

Original Research Article

Gerber-Shiu function in a discrete-time risk model with dividend strategy

ABSTRACT

In this paper, a discrete-time risk model with dividend strategy and a general premium rate is considered. Under such a strategy, once the insurer's surplus hits a constant dividend barrier b , dividends are paid off to shareholders at α instantly. Using the roots of a generalization of Lundberg's fundamental equation and the general theory on difference equations, two difference equations for the Gerber-Shiu discounted penalty function are derived and solved. The analytic results obtained are utilized to derive the probability of ultimate ruin when the claim sizes is a mixture of two geometric distributions. Numerical examples are also given to illustrate the applicability of the results obtained.

Keywords: Compound binomial model; two-step premium; defective renewal equation; Gerber-Shiu discounted penalty function; dividend strategy.

1. INTRODUCTION

Risk theory has a long development time, Lundberg [1] and Gramer [2] established the connection of risk theory. The compound binomial model that was first proposed by Gerber [3] have received considerable attention. For instance, Shiu [4], Willmot [5] and Dickson [6] have analyzed the compound binomial model, for an generalization of the classical risk model see Cossette et al.[7] and Landriault [8]. Furthermore, in the discrete time risk model, the issue related to dividend is also widely considered.

Dividend strategies for insurance risk models were first proposed by DeFinetti [9] to reflect more realistically the surplus cash flows in an insurance portfolio. Because of the certainty of ruin for a risk model with a constant dividend barrier, the calculation of the Gerber-Shiu discounted penalty function is a major problem of interest in the context. Among the class of discrete-time risk models, Tan and Yang [10] derived a recursive algorithm to compute a particular class of Gerber-Shiu penalty functions in the framework of the compound binomial model with randomized dividend payments. Landriault [11] then generalize Tan and Yang's model to consider the compound binomial model with a multi-threshold dividend structure and randomized dividend payments. In the discrete time risk model, He and Yang [12] considered that dividends are paid randomly to shareholders and policyholders in the framework of the compound binomial model. In the framework of a discrete semi-Markov risk model, a randomized dividend policy is studied by Yuen et al. [13]. Zhang and Liu [14] consider a discrete-time risk model with a mathematically tractable dependence structure between interclaim times and claim sizes in the presence of an impulsive dividend strategy.

The paper is structured as follows: a brief description of the discrete-time model and the introduction of the Gerber-Shiu discounted penalty function are considered in Section 2. In section 3, we obtain and solve a non-homogeneous difference equation satisfied by the Gerber-Shiu discounted penalty function $m(u; b)$. Closed-form solutions for $m_b(u)$ are

obtained when the claim sizes is a mixture of two geometric distributions and corresponding numerical examples are also provided in Section 4.

2. THE MODEL

Throughout, denote by N the set of nature numbers and $N^+ = N \setminus \{0\}$. In the compound binomial model, the claim number process $\{N_k, k \in N\}$ is assumed to be a renewal process with independent and identically distributed (i.i.d.) interclaim times $\{W_j, j \in N^+\}$ having probability mass function (p.m.f.) $f_W(j) = q(1-q)^{j-1}$ for $l \in N^+$. Equivalently, the probability of having a claim is p ($0 < p < 1$) and the probability of no claim is $q = 1 - p$. The individual claim amount r.v.'s (random variables) $\{X_j, j \in N^+\}$ form a sequence of strictly positive, integer-valued and i.i.d. r.v.'s. We suppose that the r.v.'s $\{X_j, j \in N^+\}$ are distributed as a generic r.v. X with p.m.f. $f(x)$, probability generating function (p.g.f.) $\tilde{f}(x)$. Moreover, it is assumed that the r.v.'s W_1, W_2, \dots and X_1, X_2, \dots are mutually independent. Let $S_k = \sum_{i=1}^{N_k} X_i$ be the total amount of settled claims up the end of the k th time period with $S_0 = 0$.

Suppose that premiums are received at the beginning of each time period, and claims are paid out at the end of each time period. Denote $u \geq 0$ to be the initial surplus, $b > 0$ the constant barrier level, and $c_1 > 0$ the annual premium. Under such a strategy, let α ($0 < \alpha \leq c_1$) be the annual dividend rate, once the insurer's surplus at time k hits or exceeds a constant dividend barrier b , dividends are paid off to shareholders at α instantly. In this case, the net premium after dividend payments is $c_2 = c_1 - \alpha \geq 0$. And the corresponding surplus of the insurer at the end of the k th time period is $U_b(k)$ for $k = 1, 2, \dots$ can be described as

$$U_b(k) = \begin{cases} U_b(k-1) + c_1 - \eta_k X_k, & U_b(k-1) \leq b \\ U_b(k-1) + c_2 - \eta_k X_k, & U_b(k-1) > b \end{cases} \quad (1)$$

where $U_b(0) = u$, $\{\eta_k, k \in N\}$ is an independent and identically distributed Bernoulli sequence, we denote by $\eta_k = 1$ the event that having a claim at the time k and denote by $\eta_k = 0$ the event that no claim at the time k . We assume that

$P(\eta_i = 1) = p$ and $P(\eta_i = 0) = 1 - p = q$ and surplus process $U_b(k)$ has a positive drift by letting $c_2 > pE[X]$ (known as the positive security loading condition in ruin theory).

Define $\tau_b = \min\{k : U_b(k) < 0\}$ to be the time of ultimate ruin. Let v be a constant annual discount rate for each period. When ruin occurs, $U_b(\tau - 1)$ is the surplus one period prior to ruin and $|U_b(\tau)|$ is the deficit at ruin. For $v \in (0, 1]$, the well-known Gerber-Shiu discounted penalty function is then defined as

$$m(u; b) = E\left\{v^{\tau_b} \omega(U_{\tau_b-1}, |U_{\tau_b}|) I_{\{\tau_b < \infty\}} | U_0 = u\right\}, \quad (2)$$

where $\omega : N \times N^+ \rightarrow R$ is a penalty function and $I_{\{Q\}}$ is the indicator function of an event Q . Also, we consider some special cases of (2) with successively simplified the penalty functions. If $\omega(n_1, n_2) = 1$ for $(n_1, n_2) \in N \times N^+$, we get the generating function of the time to ruin, i.e.

$$m_b(u) = E\left\{v^{\tau_b} I_{\{\tau_b < \infty\}} | U_0 = u\right\}$$

3. THE GERBER-SHIU DISCOUNTED PENALTY FUNCTION

In this section, we derive two difference equations for the Gerber-Shiu discounted penalty function: one for the initial surplus below the barrier level b and the other for the initial surplus above the barrier level b . Clearly, the Gerber-Shiu discounted penalty function $m(u, b)$ behaves differently, depending on whether its initial surplus u is below or above the barrier level b . Hence, we write

$$m(u, b) = \begin{cases} m_1(u), & 0 \leq u < b \\ m_2(u), & u \geq b \end{cases}$$

In order to identify the structural form of the solution for the Gerber-Shiu discounted penalty function, three cases will be considered separately.

3.1 For initial surpluses less than the barrier b

In the first scenario, the initial surplus below the barrier b , for $u = 0, 1, \dots, b - c_1 - 1$, we have

$$\begin{aligned} m_1(u) &= vq m_1(u + c_1) + vp \sum_{j=1}^{u+c_1} m_1(u + c_1 - j) f(j) + vp \sum_{j=u+c_1+1}^{\infty} \omega(u + c_1; j - u - c_1) f(j) \\ &= vq m_1(u + c_1) + vp(m_1 * f)(u + c_1) + \gamma_1(u), \end{aligned} \quad (3)$$

where

$$\gamma_1(u) = vp \sum_{j=u+c_1+1}^{\infty} \omega(u + c_1; j - u - c_1) f(j).$$

and $m_1 * f$ holds for the convolution product of m_1 and f .

To state that (3) is a non-homogeneous difference equation of order c_1 , we re-express (3) according to the forward difference operator Δ and its property (see Chapter 2 of Kelly & Peterson [15]),

$$m(u + c) = \sum_{j=0}^c \binom{c}{j} \Delta^j m(u), \quad (4)$$

substituting (4) into (3) shows

$$m_1(u) = vq \sum_{j=0}^{c_1} \binom{c_1}{j} \Delta^j m_1(u) + vp \sum_{j=0}^{c_1} \binom{c_1}{j} \Delta^j (m_1 * f)(u) + \gamma_1(u), \quad (5)$$

for $u = 0, 1, \dots, b - c_1 - 1$, (5) can be simplified to

$$\sum_{j=0}^{c_1} a_{1,j} \Delta^j m_1(u) = \sum_{j=0}^{c_1} b_{1,j} \Delta^j (m_1 * f)(u) + \gamma_1(u), \quad (6)$$

where

$$a_{1,j} = I_{\{j=0\}} - vq \binom{c_1}{j}, \quad b_{1,j} = vp \binom{c_1}{j}.$$

and $A_1(z)$, $B_1(z)$ are polynomials (in z) defined as

$$A_1(z) = \sum_{j=0}^{c_1} a_{1,j} z^j, \quad B_1(z) = \sum_{j=0}^{c_1} b_{1,j} z^j.$$

becomes

$$A_1(\Delta)m_1(u) = B_1(\Delta)(m_1 * f)(u) + \gamma_1(u), \quad u = 0, 1, 2, \dots, b - c_1 - 1, \quad (7)$$

We know from (7) that $m_1(u)$ satisfies a non-homogeneous difference equation of order c_1 . From the general theory on difference equations, every solution to a c_1 -th order difference equation can be expressed as a particular solution to this difference equation plus a linear combination of c_1 linearly independent solutions to the associated homogeneous difference equation (cf. Elaydi [16], Theorem 2.30). Therefore, for $u = 0, 1, \dots, b - 1$, the Gerber-Shiu discounted penalty function can be expressed as

$$m_1(u) = \varphi_1(u) + \sum_{j=0}^{c_1-1} \alpha_{1,j} y_{1,j}(u), \quad u = 0, 1, 2, \dots, b - 1. \quad (8)$$

where $\{y_{1,j}(u)\}_{u=0}^{\infty}$ ($j = 0, 1, \dots, c_1 - 1$) are c_1 fundamental solutions to the following homogeneous difference equation

$$A_1(\Delta)y_1(u) = B_1(\Delta)(y_1 * f)(u) \quad u \geq 0. \quad (9)$$

$\{\varphi_1(u)\}_{u=0}^{\infty}$ is a particular solution to

$$A_1(\Delta)\varphi_1(u) = B_1(\Delta)(\varphi_1 * f)(u) + \gamma_1(u) \quad u \geq 0. \quad (10)$$

Combining (3) and (9), we get

$$y_1(u) = vqy_1(u + c_1) + vp(y_1 * f)(u + c_1). \quad (11)$$

Multiplying (11) by z^{u+c_1} and then summing over u from 0 to ∞ lead to

$$\sum_{u=0}^{\infty} z^{u+c_1} y_1(u) = vq \sum_{u=0}^{\infty} z^{u+c_1} y_1(u + c_1) + vp \sum_{u=0}^{\infty} z^{u+c_1} (y_1 * f)(u + c_1), \quad (12)$$

routine calculations lead to

$$z^{c_1} \tilde{y}_1(z) = vq \left[\tilde{y}_1(z) - \sum_{u=0}^{c_1-1} z^u y_1(u) \right] + vp \left\{ \tilde{y}_1(z) \tilde{f}(z) - \sum_{u=0}^{c_1-1} z^u (y_1 * f)(u) \right\}.$$

After some algebra, one could see that (12) can be written as

$$\tilde{y}_1(z) = \frac{-v \left\{ q \sum_{u=0}^{c_1-1} z^u y_1(u) + p \sum_{u=0}^{c_1-1} z^u (y_1 * f)(u) \right\}}{z^{c_1} - vq - vp \tilde{f}(z)}. \quad (13)$$

By choosing $y_{1,j}(u) = I_{\{j=u\}}$ for $j, u \in \{0, 1, \dots, c_1 - 1\}$. According to (13), the generating function associated to the fundamental solution $\{y_{1,j}(u)\}_{u=0}^{\infty}$ is

$$\tilde{y}_{1,j}(z) = \frac{-v \left\{ qz^j + p \sum_{u=j+1}^{c_1-1} z^u f(u-j) \right\}}{z^{c_1} - vq - vp \tilde{f}(z)} = \frac{-R_{1,j}(z)}{\tilde{h}_{1,1}(z) - \tilde{h}_{1,2}(z)} \quad u \geq 0, \quad (14)$$

where

$$\tilde{h}_{1,1}(z) = z^{c_1}, \quad \tilde{h}_{1,2}(z) = vq + vp \tilde{f}(z), \quad R_{1,j}(z) = v \left\{ qz^j + p \sum_{u=j+1}^{c_1-1} z^u f(u-j) \right\}.$$

Lemma 3.1: When $v \in (0, 1)$, the denominator in (14) has exactly c_1 zeros, say $\{z_i\}_{i=1}^{c_1}$ inside the unit circle $C = \{z : |z| = 1\}$.

Lemma 3.2: When $v = 1$, the denominator in (14) has exactly $c_1 - 1$ zeros, say $\{z_i\}_{i=1}^{c_1-1}$ inside the unit circle $C = \{z : |z| = 1\}$ and another trivial root $z_{c_1} = 1$.

For the rest of the paper, we assume that all $\{z_i\}_{i=1}^{c_1}$ are distinct, since the analysis of the multiple roots of Lundberg's generalized equation leads to tedious derivations.

Let $\pi_i(z) = \prod_{j=1}^{c_i} (z - z_j)$ and $\pi_i'(z_k) = \prod_{l=1, l \neq k}^{c_i} (z_k - z_l)$, from Liu and Bao [17], we have

$$\frac{\tilde{h}_{1,1}(z) - \tilde{h}_{1,2}(z)}{\pi_1(z)} = 1 - \nu \rho T_z T_{z_{c_1}} \dots T_{z_2} T_{z_1} f(c_1), \quad (15)$$

where T_z is an operator (see Li [18]) defined as

$$T_z y(c) = \sum_{u=0}^{\infty} z^u y(u+c) = \sum_{u=c}^{\infty} z^{u-c} y(u).$$

(14) can be rewrote as

$$\tilde{y}_{1,j}(z) = \frac{-R_{1,j}(z)}{\pi_1(z)} \cdot \frac{\pi_1(z)}{\tilde{h}_{1,1}(z) - \tilde{h}_{1,2}(z)}. \quad (16)$$

Regarding the numerator in (16), partial fractions yield the equivalent representation

$$\frac{-R_{1,j}(z)}{\pi_1(z)} = \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1'(z_k)} \cdot \frac{1}{z_k - z}. \quad (17)$$

By inserting (15) and (17) into (16), we obtain

$$\tilde{y}_{1,j}(z) = \nu \rho \tilde{y}_{1,j}(z) T_z T_{z_{c_1}} \dots T_{z_2} T_{z_1} f(c_1) + \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1'(z_k)} \frac{1}{z_k - z}. \quad (18)$$

Theorem 3.1: For $j = 0, 1, \dots, c_1 - 1$, $y_{1,j}(u)$ satisfies the following defective renewal equation

$$y_{1,j}(u) = \varsigma_1 \sum_{n=0}^u y_{1,j}(u-n) \chi_1(n) + \zeta_1(u), \quad (19)$$

where

$$\varsigma_1 = \nu \rho T_1 T_{z_{c_1}} \dots T_{z_2} T_{z_1} f(c_1), \quad \chi_1(n) = \frac{T_{z_{c_1}} \dots T_{z_2} T_{z_1} f(c_1 + n)}{T_1 T_{z_{c_1}} \dots T_{z_2} T_{z_1} f(c_1)}, \quad \zeta_1(u) = \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1'(z_k)} \left(\frac{1}{z_k} \right)^{u+1}.$$

Now, we turn our attention to the calculation of the particular solutions $\{\varphi_1(u)\}_{u=0}^{\infty}$, combining (3) and (10), $\{\varphi_1(u)\}_{u=0}^{\infty}$ satisfies

$$\varphi_1(u) = \nu \rho \varphi_1(u+c_1) + \nu \rho (\varphi_1 * f)(u+c_1) + \gamma_1(u). \quad (20)$$

We use a solution procedure analogous to the fundamental solutions, we get

$$\begin{aligned} \tilde{\varphi}_1(z) &= \frac{z^{c_1} \tilde{\gamma}_1(z) - \nu \left\{ q \sum_{u=0}^{c_1-1} z^u \varphi_1(u) + \rho \sum_{u=0}^{c_1-1} z^u (\varphi_1 * f)(u) \right\}}{z^{c_1} - \nu \rho - \nu \rho \tilde{f}(z)} \\ &= \frac{z^{c_1} T_z \gamma_1(0) - Q_{1,j}(z)}{z^{c_1} - \nu \rho - \nu \rho \tilde{f}(z)}. \end{aligned} \quad (21)$$

where $Q_{1,j}(z) = \nu \left\{ q \sum_{u=0}^{c_1-1} z^u \varphi_1(u) + \rho \sum_{u=0}^{c_1-1} z^u (\varphi_1 * f)(u) \right\}$ is a polynomial of degree $c_1 - 1$ (or less) in z . It is known from (35) in Liu and Zhang [14] that

$$\frac{z^{c_1} T_z \gamma_1(0) - Q_{1,j}(z)}{\pi_1(z)} = T_z T_{z_{c_1}} \dots T_{z_2} T_{z_1} \gamma_1(0). \quad (22)$$

By substituting (15) and (22) into (21), we get

$$\tilde{\varphi}_1(z) = \frac{T_z T_{z_{q_1}} \dots T_{z_2} T_{z_1} \gamma_1(0)}{1 - \nu \rho T_z T_{z_{q_1}} \dots T_{z_2} T_{z_1} f(c_1)} = \frac{\tilde{\gamma}_1(z)}{1 - \nu \rho T_z T_{z_{q_1}} \dots T_{z_2} T_{z_1} f(c_1)}. \quad (23)$$

Theorem 3.2: For $u \in N$, it holds that

$$\varphi_1(u) = \varsigma_1 \sum_{n=0}^u \varphi_1(u-n) \chi_1(n) + \gamma_1(u), \quad (24)$$

where $\gamma_1(u) = T_{z_{q_1}} \dots T_{z_2} T_{z_1} \gamma_1(u)$.

In the second scenario, for $u = b - c_1, \dots, b - 1$,

$$m_1(u) = \nu q m_2(u + c_1) + \nu \rho (m * f)(u + c_1) + \gamma_1(u). \quad (25)$$

3.2 For initial surpluses equal to or more than the barrier b

The last scenario, for $u \geq b$,

$$m_2(u) = \nu q m_2(u + c_2) + \nu \rho (m * f)(u + c_2) + \gamma_2(u), \quad (26)$$

$$\text{where } \gamma_2(u) = \nu \rho \sum_{j=u+c_2+1}^{\infty} \omega(u + c_2; j - u - c_2) f(j).$$

The structural form (8) for $m_1(u)$ is expressed in terms of the $\alpha_{1,j}$, and also depends on $m_2(u)$ in (26). In order to drive the solutions of $m(u)$, shifting the argument u in (26) by b units, for $u \geq 0$ (26) can be rewritten as

$$\begin{aligned} m_2(u + b) &= \nu q m_2(u + b + c_2) + \nu \rho \sum_{j=b}^{u+b+c_2} m_2(j) f(u + b + c_2 - j) \\ &\quad + \nu \rho \sum_{j=0}^{b-1} m_1(j) f(u + b + c_2 - j) + \gamma_2(u + b), \end{aligned}$$

Let $\xi_2(u) \equiv m_2(u + b)$, (26) becomes

$$\xi_2(u) = \nu q \xi_2(u + c_2) + \nu \rho (\xi_2 * f)(u + c_2) + \eta(u), \quad (27)$$

where $\eta(u) = \nu \rho \sum_{j=0}^{b-1} m_1(j) f(u + b + c_2 - j) + \gamma_2(u + b)$.

We use a solution procedure analogous to that of Section 3.1, $\xi_2(u)$ satisfies

$$A_2(\Delta) \xi_2(u) = B_2(\Delta) (\xi_2 * f)(u) + \eta(u) \quad u \geq 0, \quad (28)$$

where

$$A_2(z) = \sum_{j=0}^{c_2} a_{2,j} z^j, B_2(z) = \sum_{j=0}^{c_2} b_{2,j} z^j, a_{2,j} = I_{\{j=0\}} - \nu q \binom{c_2}{j}, b_{2,j} = \nu \rho \binom{c_2}{j}.$$

From the general theory on difference equations, can be expressed as

$$m_2(u + b) \equiv \xi_2(u) = \varphi_2(u) \quad u = 0, 1, \dots$$

where $\{\varphi_2(u)\}_{u=0}^{\infty}$ satisfies

$$A_2(\Delta) \varphi_2(u) = B_2(\Delta) (\varphi_2 * f)(u) + \eta(u) \quad u \geq 0. \quad (29)$$

Some solution procedures are omitted, similar discussions can be find in Section 3.1.

Generating function of the particular solution $\varphi_2(u)$ is

$$\begin{aligned}\tilde{\varphi}_2(z) &= \frac{z^{c_2} \tilde{\eta}(z) - v \left\{ q \sum_{u=0}^{c_2-1} z^u \varphi_2(u) + \rho \sum_{u=0}^{c_2-1} z^u (\varphi_2 * f)(u) \right\}}{z^{c_2} - vq - v\rho \tilde{f}(z)} \\ &= \frac{z^{c_2} T_z \eta(0) - R_{2,j}(z)}{\tilde{h}_{2,1}(z) - \tilde{h}_{2,2}(z)},\end{aligned}\quad (30)$$

where

$$\tilde{h}_{2,1}(z) = z^{c_2}, \quad \tilde{h}_{2,2}(z) = vq + v\rho \tilde{f}(z), \quad R_{2,j}(z) = v \left\{ q \sum_{u=0}^{c_2-1} z^u \varphi_2(u) + \rho \sum_{u=0}^{c_2-1} z^u (\varphi_2 * f)(u) \right\}.$$

Theorem 3.3: For $u \in N$, it holds that

$$\varphi_2(u) = \varsigma_2 \sum_{n=0}^u \varphi_2(u-n) \chi_2(n) + \delta_2(u), \quad (31)$$

where

$$\varsigma_2 = v\rho T_1 T_{z_{c_2}} \dots T_{z_2} T_{z_1} f(c_2), \quad \chi_2(n) = \frac{T_{z_{c_2}} \dots T_{z_2} T_{z_1} f(c_2 + n)}{T_1 T_{z_{c_2}} \dots T_{z_2} T_{z_1} f(c_2)}, \quad \delta_2(u) = T_{z_{c_2}} \dots T_{z_2} T_{z_1} \eta(u).$$

So for $u \geq b$,

$$m_2(u) \equiv \xi_2(u-b) = \varphi_2(u-b), \quad (32)$$

4. TGEOMETRIC DISTRIBUTIONS

In this section, it is further assumed that $f(x)$ is a mixture of two geometric distributions with

$$f(x) = \theta(1-\rho_1)\rho_1^{x-1} + (1-\theta)(1-\rho_2)\rho_2^{x-1},$$

Obviously, probability generating function is $\tilde{f}(z) = \frac{z[(1-\rho_1)(1-\rho_2) + \beta(1-z)]}{(1-\rho_1 z)(1-\rho_2 z)}$ where

$\beta = \theta\rho_2(1-\rho_1) + (1-\theta)\rho_1(1-\rho_2)$, and mean is $\mu = \frac{\theta}{1-\rho_1} + \frac{1-\theta}{1-\rho_2}$, we rewrite (16) as

$$\tilde{y}_{1,j}(z) = \frac{-R_{1,j}(z)(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_1(z)}, \quad (33)$$

where

$$\Lambda_1(z) = z^{c_1} (1-\rho_1 z)(1-\rho_2 z) - vq(1-\rho_1 z)(1-\rho_2 z) - v\rho[z(1-\rho_1)(1-\rho_2) + \beta(1-z)]$$

Since $\Lambda_1(z)$ is a polynomial of degree $c_1 + 1$, with leading coefficient $\rho_1\rho_2$, it can be expressed as

$$\Lambda_1(z) = \rho_1\rho_2\pi_1(z) \prod_{j=1}^h (z - \xi_j),$$

where ξ_j are solutions of $\Lambda_1(z)$ on the complex plane. It is notable that ξ_j have a module larger than 1, from performing partial fraction, we have

$$\frac{\pi_1(z)(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_1(z)} = \frac{\pi_1(z)(1-\rho_1 z)(1-\rho_2 z)}{\rho_1\rho_2\pi_1(z) \prod_{j=1}^h (z - \xi_j)} = 1 + \sum_{i=1}^h \frac{\omega_i}{\xi_i - z}, \quad (34)$$

where

$$\omega_i = \frac{\prod_{k=1}^h (\rho_k^{-1} - \xi_i)}{\prod_{k=1, k \neq i}^h (\xi_k - \xi_i)}.$$

For $i = 1$, substituting (34) into (16) shows

$$\tilde{y}_{1,j}(z) = \frac{(1 - \rho_1 z)(1 - \rho_2 z)}{\Lambda_1(z)} \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1'(z_k)} \frac{\pi_1(z)}{z_k - z} = \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1'(z_k)} \left(1 + \sum_{i=1}^h \frac{\omega_i}{\xi_i - z} \right) \frac{1}{z_k - z}. \quad (35)$$

Upon inversion, we obtain from (35) that

$$\begin{aligned} y_{1,j}(u) &= \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1'(z_k)} \left[z_k^{-(u+1)} + \sum_{i=1}^h \omega_i \sum_{l=0}^u \xi_i^{-(u+1-l)} z_k^{-(l+1)} \right] \\ &= \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1'(z_k)} \left(1 - \sum_{i=1}^h \frac{\omega_i}{z_k - \xi_i} \right) z_k^{-(u+1)} + \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1'(z_k)} \sum_{i=1}^h \frac{\omega_i}{z_k - \xi_i} \xi_i^{-(u+1)}, \end{aligned} \quad (36)$$

Use the same method, we obtain from (30) and (34) that,

$$\tilde{\varphi}_i(z) = \frac{\pi_i(z)(1 - \rho_1 z)(1 - \rho_2 z)}{\Lambda_i(z)} \tilde{\delta}_i(z) = \left(1 + \sum_{j=1}^h \frac{\omega_j}{\xi_j - z} \right) \tilde{\delta}_i(z), \quad (37)$$

Upon the inversion of the generating functions, one obtains from (37) that

$$\varphi_i(u) = \delta_i(u) + \sum_{j=1}^h \omega_j \sum_{l=0}^u \xi_j^{-(u+1-l)} \delta_i(l). \quad (38)$$

Example: Suppose $c_1 = 2, c_2 = 1, \rho = 0.2, q = 0.8, v = 0.95, \rho_1 = 0.3, \rho_2 = 0.6$, from (33)

$$\Lambda_1(z) = z^2(1 - \rho_1 z)(1 - \rho_2 z) - vq(1 - \rho_1 z)(1 - \rho_2 z) - v\varphi[z(1 - \rho_1)(1 - \rho_2) + \beta(1 - z)]$$

$$\Lambda_2(z) = z(1 - \rho_1 z)(1 - \rho_2 z) - vq(1 - \rho_1 z)(1 - \rho_2 z) - v\varphi[z(1 - \rho_1)(1 - \rho_2) + \beta(1 - z)].$$

and the relatively safety loading condition $c_2 - \rho\mu > 0$ holds for all $\theta \in (0, 1)$. Hence, θ is chosen to be 0.3, 0.5, 0.7, respectively. By solving Lundberg's equation $\Lambda_1(z) = 0$, we obtain the values of z_i 's and ξ_j 's, see Table 1. By solving Lundberg's equation $\Lambda_2(z) = 0$, we obtain the values of z_1 's and ξ_j 's, see Table 2.

Table 1: Numerical results of z_i 's and ξ_j 's, for $c_1 = 2$

θ	z_1	z_2	ξ_1	ξ_2
0.3	-0.8801119231	0.9682497475	1.5521641647	3.5969801084
0.5	-0.8863334780	0.9690110686	1.5658315556	3.3514908538
0.7	-0.8925049154	0.9697383777	1.5795907333	3.3431758044

Table 2: Numerical results of z_i 's and ξ_2 's, for $c_2 = 1$.

θ	z_1	ξ_1	ξ_2
0.3	0.921092162877287	1.40497341947108	3.43393441765163
0.5	0.925035115678722	1.43171692276687	3.4032479615544
0.7	0.928664417778473	1.45973526774272	3.37160031447881

Explicit expressions for $y_{1,j}(u)$ is determined by (36), so we obtain the values of $y_{1,j}(u)$ for $\theta = 0.5, c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, \rho_1 = 0.3, \rho_2 = 0.6, v = 0.95, b = 10$.

For instance, one has for $\theta = 0.5$,

$$y_0(u) = -0.42464 \times (-0.88633)^{-u} + 0.47519 \times 0.96901^{-u} - 0.05183 \times 1.56583^{-u} + 0.00129 \times 3.40325^{-u}$$

$$y_1(u) = 0.42071 \times (-0.88633)^{-u} + 0.55572 \times 0.96901^{-u} - 0.04023 \times 1.56583^{-u} + 0.00056 \times 3.40325^{-u}$$

Then solve a system of linear equations with $\alpha_{1,j}$, Table 3 lists the values of $\alpha_{1,j}$'s.

Table 3: Numerical results of $\alpha_{1,j}$ for $b = 10$.

θ	0.3	0.5	0.7
$\alpha_{1,0}$	0.00103	0.00059	0.00027
$\alpha_{1,1}$	0.00109	0.00063	0.00029

Explicit expressions for $\phi_2(u)$ is determined by (38), so we get the values of $\phi_2(u)$ for $\theta = 0.5, c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, \rho_1 = 0.3, \rho_2 = 0.6, v = 0.95, b = 10$, see Table 4.

Table 4: Numerical results of $\phi_2(u)$ for $b = 10, \theta = 0.5$.

u	10	11	12	13	14
$\phi_2(u)$	7.22942×10^{-4}	5.02029×10^{-4}	3.49790×10^{-4}	2.44063×10^{-4}	1.70395×10^{-4}
u	15	16	17	18	19
$\phi_2(u)$	1.8993×10^{-4}	8.31054×10^{-5}	5.80441×10^{-5}	4.05411×10^{-5}	2.83162×10^{-5}

Especially, when $\omega(N_1, N_2) = 1, b = 10$, Figure 1 and Figure 2 depict the generating function of the time to ruin $m_b(u)$ as functions of u . Observing Figure 1 and Figure 2, for each fixed θ it is easy that a larger u corresponds to a smaller expected ruin time and $m_b(u)$ is a increasing function of θ when u is fixed.

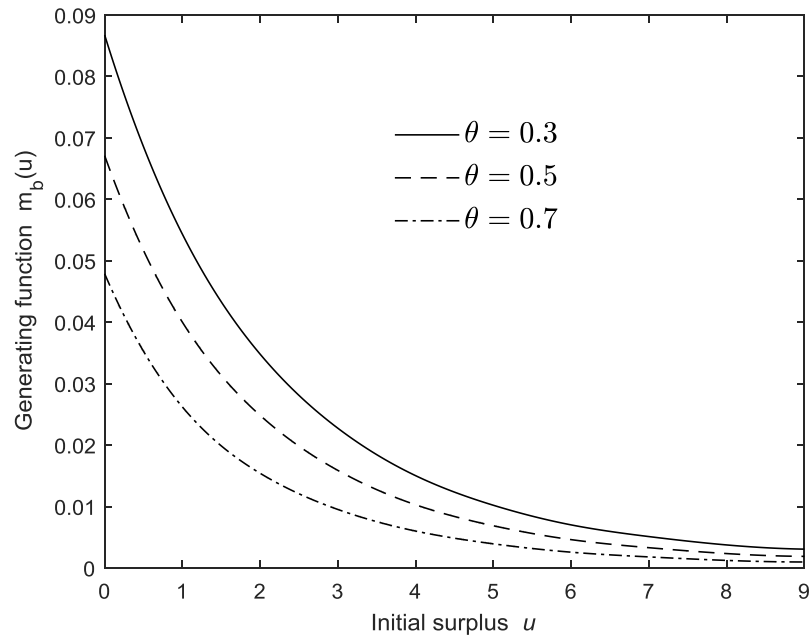


Figure 1: Numerical results of $m_b(u)$ for $b=10, u < b$.

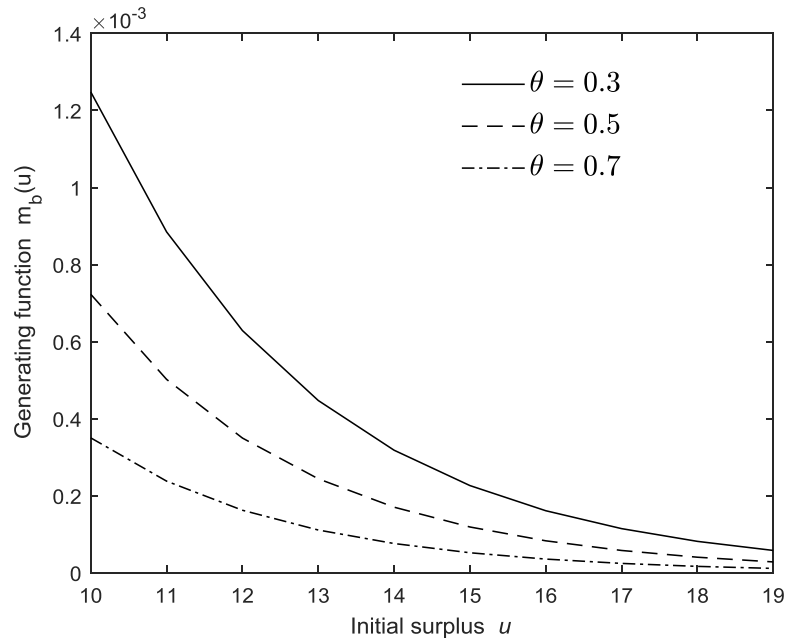


Figure 2: Numerical results of $m_b(u)$ for $b=10, u \geq b$.

5. CONCLUSION

In this paper, we consider the compound binomial model with general premium rate and a constant dividend barrier. Using the roots of a generalization of Lundberg's fundamental

equation and the general theory on difference equations, we derive an explicit expression for the Gerber-Shiu discounted penalty function up to the time of ruin. In particular, a numerical example is provided to show that the formulae are readily programmable in practice. From the numerical example given above, we can see that the barrier level has a negative effect on the total Gerber-Shiu discounted penalty function.

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