

Fixed Point Results for Rational Type Contraction in A -Metric Spaces

ABSTRACT

The goal of this paper is to define rational contraction in the context of A -metric spaces and to develop various fixed-point theorems in order to elaborate, generalize, and synthesize several previously published results. Finally, to illustrate the new theorem, an example is given.

KEYWORDS: A -metric space; rational contraction; fixed point.

MSC: Primary 47H10; secondary 54H25

1. Introduction

In science and mathematics, fixed point theory is very important. Due to its wide range of applications in domains such as nonlinear analysis, topology, and engineering challenges, this field has drawn a lot of interest from academics in the last two decades. The Banach fixed-point theorem [3] is a key tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, as well as a constructive method for finding them. Bakhtin [2] proposed the concept of a b -metric space as a generalization of metric spaces in 1989, and numerous scholars used it to fixed point theory. Many b -metric space discoveries were expanded by Czerwik [4-5] in 1993. An S -metric space was introduced by Sedghi et al. [7]. The S -metric space is a space with three dimensions. S_b -metric space was introduced by Souayah et al. [8], and several fixed point theorems were developed. The concept of A -metric space was established by Abbas et al. [1], which is a generalisation of the S -metric space.

Das and Gupta [6] derived the first fixed point theorem for rational contractive type conditions.

Theorem 1.1 (see [6]). Let (Y, d) be a complete metric space, and let $\mathcal{T}: Y \rightarrow Y$ be a self-mapping. If there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ such that

$$d(\mathcal{T}\eta, \mathcal{T}\mu) \leq \alpha d(\eta, \mu) + \beta \frac{[1 + d(\eta, \mathcal{T}\eta)]d(\mu, \mathcal{T}\mu)}{1 + d(\eta, \mu)} \tag{1.1}$$

for all $\eta, \mu \in Y$, then \mathcal{T} has a unique fixed point $\eta^* \in Y$.

The purpose of this article is to define Dass and Gupta’s rational contraction in the context of A -metric spaces and establish some fixed-point theorems to elaborate, generalize and synthesize several known results in the literature. Finally, the example is presented to support the new theorem proved.

2. Preliminaries

In this part, some useful notions and facts will be given.

In this part, some useful notions and facts will be given. In 2015, Abbas et al. [1] introduced the notion of A -metric space.

Definition 7.2.1 (see [1]) Let Y be a nonempty set. A mapping $A: Y^n \rightarrow [0, +\infty)$ is called an A -metric on Y if and only if for all $\sigma_i, a \in Y, i = 1, 2, 3, \dots, n$: the following conditions hold:

- (A1). $A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) \geq 0$,
- (A2). $A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) = 0$ if and only if $\sigma_1 = \sigma_2 = \dots = \sigma_{n-1} = \sigma_n$,
- (A3). $A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) \leq A(\sigma_1, \sigma_1, \sigma_1, \dots, (\sigma_1)_{n-1}, a)$
 $+ A(\sigma_2, \sigma_2, \sigma_2, \dots, (\sigma_2)_{n-1}, a)$
 $+ A(\sigma_3, \sigma_3, \sigma_3, \dots, (\sigma_3)_{n-1}, a) + \dots$
 $+ A(\sigma_{n-1}, \sigma_{n-1}, \sigma_{n-1}, \dots, (\sigma_{n-1})_{n-1}, a)$
 $+ A(\sigma_n, \sigma_n, \sigma_n, \dots, (\sigma_n)_{n-1}, a)]$

The pair (Y, A) is called an A -metric space.

The following is the intuitive geometric example for A -metric spaces.

Example 7.2.2 (see [1]) Let $Y = [1, +\infty)$. Define $A: Y^n \rightarrow [0, +\infty)$ by

$$A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) = \sum_{i=1}^n \sum_{i < j} |\sigma_i - \sigma_j|$$

for all $\sigma_i \in Y, i = 1, 2, \dots, n$.

Example 7.2.3 (see [1]) Let $Y = \mathbb{R}$. Define $A: Y^n \rightarrow [0, +\infty)$ by

$$\begin{aligned} A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_{n-1}, \sigma_n) &= \left| \sum_{i=2}^n \sigma_i - (n-1)\sigma_1 \right| \\ &+ \left| \sum_{i=3}^n \sigma_i - (n-2)\sigma_2 \right| + \dots \\ &+ \left| \sum_{i=n-3}^n \sigma_i - 3\sigma_{n-3} \right| \\ &+ \left| \sum_{i=n-2}^n \sigma_i - 2\sigma_{n-2} \right| \\ &+ |\sigma_n - \sigma_{n-1}| \end{aligned}$$

for all $\sigma_i \in Y, i = 1, 2, \dots, n$.

Lemma 7.2.4 (see [1]) Let (Y, A) be an A -metric space. Then for all $\sigma, \varsigma \in Y$,

$$A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \varsigma) = A(\varsigma, \varsigma, \varsigma, \varsigma, \dots, (\varsigma)_{n-1}, \sigma)$$

Lemma 7.2.5 (see [1]) Let (Y, A) be an A -metric space. Then for all $\sigma, \varsigma, \xi \in Y$,

$$\begin{aligned} A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \xi) &\leq (n-1)A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \varsigma) \\ &+ A(\xi, \xi, \xi, \xi, \dots, (\xi)_{n-1}, \varsigma) \end{aligned}$$

and

$$\begin{aligned} A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \xi) &\leq (n-1)A(\sigma, \sigma, \sigma, \sigma, \dots, (\sigma)_{n-1}, \varsigma) \\ &+ A(\varsigma, \varsigma, \varsigma, \varsigma, \dots, (\varsigma)_{n-1}, \xi) \end{aligned}$$

Lemma 7.2.6 (see [1]) Let (Y, A) be an A -metric space. Then $(Y \times Y, D_A)$ is an A -metric space on $Y \times Y$, where D_A is given by for all $\sigma_i, \zeta_j \in Y, i, j = 1, 2, \dots, n$:

$$\begin{aligned} D_A((\sigma_1, \zeta_1), (\sigma_2, \zeta_2), (\sigma_3, \zeta_3), \dots, (\sigma_n, \zeta_n)) \\ = A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n) + A(\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_n). \end{aligned}$$

Definition 7.2.7 (see [1]) Let (Y, A) be an A -metric space. Then

1. A sequence $\{\sigma_k\}$ is called convergent to σ in (Y, A) if

$$\lim_{k \rightarrow +\infty} A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, we have

$$A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma) \leq \epsilon$$

and we write $\lim_{k \rightarrow +\infty} \sigma_k = \sigma$.

2. A sequence $\{\sigma_k\}$ is called Cauchy in (Y, A) if

$$\lim_{k, m \rightarrow +\infty} A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma_m) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, m \geq n_0$, we have

$$A(\sigma_k, \sigma_k, \sigma_k, \sigma_k, \dots, (\sigma_k)_{n-1}, \sigma_m) \leq \epsilon.$$

3. (Y, A) is said to be complete if every Cauchy sequence in (Y, A) is a convergent.

Lemma 7.2.8 (see [1]) Let (Y, A) be an A -metric space. If the sequence $\{\sigma_k\}$ in Y converges to σ , then σ is unique.

Lemma 7.2.9 (see [1]) Every convergent sequence in A -metric space (Y, A) is a Cauchy sequence.

Lemma 7.2.10 (see [1]) Let (Y, A) be an A -metric space. Then for all $\sigma, \zeta \in Y$,

$$A(\sigma, \sigma, \sigma, \dots, \sigma, \zeta) = A(\zeta, \zeta, \zeta, \dots, \zeta, \sigma)$$

Lemma 7.2.11 (see [1]) Let (Y, A) be an A -metric space. Then for all $\sigma, \zeta, \xi \in Y$,

$$A(\sigma, \sigma, \sigma, \dots, \sigma, \xi) \leq (n - 1)A(\sigma, \sigma, \sigma, \dots, \sigma, \zeta) + A(\xi, \xi, \xi, \dots, \xi, \zeta)$$

and

$$A(\sigma, \sigma, \sigma, \dots, \sigma, \xi) \leq (n - 1)A(\sigma, \sigma, \sigma, \dots, \sigma, \zeta) + A(\zeta, \zeta, \zeta, \dots, \zeta, \xi)$$

Definition 7.2.12 (see [1]) The A -metric space (Y, A) is said to be bounded if there exists a constant $r > 0$ such that $A_b(\sigma, \sigma, \sigma, \dots, \sigma, \zeta) \leq r$ for all $\sigma, \zeta \in Y$. Otherwise, Y is unbounded.

Definition 7.2.13 (see [1]) Given a point σ_0 in A -metric space (Y, A) and a positive real number r , the set

$$B(\sigma_0, r) = \{\zeta \in Y : A(\zeta, \zeta, \zeta, \dots, \zeta, \sigma_0) < r\}$$

is called an open ball centered at σ_0 with radius r .

The set

$$\overline{B(\sigma_0, r)} = \{\zeta \in Y : A(\zeta, \zeta, \zeta, \dots, \zeta, \sigma_0) \leq r\}$$

is called a closed ball centered at σ_0 with radius r .

Definition 7.2.14 (see [1]) A subset G in A -metric space (Y, A) is said to be an open set if for each $\sigma \in G$ there exists an $r > 0$ such that $B(\sigma, r) \subset G$. A subset $F \subset Y$ is called closed if $Y \setminus F$ is open.

Definition 7.2.15 (see [1]) Let (Y, A) be an A -metric space with $s \geq 1$. A map $f: Y \rightarrow Y$ is said to be contraction if there exists a constant $\lambda \in [0, 1)$ such that

$$A(f\sigma_1, f\sigma_2, f\sigma_3, \dots, f\sigma_n) \leq \lambda A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n)$$

for all $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n \in Y$. In case

$$A(f\sigma_1, f\sigma_2, f\sigma_3, \dots, f\sigma_n) < A(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n)$$

for all $\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_n \in Y, \sigma_i \neq \sigma_j$ for some $i \neq j, i, j \in \{1, 2, \dots, n\}$, f is called contractive mapping.

3. Main Results

We define rational contraction in A -metric space in this section, and then prove a fixed-point theorem.

Definition 3.1 Assume that (Y, A) be an A -metric space. A self mapping T on Y is said to be Dass and Gupta's rational contraction, if there exist $\lambda_1, \lambda_2 \in [0, 1)$ with $\lambda_1 + \lambda_2 < 1$ such that

$$A(T\eta, T\eta, \dots, T\eta, T\mu) \leq \lambda_1 \frac{[1+A(\eta, \eta, \dots, \eta, T\eta)]A(\mu, \mu, \dots, \mu, T\mu)}{1+A(\eta, \eta, \dots, \eta, \mu)} + \lambda_2 A(\eta, \eta, \dots, \eta, \mu) \tag{3.1}$$

for all $\eta, \mu \in Y$.

Theorem 3.1 Let (Y, A) be a complete A -metric space and let $T: Y \rightarrow Y$ be a Dass and Gupta's contraction satisfy the condition, then T has a unique common fixed point $\eta^* \in Y$. Moreover, for any $\eta_0 \in Y$, the sequence $\{\eta_k\} \subset Y$ defined by

$$\eta_{k+1} = T\eta_k, k \in \mathbb{N}, \tag{3.2}$$

is convergent to η^* .

Proof. First, we observe that T has at least one most one fixed point. In fact, $u, v \in Y$ are two fixed points of T with $u \neq v, i. e.$

$$A(u, u, u, \dots, v) > 0, Tu = u, Tv = v.$$

As $T: Y \rightarrow Y$ is a Dass and Gupta's contraction, so

$$\begin{aligned} A(u, u, \dots, u, v) &= A(Tu, Tu, \dots, Tu, Tv) \\ &\leq \lambda_1 \frac{[1+A(u, u, \dots, u, Tu)]A(v, v, \dots, v, Tv)}{[1+A(u, u, \dots, u, v)]} + \lambda_2 A(u, u, \dots, u, v) \\ &\leq \lambda_1 \frac{[1+A(u, u, \dots, u, u)]A(v, v, \dots, v, v)}{[1+A(u, u, \dots, u, v)]} + \lambda_2 A(u, u, \dots, u, v) \end{aligned}$$

$$< \lambda_2 A(u, u, \dots, u, v)$$

which is a contradiction

Let $\eta_0 \in Y$ be an arbitrary element and construct a sequence $\{\eta_k\}$ by the rule (3.2). As $T: Y \rightarrow Y$ is a Dass and Gupta's contraction, so

$$\begin{aligned} A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) &= A(T\eta_{k-1}, T\eta_{k-1}, \dots, T\eta_{k-1}, T\eta_k) \\ &\leq \lambda_1 \frac{[1+A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, T\eta_{k-1})]A(\eta_k, \eta_k, \dots, \eta_k, T\eta_k)}{1+A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k)} + \lambda_2 A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k) \\ &\leq \lambda_1 \frac{[1+A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k)]A_b(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1})}{1+A_b(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k)} + \lambda_2 A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k) \\ &= \lambda_1 A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + \lambda_2 A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k) \end{aligned}$$

which further yields that

$$A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) \leq \frac{\lambda_2}{1-\lambda_1} A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k) \tag{3.3}$$

Similarly,

$$\begin{aligned} A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k) &= A(T\eta_{k-2}, T\eta_{k-2}, \dots, T\eta_{k-2}, T\eta_{k-1}) \\ &\leq \lambda_1 \frac{[1+A(\eta_{k-2}, \eta_{k-2}, \dots, \eta_{k-2}, T\eta_{k-2})]A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, T\eta_{k-1})}{1+A(\eta_{k-2}, \eta_{k-2}, \dots, \eta_{k-2}, \eta_{k-1})} + \lambda_2 A(\eta_{k-2}, \eta_{k-2}, \dots, \eta_{k-2}, \eta_{k-1}) \\ &\leq \lambda_1 \frac{[1+A(\eta_{k-2}, \eta_{k-2}, \dots, \eta_{k-2}, \eta_{k-1})]A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k)}{1+A(\eta_{k-2}, \eta_{k-2}, \dots, \eta_{k-2}, \eta_{k-1})} + \lambda_2 A(\eta_{k-2}, \eta_{k-2}, \dots, \eta_{k-2}, \eta_{k-1}) \\ &= \lambda_1 A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k) + \lambda_2 A(\eta_{k-2}, \eta_{k-2}, \dots, \eta_{k-2}, \eta_{k-1}) \end{aligned}$$

The last inequality gives

$$A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k) \leq \frac{\lambda_2}{1-\lambda_1} A(\eta_{k-2}, \eta_{k-2}, \dots, \eta_{k-2}, \eta_{k-1}) \tag{3.4}$$

Let $\lambda = \frac{\lambda_2}{1-\lambda_1}$. Then from (3.3) and (3.4) and continuing the process, we get

$$A(\eta_{k-1}, \eta_{k-1}, \dots, \eta_{k-1}, \eta_k) \leq \lambda^k A(\eta_0, \eta_0, \dots, \eta_0, \eta_1), k \in \mathbb{N} \tag{3.5}$$

For $m, k \in \mathbb{N}$ with $m > k$, we have by repeated use of (A3)

$$\begin{aligned}
 A(\eta_k, \eta_k, \dots, \eta_k, \eta_m) &\leq (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + A(\eta_m, \eta_m, \dots, \eta_m, \eta_{k+1}) \\
 &\leq (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_m) \\
 &\leq (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + (n-1)A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_{k+2}) \\
 &\quad + A(\eta_m, \eta_m, \dots, \eta_m, \eta_{k+2}) \\
 &\leq (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + (n-1)A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_{k+2}) \\
 &\quad + A(\eta_{k+2}, \eta_{k+2}, \dots, \eta_{k+2}, S\eta_m) \\
 &\leq (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + (n-1)A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_{k+2}) \\
 &\quad + (n-1)A_b(\eta_{k+2}, \eta_{k+2}, \dots, \eta_{k+2}, \eta_{k+3}) + A(\eta_m, \eta_m, \dots, \eta_m, \eta_{k+3}) \\
 &\leq (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + (n-1)A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_{k+2}) \\
 &\quad + (n-1)A(\eta_{k+2}, \eta_{k+2}, \dots, \eta_{k+2}, \eta_{k+3}) \\
 &\quad + (n-1)A(\eta_{k+3}, \eta_{k+3}, \dots, \eta_{k+3}, \eta_{k+4}) + \dots \\
 &\quad + (n-1)A(\eta_{m-2}, \eta_{m-2}, \dots, \eta_{m-2}, \eta_{m-1}) \\
 &\quad + A(\eta_{m-1}, \eta_{m-1}, \dots, \eta_{m-1}, \eta_m) \\
 &\leq (n-1)[\lambda^k + \lambda^{k+1} + \dots + \lambda^{m-2}]A(\eta_0, \eta_0, \dots, \eta_0, \eta_1) \\
 &\quad + \lambda^{m-1}A(\eta_0, \eta_0, \dots, \eta_0, \eta_1) \\
 &= (n-1)\lambda^k[1 + \lambda + \lambda^2 + \dots + \lambda^{m-k-2}]A(\eta_0, \eta_0, \dots, \eta_0, \eta_1) \\
 &\quad + \lambda^{m-k-1}A_b(\eta_0, \eta_0, \dots, \eta_0, \eta_1) \\
 &\leq (n-1)\lambda^k[1 + \lambda + \lambda^2 + \lambda^3 + \dots]A(\eta_0, \eta_0, \dots, \eta_0, \eta_1) \\
 &\leq (n-1)\frac{\lambda^k}{1-\lambda}A(\eta_0, \eta_0, \dots, \eta_0, \eta_1) \tag{3.6}
 \end{aligned}$$

Since $\lambda < 1, \lambda \in [0, 1)$. By taking limit as $k, m \rightarrow +\infty$ in above inequality, we get

$$\lim_{k,m \rightarrow +\infty} A(\eta_k, \eta_k, \dots, \eta_k, \eta_m) = 0.$$

for all $m > k$. Therefore, $\{\eta_k\}$ is a Cauchy sequence in Y . But Y being complete, then there exists $u \in Y$ such that $\eta_k \rightarrow u$. We shall that u is a fixed point of T . We declare contradiction that $A(u, u, \dots, u, Tu) > 0$. By (3.1), we have

$$\begin{aligned} A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, Tu) &= A(T\eta_k, T\eta_k, \dots, T\eta_k, Tu) \\ &\leq \lambda_1 \frac{[1+A(\eta_k, \eta_k, \dots, \eta_k, T\eta_k)]A(u, u, \dots, u, Tu)}{1+A(\eta_k, \eta_k, \dots, \eta_k, u)} + \lambda_2 A(\eta_k, \eta_k, \dots, \eta_k, u) \\ &\leq \lambda_1 \frac{[1+A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1})]A(u, u, \dots, u, Tu)}{1+A(\eta_k, \eta_k, \dots, \eta_k, u)} + \lambda_2 A(\eta_k, \eta_k, \dots, \eta_k, u) \end{aligned}$$

Taking $k \rightarrow \infty$ in the above inequality, we get

$$A(u, u, \dots, u, Tu) \leq \lambda_1 A(u, u, \dots, u, Tu) < A(u, u, \dots, u, Tu)$$

This is contradiction. Hence, we have $A(u, u, \dots, u, Tu) = 0$, *i.e.* $Tu = u$. Therefore it concludes that $u \in Y$ is the unique fixed point of T .

The following Corollary is the A -metric version of Banach's contraction principle.

Corollary 3.1 Let (Y, A) be a complete A -metric space and let $T: Y \rightarrow Y$ be a mapping such that

$$A(T\eta, T\eta, \dots, T\eta, T\mu) \leq \lambda A(\eta, \eta, \dots, \eta, \mu) \tag{3.7}$$

for all $\eta, \mu \in Y$, where $\lambda < 1$. Then T has a unique fixed point $u \in Y$. Moreover, for any $\eta_0 \in Y$, the sequence $\{\eta_k\} \subset Y$ defined by

$$\eta_{k+1} = T\eta_k, k \in \mathbb{N},$$

is convergent to u .

Proof. It follows by $\lambda_1 = 0$ and $\lambda_2 = \lambda$ in Theorem 3.1.

Corollary 3.2 Let (Y, A) be a complete A -metric space and let $T: Y \rightarrow Y$ be a mapping such that

$$A(T\eta, T\eta, \dots, T\eta, T\mu) \leq \lambda \frac{[1+A(\eta, \eta, \dots, \eta, T\eta)]A(\mu, \mu, \dots, \mu, T\mu)}{1+A(\eta, \eta, \dots, \eta, \mu)} \tag{3.8}$$

for all $\eta, \mu \in Y$, where $\lambda < 1$. Then T has a unique fixed point $u \in Y$. Moreover, for any $\eta_0 \in Y$, the sequence $\{\eta_k\} \subset Y$ defined by

$$\eta_{k+1} = T\eta_k, k \in \mathbb{N},$$

is convergent to u .

Proof. Taking $\lambda_1 = \lambda$, $\lambda_2 = 0$ in Theorem 3.1, we obtain the desired result.

We conclude with an example.

Example 3.1 Let $Y = [0,1]$. Define $A: Y^n \rightarrow [0,+\infty)$ by

$$A_b(\eta_1, \eta_2, \eta_3, \dots, \eta_{n-1}, \eta_n) = \sum_{i=1}^n \sum_{i < j} |\eta_i - \eta_j|$$

for all $\eta_i \in Y, i = 1, 2, \dots, n$. Therefore, (Y, A) is an A -metric space. Let us define $T: Y \rightarrow Y$ as

$T\eta = \frac{\eta}{3}$ for all $\eta \in Y$. Then, for every $\eta, \mu \in Y$, the condition (3.1) holds for $\lambda_1 = \frac{1}{4}$, $\lambda_2 = \frac{1}{9}$.

Thus, T fulfill all assumptions of Theorem 3.1. Therefore T has a unique fixed point $0 \in Y$.

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