Original Research Article

A New Trivariate Semicopula Using Rüschendorf Method

Abstract

In this paper we introduced semi copula function using Rüschendorf method, semicopula interested in correlate two or more random variables and it is more flexibile than traditional correlation approaches and dependency between variables. Every semicopula has density associated with it, which is similar to the probability density of a multivariate distribution. we aim to develop new trivariate semicopula under conditions which is trivariate cumulative distribution with uniform marginal distribution on the interval [0,1].

Keywords

Copula, Semicopula, Rüschendorf Technique, Superharmonic, Subharmonic, concave, Multivariate Kendall's Tau, Blomqvist.

- **Introduction**

On the other hand, any random variables is a multivariate copula, which is a d-variate density function whose univariate marginals are uniformly distributed on. whose constituents are spread uniformly on I is distributed according to some copula.

An m-dimensional copula (or m -copula) is a function C from the unit m-cube [0,1]m to the unit interval [0,1] according to Sklar's Theorem; a function is a copula if and only if the following properties hold:

(1) $C(1,...,1, a_n,1,...,1)$ = an for every $n \leq m$ and all an in [0,1] ;
(2) $C(a_1,..., a_m) = 0$ if $a_n = 0$ for any $n \leq m$; (Property 2 says that the joint probability of all outcomes is zero if the marginal probability of any outcome is zero).
(3) C is becoming more m-increasing. (Any m-dimensional interval's C-volume is non-negative, according to Property 3).

Property 2 and 3 are multivariate cdfs' general properties.

Copula modelling has gained popularity in recent years because it can model multivariate distributions without having to worry about the variables' dependence structure; where is a copula parameter called the dependence parameter, which measures reliance between the marginals.

There are numerous methods for creating multivariate distributions with uniform marginals. In the Rüschendorf technique, an arbitrary bivariate function that is integrable on $[0,1]^2$ is chosen, but we employed an arbitrary trivariate function that is also integrable on $[0,1]^3$ Pauline and Mahendran are two people who you should look into (2014) Given the use of the Rüschendorf approach and the flexibility of being able to select arbitrary functions to construct new copulas, we decided to focus on building a copula and semicopula using the Rüschendorf method and selecting an acceptable arbitrary trivariate function. This arbitrary trivariate function was chosen by us.

$$f(x,y,z) = (1-2z)\left(\sqrt{xy}\right)^{-1} \qquad (1)$$

$C(u,v,z)$ a trivariate function that maps $[0,1]^3$ to [0,1], is a copula if it meets the following criteria:

- ***Conditions of the boundary:***

$$C(u,v,0)=0 \quad ,C(u,0,z)=0, \quad C(0,v,z)=0$$

$$C(u,1,1)=\mathrm{u} \quad ,C(1,1,z)=\mathrm{z} \quad C(1,v,1)=v$$

- ***m-increasing propriety:*** m-increasing propriety is a term that refers to the process.
  Sklar (1959) invented copula functions, which can link multivariate marginal functions that are uniformly distributed on $[0, 1]^m$ to form trivariate functions, where fx, fy ,fz, and c are the density functions of F_x, F_y ,F_z, and C respectively

- **Rüschendorf Method for Building a New Trivariate Copula**

Assume that $f^1(x,y,z)$ has integral zero on the $[0,1]^3$ and that its three marginals are all equal to zero.

$$\int_0^1\int_0^1\int_0^1 f^1(x,y,z)\,dxdydz = 0$$

and

$$\int_0^1\int_0^1 f^1(x,y,z)\,dxdy = \int_0^1\int_0^1 f^1(x,y,z)\,dydz = \int_0^1\int_0^1 f^1(x,y,z)\,dxdz = 0$$

The density of a copula is then $1+f^1(x,y,z)$. There is, however, a need that the expression be non-negative. If this is not the case, but $1+f^1(x,y,z)$is bounded, then we can obtain a constant θ. The following steps can be used to create the function

$1+f^1(x,y,z)$:

the initial step:

$$f(x) = \int_0^1 \int_0^1 f(x, y, z) dydz,$$

$$f(y) = \int_0^1 \int_0^1 f(x, y, z) dxdz,$$

$$f(x) = \int_0^1 \int_0^1 f(x, y, z) dxdy$$

$$A = \int_0^1 \int_0^1 \int_0^1 f(x, y, z) dxdy\, dz$$

The second stage is to:

$$f^1(x, y, z) = f(x, y, z) - f(x) - f(y) - f(z) + A$$

Our new trivariate copula is derived from the previous phase. We choose a function

$$f(x, y, z) = (1 - 2z)\left(\sqrt{xy}\right)^{-1}$$

where *x*, *y*, and *z* are all in the range $[0,1]^3$. Following that, we compute $f(x), f(y), f(z)$and *A* as follows:

$$f(x) = \int_0^1 \int_0^1 (1 - 2z)\left(\sqrt{xy}\right)^{-1} dydz = 0$$

$$f(y) = \int_0^1 \int_0^1 (1 - 2z)\left(\sqrt{xy}\right)^{-1} dxdz = 0$$

$$f(z) = \int_0^1 \int_0^1 (1 - 2z)\left(\sqrt{xy}\right)^{-1} dxdy = (4 - 8z)$$

and

$$A = \int_0^1 \int_0^1 \int_0^1 (1 - 2z)\left(\sqrt{xy}\right)^{-1} dxdy = 0$$

We are going to build.

$$f^1(x, y, z) = f(x, y, z) - f(x) - f(y) - f(z) + A$$

because, in our situation,

$$f^1(x, y, z) = (1 - 2z)\left(\sqrt{xy}\right)^{-1} - (4 - 8z) \quad (4)$$

It is critical to verify that the double integral of function (4) on the unit square equals zero, which can be demonstrated in one of two ways.

$$\int_0^1\int_0^1 f^1(x,y,z)dxdy = \int_0^1\int_0^1 \left\{(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right\} dxdy = 0$$

$$\int_0^1\int_0^1 f^1(x,y,z)dydz = \int_0^1\int_0^1 \left\{(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right\} dydz = 0$$

and

$$\int_0^1\int_0^1 f^1(x,y,z)dxdz = \int_0^1\int_0^1 \left\{(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right\} dxdz = 0$$

Obviously, the integer

$$\int_0^1\int_0^1\int_0^1 f^1(x,y,z)dxdydz = \int_0^1\int_0^1\int_0^1 \left\{(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right\} dxdydz = 0$$

The density of copula is $g(x,y,z)$ as shown in the following stages.

$$g(x,y,z) = 1 + \theta\left[(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right], \theta \in [-1,1]$$

Proposition 1.1 is a phrase that can be written in a variety of ways.

$$g(x,y,z) = 1 + \theta\left[(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right], \theta \in [-1,1] \quad (5)$$

is the density of a copula.

Proof

It is sufficient to establish that $g(x,y,z) > 0$ to prove that $g(x,y,z)$ is the density of a copula.

The first and second order partial derivatives of $g(x,y,z)$ are found first, and they are

$$\frac{\partial g(x,y,z)}{\partial x} = -\frac{y(1-2z)}{2(xy)^{3/2}}\theta \qquad (6)$$

$$\frac{\partial^2 g(x,y,z)}{\partial x^2} = \frac{3y^2(1-2z)}{4(xy)^{5/2}}$$

$$\frac{\partial g(x,y,z)}{\partial y} = -\frac{x(1-2z)}{2(xy)^{3/2}}$$

$$\frac{\partial^2 g(x,y,z)}{\partial y^2} = \frac{3x^2(1-2z)}{4(xy)^{5/2}}$$

$$\frac{\partial g(x,y,z)}{\partial z} = 8 - \frac{2}{\sqrt{xy}}$$

$$\frac{\partial^2 g(x,y,z)}{\partial z^2} = 0$$

This paper is organized as follows :

- **The Copula of the Future**

To build the new copula, we must first extract the triple integral of *g*(*x*,*y*,*z*), which is *g*(*x*,*y*,*z*).

$$C(u,v,w) = \int_0^u \int_0^v \int_0^w g(x,y,z)\, dxdydz$$

$$C(u,v,w) = v\big(-4(-1+v)\sqrt{uw}\theta + u\big(w + 4(-1+v)\big)w\theta\big)$$

Conditions along the border

- **1st step:** Boundary conditions

$$C(0,v,w) = 0, \quad C(u,0,w) = 0 \text{ and } C(u,v,0) = 0$$

, as well as

$$C(u,1,1) = u, \quad C(1,v,1) = v \quad \text{and } C(1,1,z) = z$$

- **Step two**: increasing the value of your home

To show that *C*(*u*,*v*,*w*) in the cubic *C*(*u*,*v*,*w*) is rising satisfies the 3-increasing property, In reality, the mixed third partial derivative of *C* may be computed, because the mixed third partial derivative of *C* is independent of the order of the derivatives and is given by

$$\frac{\partial^3 C(u,v,w)}{\partial u \partial v \partial w} = 1 + \frac{(-1+2v)\big(-1+4\sqrt{uw}\theta\big)}{\sqrt{uw}}$$

We notice that *C* is unknown if *u=0*, *v=0*, or both of them equal zero at the same time. Whether *C* is an increasing or decreasing function depends on the parameterθ *;* if *C* is an increasing function, the function *C* is called copula; if $\theta \in (0,1]$,C is a decreasing function (so it does not satisfy the increasing property), the function *C* is called semicopula, $\theta \in [-1,0)$.

Semicopula function generalises both the concept of copula and the concept of triangular norm see Klement et al., semicopula function has all the principle properties of a copula except increasing property see F. Durante (2006), semicopula function has all the principle properties of a copula except increasing property see F. Durante (2006), semicopula function has all the principle properties of a copula except increasing property see F. Durante (2006), semicopul (2000).

- **Semicopulas Properties**

- **Semicopulas in Superharmonic and Subharmonic Modes**

Let C be a semicopula with second-order partial derivatives that are continuous on$(0,1)^2$. If C satisfies Laplace's equation inI^2, then C is harmonic

$$\nabla^2 C(u) = \frac{\partial^2}{\partial u_1^2} C(u, v) + \frac{\partial^2}{\partial u_2^2} C(u) + \ldots + \frac{\partial^2}{\partial v_m^2} C(u) = 0$$

Π It is the only harmonic semicopula.

A semicopula C is subharmonic if$\nabla^m C(u, v) \geq 0$ and superharmonic if $\nabla^2 C(u, v) \leq 0$, see Axler et al. for additional information on harmonic function theory (2001). It's a simple calculus problem to prove that our semicopula function is superharmonic since, $\nabla^m C(u, v, w) \leq 0$,and

$$\frac{\partial^2 C(u, v, w)}{\partial u^2} = \frac{(v-1) v w^2 \theta}{(uw)^{3/2}}$$

$$\frac{\partial^2 C(u, v, w)}{\partial v^2} = -8\left(-uw + \sqrt{uw}\right)\theta$$

and

$$\frac{\partial^2 C(u, v, w)}{\partial w^2} = uv\left(2 + \frac{(v-1)u}{(uw)^{3/2}}\right)\theta$$

then

$$\frac{\partial^2 C(u, v, w)}{\partial u^2} + \frac{\partial^2 C(u, v, w)}{\partial v^2} + \frac{\partial^2 C(u, v, w)}{\partial w^2} =$$
$$\frac{(v-1) v w^2 \theta}{(uw)^{3/2}} + -8\left(-uw + \sqrt{uw}\right)\theta + uv\left(2 + \frac{(v-1)u}{(uw)^{3/2}}\right)\theta \leq 0$$

It is undeniable that

$$\left[\frac{(v-1)v\sqrt{uw}}{u^2} + 2u(v + 4w) + \frac{\sqrt{uw}\left((v-1)v - 8w^2\right)}{w^2}\right]\theta \leq 0$$

$$\frac{\partial^2 C(u, v, w)}{\partial u^2} + \frac{\partial^2 C(u, v, w)}{\partial v^2} + \frac{\partial^2 C(u, v, w)}{\partial w^2} \leq 0$$

so $\quad C(u, v, w) = v\left(-4(-1 + v)\sqrt{uw}\theta + u\left(w + 4(-1 + v)\right)w\theta\right)$

A superharmonic semicopula is the same way.

If function C is semicopula, the following sentences are equivalent:

1. C has a concave shape.
2. For all and any, C is super-homogeneous, $\forall u, v, w \in [0,1]$and $\lambda \in [0,1]$
3. C is an idempotent variable.
4. M is concave, therefore C=M.

All semicopulas' class C is a convex and log-convex subclass of all copulas' class C. Furthermore, C is a complete lattice since it is closed under suprema and infima and compact with regard to the topology of uniform convergence.

For a semicopula, the following qualities are equivalent:

1- Semicopula S is positively homogeneous.
2- In both variables, S has linear sections.
3- Π = S.

When our function satisfying the increasing property for d observed variables in terms of *p* latent variables is presented in terms of bivariate copulas that link observed data to latent variables, general conditional independence models for copula function *C* are presented in terms of bivariate copulas that link observed data to latent variables See Nelsen, R. B. (1991, 2006) and Jogdeo (1991, 2006) for examples of how we analyse the reliance of our copula or the link between random variables (1982).

- **The properties of a measure of random variable dependence are as follows:**

- Multivariate Copula Kendall's Tau

$$\tau_d = \frac{1}{2^{d-1}-1}\left\{-1 + 2^d \int_{[0,1]^d} C(u)dC(u)\right\}$$

d=3 is our choice.

$$\tau_3 = \frac{1}{2^2-1}\left\{-1 + 2^3 \int_0^1 \int_0^1 \int_0^1 v\big(-4(-1+v)\sqrt{uw}\theta + u\big(w + 4(-1+v)\big)w\theta\big)\right\} dudvdw$$

$$\tau_3 = \frac{28}{81}\theta$$

- **Blomqvist**

Blomqvist's β or the medial correlation coefficient is the name given to this statistic. Blomqvist's β has certain advantages over competing measures such as Spearman's ρ or Kendall's τ as a measure of multivariate dependence. Blomqvist's β is the low computational complexity of its estimation, which is often not possible for the previously mentioned dependence measures, see Joe (1997), Chapter 5; Blomqvist's β is the explicit derivation of its whenever the copula is of explicit form, which is often not possible for the previously mentioned dependence measures, see Joe (1997), Chapter 5; Blomqvist's β is the low computational complexity of its estimation. While most Spearman's ρ and Kendall's τ implementations have a complexity of n^2, Blomqvist's β estimation has a complexity of n. This is due to the fact that just the median of the univariate margins must be established (see Schöning) (1997).

$$\beta = \frac{2^d C(1/2) - 1}{2^{d-1} - 1}$$

$$\beta = \frac{2^3 C(1/2) - 1}{2^2 - 1} = \frac{2}{3}\theta$$

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