35.1947	-32.4711
-log	-27.5518	-26.023	-27.676	-26.1004	-24.7385






Beta regression
Expected Probability
Observed Probability
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
Unit Weibull regression
Expected Probability
Observed Probability
Kumaraswamy regression
Expected Probability
Observed Probability

Reference

[1] Abd El-Monsef, M.M.E; (2021). *Erlang mixture distribution with application on COVID-19 cases in Egypt. Intrentional Journal of Biomathematics.*

[2] Alzaatreh, A; Lee, C; Famoye, F (2013). *A new method for generating families of continuous distributions*. METRON. 71, 63-79.

[3] Andrews, D. F; Herzberg, A. M. (1985). Data: *A Collection of Problems from Many Fields for the Student and Research Worker*. New York: Springer Series in Statistics.

[4] Bayes, C.L; Bazán, J. L; De Castro, M. (2017). *A quntile parametric mixed regression model for bounded response variables*. Statistics and its interface. 10(3):483—493.

[5] *Dey, S; Raheem, E; Mukherjee, S (2017). Statistical Properties and Different Methods of Estimation of Transmuted Rayleigh Distribution.* Revista Colombiana de Estadística. 40(1):165-203.

[6] Ferrari, S; Cribari-Neto,F. (2004). *Beta regression for modeling rates and proportions.* Journal of applied statistics. 31(7):799—815.

[7] Grassia, A. (1977). *On a family of distributions with argument between 0 and 1 obtained by transformation of the Gamma distribution and derived compound distributions*. Aust.J. Stat.19:108–114.

[8] Korkmaz, M.Ç., Chesneau, C.; (2021). On the unit burr-xii distribution with the quantile regression modeling and applications. Computational and Applied Mathematics. 40(1):1—26.

[9] Kumarasamy, P.; (1980). *A generalized probability density function for double random processes.* Journal of hydrology.46 (1-2):79—88.

[10] Mazucheli, J; Menezesa, A. F.; Fernandes, L. B; de Oliveira, R. P. ; M. E. Ghitany, M. E.(2019). *The Unit-Weibull distribution as an alternative to the Kumaraswamy distribution for the modeling of quantiles conditional on covariates.* APPLIED STATISTICS.

[11] Mazucheli, J; Menezesa, A. F.; Fernandes, Chakraborty, S. (2018). *On the one parameter Unit-Lindley distribution and its associated regression model for proportion data.* Applied Statistics, 46:4, 700-714.

[12] Mitnik,P. A; Baek, S.(2013). *The Kumaraswamy distribution: median-dispersion re-parametrizations for regression modeling and simulation-based estimation*. Statistical Papers. 54(1):177—192.

[13] Mousa, A. M., El-Sheikh, A. A., Abdel-Fttah, M. A. (2016). A gamma regression for bounded continuous variables. Advanced and Applications in Statistics. 49 (4):305—326.

[14] Swain. James. J; Venkatraman, Sekhar; Wilson, James. R. (1988). *Least-squares estimation of distribution functions in johnson's translation system.* Journal of Statistical Computation. 29:4, 271-297.

[15] Tadikamalla, P. R.; Johnson, N. L;(1982). *Systems of frequency curves generated by transformations of logistic variables*. Biometrika. 69: 461–465.