Original Research Article

Symmetry analysis, one-dimensional optimal system, and group invariant solutions for the shallow water waves of finite amplitude in (1 + 1)-dimensions

Abstract

The Lie group theoretic approach is employed here to obtain inequivalent group invariant solutions of a system of nonlinear partial differential equations (SNLPDEs) in (1 + 1)-dimensions appearing in the mathematical analysis of shallow water waves of finite amplitude, tsunami waves in particular. It is found that the system of equations admits a five-parameter Lie group of symmetry transformations with a Lie algebra $\mathscr{G}$ of infinitesimal generators. An optimal set $\mathscr{O}\mathscr{G}_1$ of eighteen one-dimensional subalgebras of $\mathscr{G}$ has been obtained. Similarity variables corresponding to each member of $\mathscr{O}\mathscr{G}_1$ have been determined and used to reduce the SNLPDEs to the system of ordinary differential equations. Some constants of the motion and inequivalent group invariant solutions have been provided whenever the system of reduced equations is solvable. The solutions to some of the reduced equations and the constants of motion provided here seem novel. Results obtained here may be used to check the accuracy of approximate solutions (grid formation) in the approximation (numerical) scheme.

Keywords: Shallow water waves of finite amplitude, optimal one-dimensional subalgebras of $A_{5,33}^{2,-2}$, group-invariant solutions, Casimir- and Killing polynomials, invariants of Int$\mathscr{G}$ on $A_{5,33}^{2,-2}$.

List of abbreviations:

SNLPDEs	System of nonlinear partial differential equations
NPDEs	Nonlinear partial differential equations
KP	Kadomtsev-Petviashvili
HBM	Hirota bilinear method
gCBS-BK	Generalized Calogero-Bogoyavlenskii-Schiff-Bogoyavlensky-Konopelchenko
EADM	Elzaki Adomian decomposition method
LGSTs	Lie group of symmetry transformations
IGs	Infinitesimal generators
PDEs	Partial differential equations
ODEs	Ordinary differential equations
LGPTs	Lie group of point transformations
GISs	Group invariant solutions
SWWFAVSB	Shallow water waves of finite amplitude on the variable sea bed

1 Introduction

The study of the behavior of actual physical problems falls under nonlinear sciences. These problems are modeled in terms of nonlinear partial differential equations (NPDEs), and their solutions are the fundamental ingredients for understanding the behavioral pattern of the problem. Recently, many attempts in the literature have been made to obtain various soliton profiles and their interaction phenomenon for various model equations by employing powerful numerical and analytical techniques and with the aid of potential symbolic computer programming. Ilhan et al. [1] investigated the lump interactions with the exponential- and hyperbolic types solutions for the variable-coefficient Kadomtsev-Petviashvili (KP) equation by applying the Hirota bilinear method (HBM) and also reported the kinky breather- and kinky periodic-soliton solutions. In [2], Zhang et al. have employed the bilinear Hirota polynomial scheme to the generalized KP equation, and the dynamical behavior of new types of rogue waves and multi-soliton solutions are obtained using the corresponding bilinear equation. Also, Hirota's bilinearization and multi-dimensional Bell polynomials are used to obtain the interaction between lump and kink solutions for the $(3+1)$-D Burger system [3]. Several authors utilized the multiple Exp-function scheme in their investigations [4, 5] on the generalized Calogero-Bogoyavlenskii-Schiff-Bogoyavlensky-Konopelchenko (gCBS-BK) type equation and other notable NPDEs. They exhibited the dynamics of novel multi-soliton solutions. Nisar et al. [6] considered the fractional Bogoyavlenskii equation and showed various types of soliton profiles using the $\exp(-K(\rho))$ expansion and $\tan(K(\rho))$ methods. Many other interesting physical models and their solitary wave interactions using HBM, $\tan\left(\frac{\Phi(\rho)}{2}\right)$-expansion technique, multiple *Exp*-function method can be found in the recent works [7, 8, 9, 10].

The investigations on various types of wave propagation generated by some unavoidable geophysical disturbance are parts of extensive mathematical modeling and analysis. It includes the propagation and run-up of tsunami waves caused by the exertion of enormous impulsive forces, such as underwater earthquakes, landslides, submarine slumps, meteor clashes, and volcanic eruptions, to name a few. These phenomena can be well comprehended scientifically by modeling the nonlinear shallow water wave equations neglecting frequency dispersion and assuming the wavelength is much larger than the water depth in consideration [11, 12, 13, 14, 15, 16]. Without the nonlinear effects and linearizing the shallow water equation, Synolakis investigated various waveforms [12]. However, the nonlinear terms must be addressed to analyze the wave behavior near the coast, and fully nonlinear shallow water wave equations must be considered.

In their attempts, Carrier and Greenspan [11], and Synolakis [12] obtained solutions by using the method of characteristics where the dependent and independent variables u, η, x, t play the role of unknown functions of the parameters α, β (characteristics) of the SNLPDEs of hyperbolic type for various initial conditions. Due to the intricate dependence of dependent and independent variables on the parameters, their use for analyzing the phenomena for different realistic initial conditions is not straightforward. In their recent studies, Archana et al. [17] have applied the Elzaki Adomian decomposition method (EADM) and found a series solution of (1+1)-dimensional model for different coastal slopes and ocean depths.

In the late 19th century, Norwegian mathematician Marius Sophus Lie invented a distinctive approach for solving ordinary and partial differential equations using their admitted group of symmetry transformations [18, 19, 20]. When a system of differential equations admits a Lie group of symmetry transformations (LGSTs), there is an associated Lie algebra of infinitesimal generators (IGs). Similarity(canonical) variables corresponding to some elements of Lie algebra are then used to transform the system of partial differential equations (PDEs) to another system with fewer independent variables (of reduced order in the case of ordinary differential equations (ODEs)). For a system of PDEs admitting a Lie algebra $\mathscr{G}$, a one-dimensional subalgebra of $\mathscr{G}$ may be used to reduce the number of independent variables by one. In this way, a system of PDEs with two independent variables may be reduced to a system of ODEs in a much simpler form. Any solution to this reduced system of ODEs provides a solution to the original system of PDEs, which are invariant under the one-parameter Lie group of point transformations (LGPTs) generated by the corresponding one-dimensional subalgebra. Construction of these one-parameter group invariant solutions (GISs) of PDEs using subalgebras of the admitted Lie algebra generalizes many ad-hoc techniques of transformations and provides a guaranteed similarity reduction in not only traveling coordinate form but in several other forms like cnoidal waves, Galilean invariance, translational and rotational invariant solutions, reduction into the class of Painlevé transcendent, etc. [21, 22, 23, 24, 25, 26, 27].

The collection of all one-dimensional subalgebras may provide infinitely many GISs. However, they may not be essentially different. One requires the optimal set of subalgebras to find inequivalent GISs. The advantage of the construction of a one-dimensional optimal system $\mathscr{O}\mathscr{G}_1$ is that the list of GISs obtained from the invariance under the IGs in $\mathscr{O}\mathscr{G}_1$ provides an optimal system of one-parameter GISs, i.e., GISs obtained in this way are all inequivalent. The concept of an optimal system of subalgebras and systematic approaches for constructing a one-dimensional optimal system is available in classical texts [28, 21, 29, 30] and in some recent works by Kötz and Hu et al. [31, 32, 33].

The optimal system of subalgebras has been used recently to obtain exact solutions of PDEs appearing as a mathematical model for various physical systems [34, 35, 36, 37, 38, 39, 40]. This work attempts to obtain some inequivalent GIS for SNLPDEs appearing in the mathematical analysis of shallow water waves of finite amplitude on the variable sea bed (SWWFAVSB), particularly tsunami waves around the coastal region. In Sec.2, a brief description of the SNLPDEs appearing as the model of SWWFAVSB in the (1+1)-dimension [11, 12, 13, 15] has been presented. In Sec.3, IGs for LGSTs admitted by the SNLPDEs, their Lie algebra, adjoint representation, and invariants for the inner automorphism, etc., are derived. The optimal system of one-dimensional subalgebra is obtained here. Invariants for the propagation of waves and some group invariant solutions of the SNLPDEs considered here are provided in Sec.4. Some remarks on our investigation have been presented in the concluding section, Sec.5.

2 The model

Propagation of tsunami waves and runup of long waves incident upon a variable bottom topography is modeled using the shallow water wave equation, assuming the ratio between the depth of the water and the wavelength of the propagating wave under investigation is very small. Further assumptions are that the water is incompressible and the flow is irrotational. The derivation of the SNLPDEs modeling the equations for SWWFAVSB has been presented by Stoker in brief in his seminal work [41], and Dutykh in his thesis [15].

The $(1+1)$-dimensional system of equations for SWWFAVSB is given by [11, 12, 13, 14, 15, 16]

$$\begin{aligned} u_t + uu_x + g\omega_x &= 0 \\ \omega_t + (u(h+\omega))_x &= 0, \end{aligned} \tag{1}$$

where, ω =wave amplitude, u =depth average velocity, h =variable water depth, g = acceleration due to gravity.

For the mathematical analysis of the model (1) we adopt here the dimensionless parameters and variables suggested by Dutykh [15] as

$$\alpha = \frac{a}{d} << 1,\ \beta = \frac{d^2}{l^2} << 1, \tag{2a}$$

$$\tilde{x} = \frac{1}{l}x,\ \tilde{y} = \frac{1}{d}y,\ \tilde{t} = \frac{c_0}{l}t, \tag{2b}$$

$$\tilde{\omega} = \frac{1}{a}\omega,\ \tilde{h} = \frac{1}{d}h,\ \tilde{u} = \frac{c_0}{gal}u. \tag{2c}$$

Here, d, a, and l are the typical water depth, wave amplitude, and wavelength. Also, $c_0 = \sqrt{gd}$ is the wave speed in the open ocean.

In terms of new variables and parameters introduced in (2a)-(2c), Eq. (1) can be reformulated as

$$\begin{aligned} \tilde{u}_{\tilde{t}} + \alpha\,\tilde{u}\tilde{u}_{\tilde{x}} + \tilde{\omega}_{\tilde{x}} &= 0, \\ \tilde{\omega}_{\tilde{t}} + \left(\tilde{u}\left(\tilde{h} + \alpha\,\tilde{\omega}\right)\right)_{\tilde{x}} &= 0, \end{aligned} \tag{3}$$

involving a parameter α (the height of the wave's crest to the ocean's depth below the free surface in equilibrium) given in Eq. (2a).

Our objective here is to obtain reduced equation/GISs to Eq. (3) in the following sections.

3 Lie algebra of admitted Lie group of transformations and optimal one-dimensional subalgebras

Lie algebra $\mathscr{G}$ of IGs of LGSTs effectively leads to integrating a system of ODE/PDEs. Since a Lie algebra of dimensions, $r \geq 2$, may contain infinitely many subalgebras, partitioning $\mathscr{G}$ into equivalence classes is desirable. To attain the goal, we first obtain the Lie algebra $\mathscr{G}$ of IGs of LGSTs admitted by SNLPDEs (3) in the following subsections.

3.1 Admitted Lie algebra

We denote the SNLPDEs (3) by $\Delta = 0$. In the rest of our discussion, we will omit the symbol $\sim$ over (dependent and independent) variables for the simplicity of presentation. We write the one-parameter LGSTs, in their infinitesimal form, for the system $\Delta = 0$ [21, 23, 24] as

$$\begin{aligned} x^* &= x + \varepsilon\,\xi(x,t,u,\omega,h) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\,\tau(x,t,u,\omega,h) + O(\varepsilon^2), \\ u^* &= u + \varepsilon\,\eta(x,t,u,\omega,h) + O(\varepsilon^2), \\ \omega^* &= \omega + \varepsilon\,\theta(x,t,u,\omega,h) + O(\varepsilon^2), \\ h^* &= h + \varepsilon\,\mu(x,t,u,\omega,h) + O(\varepsilon^2). \end{aligned} \tag{4}$$

The symbols x, t and u, ω, h appearing in the formula (4) are dimensionless independent and dependent variables, respectively which corroborate the same role of the corresponding symbols introduced in Eqs. (2b) and (2c). We write the Infinitesimal generator (IG) for the LGSTs admitted by Eq. (3) as

$$\hat{T} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \theta\frac{\partial}{\partial \omega} + \mu\frac{\partial}{\partial h}. \tag{5}$$

The prolongation of $\hat{T}$ into the first jet space is given by [24]

$$\hat{T}^{(1)} = \hat{T} + \eta^x \frac{\partial}{\partial u_x} + \eta^t \frac{\partial}{\partial u_t} + \theta^x \frac{\partial}{\partial \omega_x} + \theta^t \frac{\partial}{\partial \omega_t} + \mu^x \frac{\partial}{\partial h_x},$$

where the expressions for $\eta^x, \eta^t, \theta^x, \theta^t, \mu^x$ can be obtained using the formula

$$\phi^a = D_a[\phi] - (D_a[\xi])u_x - (D_a[\tau])u_t, \tag{6}$$

where $a \in \{x,\, t\}$, $\phi \in \{\eta,\, \theta,\, \mu\}$. Here, the symbols D_a represent the total derivative, viz.,

$$D_a = \frac{\partial}{\partial a} + u_a \frac{\partial}{\partial u} + \omega_a \frac{\partial}{\partial \omega} + h_a \frac{\partial}{\partial h}.$$

The infinitesimal invariance criterion ([24], pp-330) reads

$$\hat{T}^{(1)} \left[u_t + \alpha\, u u_x + \omega_x\right]\Big|_{\Delta=0} = 0 \tag{7}$$

and,

$$\hat{T}^{(1)} \left[\omega_t + u(h_x + \alpha\, \omega_x) + u_x(h + \alpha\, \omega)\right]\Big|_{\Delta=0} = 0. \tag{8}$$

Algebraic manipulation of Eqs. (7) and (8) provides a pair of algebraic relation in powers of u_x, u_t, ω_x, ω_t and h_x (regarded as independent symbols). Equating the coefficients of several powers of these symbols to zero provides an overdetermined system of determining equations (coupled linear PDEs). Solving these determining equations one can get the x, t, u, ω, h dependence of $\xi, \tau, \eta, \theta, \mu$ as

$$\begin{aligned}
\xi &= c_1 x + c_2,\\
\tau &= c_3 t + c_4,\\
\eta &= (c_1 - c_3)u,\\
\theta &= 2(c_1 - c_3)\omega + c_5,\\
\mu &= 2(c_1 - c_3)h - \alpha\, c_5,
\end{aligned}$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary constants.

Therefore, the system of PDEs (3) admits a five-parameter LGSTs generated by the elements $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ of a five-dimensional Lie algebra $\mathscr{G}$ with

$$\begin{gathered}
\mathbf{v}_1 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2\omega\frac{\partial}{\partial \omega} + 2h\frac{\partial}{\partial h},\ \mathbf{v}_2 = \frac{\partial}{\partial x},\ \mathbf{v}_3 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2\omega\frac{\partial}{\partial \omega} - 2h\frac{\partial}{\partial h},\\
\mathbf{v}_4 = \frac{\partial}{\partial t},\ \mathbf{v}_5 = \frac{\partial}{\partial \omega} - \alpha\,\frac{\partial}{\partial h}.
\end{gathered} \tag{9}$$

The commutator table (whose (i,j)-th entry is $[\mathbf{v}_i, \mathbf{v}_j] = \mathbf{v}_i\mathbf{v}_j - \mathbf{v}_j\mathbf{v}_i$) for the Lie algebra $\mathscr{G}$ of IGs in Eq. (9) can be found as

Table 1: Commutator table for IGs in $\mathscr{G}$ of LGSTs admitted by the SNLPDEs (3).

$[\cdot,\cdot]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
$\mathbf{v}_1$	0	$-\mathbf{v}_2$	0	0	$-2\mathbf{v}_5$
$\mathbf{v}_2$	$\mathbf{v}_2$	0	0	0	0
$\mathbf{v}_3$	0	0	0	$-\mathbf{v}_4$	$2\mathbf{v}_5$
$\mathbf{v}_4$	0	0	$\mathbf{v}_4$	0	0
$\mathbf{v}_5$	$2\mathbf{v}_5$	0	$-2\mathbf{v}_5$	0	0

It may be observed from Table 1 that, $\{0\} = \mathscr{G}^{(2)} \subset \mathscr{G}^{(1)} \subset \mathscr{G}^{(0)} = \mathscr{G}$, where $\mathscr{G}^{(i+1)} = \left[\mathscr{G}^{(i)}, \mathscr{G}^{(i)}\right], i = 0, 1$. So, $\mathscr{G}$ is solvable and isomorphic to the element $A_{5,33}^{2,-2}$ in Table II of all five-dimensional Lie algebra classified by Patera et al. in [42] and $A_{5,33}$ in Appendix I of Basarab-Horwath et al. in [43]. The information in this table is the ingredient for obtaining an adjoint representation of elements in $\mathscr{G}$ as follows.

3.2 Adjoint transformation equation

The invertible linear inner automorphisms $\mathrm{Ad}\,(e^{\varepsilon\,\mathbf{v}}) : \mathscr{G} \to \mathscr{G}$ for $\mathbf{v} \in \mathscr{G}$ and $\varepsilon \in \mathbb{R}$, defined by

$$\mathrm{Ad}(e^{\varepsilon\,\mathbf{v}})(\mathbf{u}) = e^{\varepsilon\,\mathrm{ad}(\mathbf{v})}(\mathbf{u}) = \mathbf{u} + \varepsilon\,[\mathbf{u},\mathbf{v}] + \frac{\varepsilon^2}{2!}\,[[\mathbf{u},\mathbf{v}],\mathbf{v}] + \cdots \tag{10}$$

while for fixed $\mathbf{v} \in \mathscr{G}$, the inner derivation $\mathrm{ad}(\mathbf{v}) : \mathscr{G} \to \mathscr{G}$ is a linear mapping defined by

$$\mathrm{ad}(\mathbf{v})(\mathbf{u}) := [\mathbf{u},\mathbf{v}]. \tag{11}$$

Two elements $\mathbf{u}$ and $\mathbf{v}$ of $\mathscr{G}$ are said to be conjugate, if there exists $a(\neq 0), \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_5 \in \mathbb{R}$, such that

$$\mathbf{v} = a\,\mathrm{Ad}\,(e^{\varepsilon_{i_1}\mathbf{v}_{i_1}}) \circ \mathrm{Ad}\,(e^{\varepsilon_{i_2}\mathbf{v}_{i_2}}) \circ \cdots \circ \mathrm{Ad}\,(e^{\varepsilon_{i_r}\mathbf{v}_{i_5}})\,(\mathbf{u}), \tag{12}$$

where, $i_1, i_2, \ldots, i_5 \in \{1, 2, \ldots, 5\}$. Two subalgebras $\mathscr{H}(\mathbf{v})$, $\mathscr{H}(\mathbf{u}) \subset \mathscr{G}$ are said to be conjugate to each other if their generators $\mathbf{v}$ and $\mathbf{u}$ are conjugate. It can be checked that the binary relation $\rho \subset \mathscr{G} \times \mathscr{G}$ defined by $\mathbf{u}\ \rho\ \mathbf{v} \Leftrightarrow$ "$\mathbf{u}$ is conjugate to $\mathbf{v}$" is an equivalence relation. So ρ may be used to partition $\mathscr{G}$.

For the technical simplicity of algebraic operations, we use the tuples $(\alpha_1, \alpha_2, \ldots, \alpha_5)$ and $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_5)$ to represent the elements $\mathbf{u} = \sum_{i=1}^{5} \alpha_i \mathbf{v_i}$ and $\mathbf{v} = \sum_{i=1}^{5} \tilde{\alpha}_i \mathbf{v}_i$ in $\mathscr{G}$. Then the equation Eq. (12) may be written (for $a = 1$) as

$$(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_5) = (\alpha_1, \alpha_2, \ldots, \alpha_5)\,A. \tag{13}$$

Eq. (13) is the adjoint transformation equation equivalent to an algebraic equation system.

Table 2: Adjoint representation table for Lie algebra of LGSTs admitted by the model of SWWFAVSB (3).

$\mathrm{Ad}(\cdot)(\cdot)$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
$e^{\varepsilon_1\mathbf{v}_1}$	$\mathbf{v}_1$	$e^{\varepsilon_1}\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$e^{2\varepsilon_1}\mathbf{v}_5$
$e^{\varepsilon_2\mathbf{v}_2}$	$\mathbf{v}_1 - \varepsilon_2\mathbf{v}_2$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$
$e^{\varepsilon_3\mathbf{v}_3}$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$e^{\varepsilon_3}\mathbf{v}_4$	$e^{-2\varepsilon_3}\mathbf{v}_5$
$e^{\varepsilon_4\mathbf{v}_4}$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 - \varepsilon_4\mathbf{v}_4$	$\mathbf{v}_4$	$\mathbf{v}_5$
$e^{\varepsilon_5\mathbf{v}_5}$	$\mathbf{v}_1 - 2\varepsilon_5\mathbf{v}_5$	$\mathbf{v}_2$	$\mathbf{v}_3 + 2\varepsilon_5\mathbf{v}_5$	$\mathbf{v}_4$	$\mathbf{v}_5$

It may be easily verified that the use of the entries provided in Table 1 in the formula (10) provides the adjoint representations given in Table 2 of $\mathscr{G}$ in the basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5\}$. Results of Table 2 provides the adjoint representation of individual IGs $\mathbf{v}_i$ $(i = 1, 2, \ldots, 5)$ as

$$\mathrm{Ad}\,(e^{\varepsilon_1\mathbf{v}_1}) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{\varepsilon_1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{2\varepsilon_1} \end{pmatrix}^T,\ \mathrm{Ad}\,(e^{\varepsilon_2\mathbf{v}_2}) \equiv \begin{pmatrix} 1 & -\varepsilon_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T,\ \mathrm{Ad}\,(e^{\varepsilon_3\mathbf{v}_3}) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{\varepsilon_3} & 0 \\ 0 & 0 & 0 & 0 & e^{-2\varepsilon_3} \end{pmatrix}^T,$$

$$\mathrm{Ad}\,(e^{\varepsilon_4\mathbf{v}_4}) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\varepsilon_4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T,\ \mathrm{Ad}\,(e^{\varepsilon_5\mathbf{v}_5}) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & -2\varepsilon_5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2\varepsilon_5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

Then the general adjoint transformation matrix A in Eq. (13), which may be taken in any order, we choose it as $A \equiv \left(\mathrm{Ad}\,(e^{\varepsilon_5\mathbf{v}_5}) \circ \mathrm{Ad}\,(e^{\varepsilon_4\mathbf{v}_4}) \circ \cdots \circ \mathrm{Ad}\,(e^{\varepsilon_1\mathbf{v}_1})\right)^T$, becomes

$$A = \begin{pmatrix} 1 & -\varepsilon_2 & 0 & 0 & -2\varepsilon_5 \\ 0 & e^{\varepsilon_1} & 0 & 0 & 0 \\ 0 & 0 & 1 & -\varepsilon_4 & 2\varepsilon_5 \\ 0 & 0 & 0 & e^{\varepsilon_3} & 0 \\ 0 & 0 & 0 & 0 & e^{2(\varepsilon_1-\varepsilon_3)} \end{pmatrix}. \tag{14}$$

Use of the matrix A in the transpose of the adjoint transformation (13) gives

$$\begin{pmatrix}\tilde{\alpha}_1\\ \tilde{\alpha}_2\\ \tilde{\alpha}_3\\ \tilde{\alpha}_4\\ \tilde{\alpha}_5\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ -\varepsilon_2 & e^{\varepsilon_1} & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -\varepsilon_4 & e^{\varepsilon_3} & 0\\ -2\varepsilon_5 & 0 & 2\varepsilon_5 & 0 & e^{2(\varepsilon_1-\varepsilon_3)}\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix} \tag{15}$$

and

$$\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ \varepsilon_2 e^{-\varepsilon_1} & e^{-\varepsilon_1} & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & \varepsilon_4 e^{-\varepsilon_3} & e^{-\varepsilon_3} & 0\\ 2\varepsilon_5 e^{2(\varepsilon_3-\varepsilon_1)} & 0 & -2\varepsilon_5 e^{2(\varepsilon_3-\varepsilon_1)} & 0 & e^{2(\varepsilon_3-\varepsilon_1)}\end{pmatrix}\begin{pmatrix}\tilde{\alpha}_1\\ \tilde{\alpha}_2\\ \tilde{\alpha}_3\\ \tilde{\alpha}_4\\ \tilde{\alpha}_5\end{pmatrix}. \tag{16}$$

3.3 Invariants

The essential ingredients for partitioning a Lie algebra into disjoint classes are invariants of $\mathrm{Int}(\mathscr{G})$, the set of all inner automorphisms defined in Eq. (10). A real-valued function $\phi : \mathscr{G} \to \mathbb{R}$ satisfying

$$\phi(\mathbf{u}) = \phi\left(\mathrm{Ad}(e^{\varepsilon\,\mathbf{v}})(\mathbf{u})\right) \quad \text{for every} \quad \mathrm{Ad}(e^{\varepsilon\,\mathbf{v}}) \in \mathrm{Int}(\mathscr{G}),\ \mathbf{u} \in \mathscr{G} \tag{17}$$

is called an invariant of $\mathrm{Int}(\mathscr{G})$ [28, 31]. Their specific forms may be suggested with the help of Casimir polynomial $C_{\mathscr{G}}(\mathbf{u})$ and the non-zero coefficients of the Killing polynomial $k_{\mathscr{G}}(\lambda; \mathbf{u})$ defined as

$$C_{\mathscr{G}}(\mathbf{u}) = tr\left(\mathrm{ad}(\mathbf{u}) \circ \mathrm{ad}(\mathbf{u})\right), \tag{18}$$

$$k_{\mathscr{G}}(\lambda; \mathbf{u}) = det\left(\lambda\,\mathbb{I}_{\mathscr{G}} - \mathrm{ad}(\mathbf{u})\right). \tag{19}$$

The results presented in Table 1 may be used to get the representation of inner derivation $\mathrm{ad}(\mathbf{u})(\cdot)$ in the basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5\}$ as

$$\mathrm{ad}(\mathbf{u}) \equiv \begin{pmatrix}0 & 0 & 0 & 0 & 0\\ -\alpha_2 & \alpha_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\alpha_4 & \alpha_3 & 0\\ -2\alpha_5 & 0 & 2\alpha_5 & 0 & 2(\alpha_1-\alpha_3)\end{pmatrix}. \tag{20}$$

Use of this matrix in the formula (18) for Casimir polynomial gives

$$C_{\mathscr{G}}(\mathbf{u}) = 5\alpha_1^2 + 5\alpha_3^2 - 8\alpha_1\alpha_3 \tag{21}$$

so that $\phi_1^{CP}(\alpha_1, \alpha_3) = 5\alpha_1^2 + 5\alpha_3^2 - 8\alpha_1\alpha_3$ may be regarded as an invariant of $\mathrm{Int}(\mathscr{G})$. On the other hand use of the matrix (20) in the formula (19) for Killing polynomial $k_{\mathscr{G}}(\lambda; \mathbf{u})$ gives

$$k_{\mathscr{G}}(\lambda; \mathbf{u}) = \lambda^5 + (\alpha_3 - 3\alpha_1)\lambda^4 + \left\{\alpha_1\alpha_3 - 2(\alpha_3^2 - \alpha_1^2)\right\}\lambda^3 + 2\alpha_1\alpha_3(\alpha_3 - \alpha_1)\lambda^2. \tag{22}$$

This polynomial provides three more independent invariants

$$\phi_2^{KP}(\alpha_1, \alpha_3) = \alpha_3 - 3\alpha_1,\ \phi_3^{KP}(\alpha_1, \alpha_3) = \alpha_1\alpha_3 - 2(\alpha_3^2 - \alpha_1^2),\ \phi_4^{KP}(\alpha_1, \alpha_3) = 2\alpha_1\alpha_3(\alpha_3 - \alpha_1). \tag{23}$$

Interestingly, invariants obtained with the help of Casimir- and Killing- polynomials are homogeneous algebraic functions of α_1 and α_3. It is a natural curiosity to investigate whether more invariants for the adjoint transformations may exist. We follow the alternative scheme described below to attain the goal.

Here, one may use the formula (10), with $\mathbf{v} \equiv (\beta_1, \beta_2, \cdots, \beta_5)$, into the argument of ϕ in the right-hand side of the condition (17) and its rearrangement into the form

$$\phi\left(\alpha_1, \alpha_2, \cdots, \alpha_5\right) = \phi\left(\alpha_1 + \varepsilon\,\Theta_1 + O(\varepsilon^2), \alpha_2 + \varepsilon\,\Theta_2 + O(\varepsilon^2), \cdots, \alpha_5 + \varepsilon\,\Theta_5 + O(\varepsilon^2)\right), \tag{24}$$

containing $\Theta_i(\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots, \beta_5)$, $i = 1, 2, \ldots, 5$ provides

$$\Theta_1 = 0, \quad \Theta_2 = -\alpha_1\beta_2 + \alpha_2\beta_1, \quad \Theta_3 = 0, \quad \Theta_4 = -\alpha_3\beta_4 + \alpha_4\beta_3, \quad \Theta_5 = 2\alpha_5\beta_1 - 2\alpha_5\beta_3 + 2\alpha_3\beta_5 - 2\alpha_1\beta_5. \tag{25}$$

Then differentiation of both sides of Eq. (24) with respect to ε at $\varepsilon = 0$ leading to the relation

$$\frac{\partial\phi}{\partial\alpha_1}\Theta_1 + \frac{\partial\phi}{\partial\alpha_2}\Theta_2 + \cdots + \frac{\partial\phi}{\partial\alpha_5}\Theta_5 = 0 \quad \forall\,\beta_i \in \mathbb{R},\ i = 1, 2, \ldots, 5. \tag{26}$$

The α_i, β_i, $i = 1, 2, \ldots, 5$ dependence of Θ_i given in Eq. (25) exhibits that Eq. (26) is a set of algebraic-differential relations involving ten-parameters $\alpha_1, \ldots, \alpha_5, \beta_1, \ldots, \beta_5$ and derivatives of invariant ϕ with respect to $\alpha_i^{'}s$. In finding unknown invariant ϕ, it is customary to split the relation (26) as an algebraic polynomial in $\beta_1, \beta_2, \ldots, \beta_5$ and set coefficients of each power of β_i to zero. Following this scheme, one gets an overdetermined system of linear PDEs for the invariant ϕ as

$$\begin{aligned} \alpha_2 \frac{\partial \phi}{\partial \alpha_2} + 2\alpha_5 \frac{\partial \phi}{\partial \alpha_5} &= 0, \\ -\alpha_1 \frac{\partial \phi}{\partial \alpha_2} &= 0, \\ \alpha_4 \frac{\partial \phi}{\partial \alpha_4} - 2\alpha_5 \frac{\partial \phi}{\partial \alpha_5} &= 0, \\ -\alpha_3 \frac{\partial \phi}{\partial \alpha_4} &= 0, \\ 2(\alpha_3 - \alpha_1) \frac{\partial \phi}{\partial \alpha_5} &= 0. \end{aligned} \tag{27}$$

The second, fourth, and fifth equations suggest that ϕ is free from α_2, α_4, and α_5. So, $\phi = \phi(\alpha_1, \alpha_3)$. Thus, any arbitrary function of α_1 and α_3 is an invariant of $\mathrm{Int}(\mathscr{G})$. It is remarkable to observe that the general solution of the system of PDEs in Eq. (27) contains the invariants ϕ_1^{CP}, ϕ_2^{KP}, ϕ_3^{KP} and ϕ_4^{KP} as particular cases. We will use two invariants

$$\phi_1 = \alpha_3 \text{ and } \phi_2 = \alpha_1 \tag{28}$$

to classify one-dimensional subalgebras.

3.4 Optimal one-dimensional subalgebras

The classification of one-dimensional subalgebras of the Lie algebra $\mathscr{G}$ into an optimal system is defined as follows: A list of one-dimensional subalgebras $\{\mathscr{H}(\mathbf{u}_\gamma) : \gamma \in \Lambda\}$, Λ being an index set, is called an optimal system if **i)** for any two subalgebras $\mathscr{H}(\mathbf{u}_i)$, $\mathscr{H}(\mathbf{u}_j)$ with $i, j \in \Lambda$ and $i \neq j$- their basis element $\mathbf{u}_i$ and $\mathbf{u}_j$ are not conjugate (by the relation ρ defined in Sec.3.2) to each other, and **ii)** for any one dimensional subalgebra $\mathscr{H}(\mathbf{u})$ of $\mathscr{G}$, with basis element $\mathbf{u}$ -we have $\mathbf{u}$ is conjugate to precisely one member $\mathbf{u}_i$ of that list, for some $i \in \Lambda$.

The partitioning of $\mathscr{G}$ into an optimal set of one-dimensional subalgebras is thus reduced to finding the conjugacy classes of the relation ρ. Here, we adopt the algorithm adopted by Olver in the classic treatise [21], and Hu et al. [33] for the construction of a one-dimensional optimal system based on the degree of homogeneity of the homogeneous invariant functions of $\mathrm{Int}(\mathscr{G})$.

Theorem 1. One-dimensional optimal subalgebra of the Lie algebra $\mathscr{G} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ of IGs given in Eq. (9) of LGSTs admitted by the system of equations in Eq. (3) are

$$\begin{array}{ccc} \mathscr{H}(\mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_5), & \mathscr{H}(\mathbf{v}_1 + \mathbf{v}_3 - \mathbf{v}_5), & \mathscr{H}(a\mathbf{v}_1 + \mathbf{v}_3),\ a \in \mathbb{R} \\ \mathscr{H}(\mathbf{v}_3 + \mathbf{v}_2), & \mathscr{H}(\mathbf{v}_3 - \mathbf{v}_2), & \mathscr{H}(\mathbf{v}_1 + \mathbf{v}_4), \\ \mathscr{H}(\mathbf{v}_1 - \mathbf{v}_4), & \mathscr{H}(\mathbf{v}_2 + \mathbf{v}_4 + a\mathbf{v}_5),\ a \in \mathbb{R}, & \mathscr{H}(\mathbf{v}_2 - \mathbf{v}_4 + a\mathbf{v}_5),\ a \in \mathbb{R} \\ \mathscr{H}(\mathbf{v}_2 + \mathbf{v}_5), & \mathscr{H}(\mathbf{v}_2 - \mathbf{v}_5), & \mathscr{H}(\mathbf{v}_4 + \mathbf{v}_5), \\ \mathscr{H}(\mathbf{v}_4 - \mathbf{v}_5), & \mathscr{H}(\mathbf{v}_1), & \mathscr{H}(\mathbf{v}_2), \\ \mathscr{H}(\mathbf{v}_3), & \mathscr{H}(\mathbf{v}_4), & \mathscr{H}(\mathbf{v}_5). \end{array} \tag{29}$$

Proof. Since the degree of ϕ_1 and ϕ_2 in Eq. (28) are odd, admissible qualitative values of ϕ_1 and ϕ_2 are $\{0, 1\}$ each. So, we have the following cases:

Case I. $\phi_1 = 1$. We are considering here the partitioning of $\mathscr{G}_1^1 = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \mathbf{v}_3 + \alpha_4\mathbf{v}_4 + \alpha_5\mathbf{v}_5 : \alpha_1,\ \alpha_2,\ \alpha_4,\ \alpha_5 \in \mathbb{R}\} \subset \mathscr{G}_1^{\neq} (\equiv \{\mathbf{u} \in \mathscr{G} : \phi_1(\mathbf{u}) \neq 0\})$.

This case may be subdivided (depending upon the mutually disjoint values of the free coefficient α_1) into the following three sub-cases: viz., **I.i)** $\alpha_1 = 1$, **I.ii)** $\alpha_1 \in \mathbb{R} - \{0, 1\}$, **I.iii)** $\alpha_1 = 0$.

Case I.i) ($\alpha_1 = 1$) Here α_2, α_4, and α_5 are free coefficients in $\mathscr{G}_1^1$. For $\alpha_5 > 0$ and $\alpha_5 < 0$ we select the respective representatives as $\mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_5$ and $\mathbf{v}_1 + \mathbf{v}_3 - \mathbf{v}_5$ so that target coefficients $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_5)$ in the adjoint map $\mathrm{Ad}(e^{\sum_{i=1}^5 \varepsilon_i \mathbf{v}_i})(\mathscr{G}_1^1)$ becomes

$$(1, 0, 1, 0, 1) \quad \text{and} \quad (1, 0, 1, 0, -1) \tag{30}$$

respectively. Then the system of Eqs. (15) with the inhomogeneous term prescribed in Eq. (30) provides the domain

$$\varepsilon_2 = \alpha_2 e^{\varepsilon_1},\ \varepsilon_3 = \varepsilon_1 + \ln\sqrt{|\alpha_5|},\ \varepsilon_4 = \alpha_4\sqrt{|\alpha_5|}e^{\varepsilon_1},\ \varepsilon_1,\ \varepsilon_5 \in \mathbb{R},$$

in the parameter space $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_5)$ for the admissible adjoint transformation

$$\alpha_2 = \varepsilon_2 e^{-\varepsilon_1},\ \alpha_4 = \varepsilon_4 e^{-\varepsilon_3},\ \alpha_5 = \pm e^{2(\varepsilon_3-\varepsilon_1)}$$

among the elements in the class $[\mathbf{v}_1+\mathbf{v}_3+\mathbf{v}_5]$ and $[\mathbf{v}_1+\mathbf{v}_3-\mathbf{v}_5]$ of the elements in $\mathscr{G}_1^1$. Hence, $\mathbf{v}_1+\mathbf{v}_3\pm\mathbf{v}_5$ are admissible representative for the conjugacy class $\{\mathbf{v}_1+\alpha_2\mathbf{v}_2+\mathbf{v}_3+\alpha_4\mathbf{v}_4+\alpha_5\mathbf{v}_5 :\ |\alpha_5|\neq 0,\ \alpha_2,\ \alpha_4\in\mathbb{R}\}\subset\mathscr{G}_1^1$ accordingly.

For $\alpha_5 = 0$, $\mathbf{v}_1+\mathbf{v}_3$ may be selected as a representative for the class $\{\mathbf{v}_1+\alpha_2\mathbf{v}_2+\mathbf{v}_3+\alpha_4\mathbf{v}_4 : \alpha_2,\ \alpha_4\in\mathbb{R}\}\subset\mathscr{G}_1^1$. Then, using $(1,0,1,0,0)^T$ in the RHS of Eq. (15) gives the admissible domain in the parameter space for the admissible adjoint map

$$\varepsilon_2 = \alpha_2 e^{\varepsilon_1},\ \varepsilon_4 = \alpha_4 e^{\varepsilon_3},\ \varepsilon_1,\ \varepsilon_3,\ \varepsilon_5\in\mathbb{R}.$$

The corresponding rule among elements in the conjugacy class $[\mathbf{v}_1+\mathbf{v}_3]$ are of the form

$$\alpha_2 = \varepsilon_2 e^{-\varepsilon_1},\ \alpha_4 = \varepsilon_4 e^{-\varepsilon_3}.$$

Thus the elements in $\{\mathbf{v}_1+\alpha_2\mathbf{v}_2+\mathbf{v}_3+\alpha_4\mathbf{v}_4+\alpha_5\mathbf{v}_5\ :\ \alpha_2,\ \alpha_4,\ \alpha_5\in\mathbb{R}\}\subset\mathscr{G}_1^1$ can be partitioned into three inequivalent conjugacy classes $[\mathbf{v}_1+\mathbf{v}_3+\mathbf{v}_5]$, $[\mathbf{v}_1+\mathbf{v}_3-\mathbf{v}_5]$, and $[\mathbf{v}_1+\mathbf{v}_3]$.

Case I.ii) $(\alpha_1\in\mathbb{R}-\{0,1\})$

Here, for fixed $\alpha_1\notin\{0,1\}$ any element in $\{\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\mathbf{v}_3+\alpha_4\mathbf{v}_4+\alpha_5\mathbf{v}_5 : \alpha_2,\ \alpha_4,\ \alpha_5\in\mathbb{R}\}\subset\mathscr{G}_1^1$ is conjugate to $\alpha_1\mathbf{v}_1+\mathbf{v}_3$ through the corresponding rule (from Eq. (16))

$$\alpha_2 = \alpha_1\varepsilon_2 e^{-\varepsilon_1},\ \alpha_4 = \varepsilon_4 e^{-\varepsilon_3},\ \alpha_5 = 2(\alpha_1-1)\varepsilon_5 e^{2(\varepsilon_3-\varepsilon_1)}.$$

for the group parameter $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_5)$ in the admissible domain

$$\varepsilon_2 = \frac{\alpha_2}{\alpha_1}e^{\varepsilon_1},\ \varepsilon_4 = \alpha_4 e^{\varepsilon_3},\ \varepsilon_5 = \frac{\alpha_5}{2(\alpha_1-1)}e^{2(\varepsilon_1-\varepsilon_3)},\ \varepsilon_1,\ \varepsilon_3\in\mathbb{R}.$$

Therefore, all elements in $\{\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\mathbf{v}_3+\alpha_4\mathbf{v}_4+\alpha_5\mathbf{v}_5 : \alpha_1\notin\{0,1\}$ and $\alpha_2,\ \alpha_4,\ \alpha_5\in\mathbb{R}\}\subset\mathscr{G}_1^1$ can be partitioned into the conjugacy classes $[a\mathbf{v}_1+\mathbf{v}_3]$, where $a\in\mathbb{R}-\{0,1\}$.

Case I.iii) $(\alpha_1 = 0)$

Here we select the representatives as $\mathbf{v}_2+\mathbf{v}_3$ and $-\mathbf{v}_2+\mathbf{v}_3$ respectively for $\alpha_2>0$ and $\alpha_2<0$. Following similar steps, one can obtain the admissible domain of the parameters for the solvability of adjoint transformation equation Eq. (15) as

$$\varepsilon_1 = \ln\frac{1}{|\alpha_2|},\ \varepsilon_4 = \alpha_4 e^{\varepsilon_3},\ \varepsilon_5 = -\frac{1}{2}\frac{\alpha_5}{\alpha_2^2}\,e^{-2\varepsilon_3},\ \varepsilon_2,\ \varepsilon_3\in\mathbb{R},$$

and the corresponding rule among the coefficients $(0,\alpha_2,1,\alpha_3,\alpha_5)$ and the target coefficients $(0,\pm1,1,0,0)$ are given by

$$\alpha_2 = \pm e^{-\varepsilon_1},\ \alpha_4 = \varepsilon_4 e^{-\varepsilon_3},\ \alpha_5 = -2\varepsilon_5 e^{2(\varepsilon_3-\varepsilon_1)}.$$

For $\alpha_2 = 0$ target element in the conjugacy class is $\mathbf{v}_3$ so that Eqs. (15) and (16) provide the admissible domain of group parameters and the rule of correspondence among coefficients as

$$\varepsilon_4 = \alpha_4 e^{\varepsilon_3},\ \varepsilon_5 = -\frac{1}{2}\alpha_5 e^{2(\varepsilon_1-\varepsilon_3)},\ \varepsilon_1,\ \varepsilon_2,\ \varepsilon_3\in\mathbb{R}.$$
$$\alpha_4 = \varepsilon_4 e^{-\varepsilon_3},\ \alpha_5 = -2\varepsilon_5 e^{2(\varepsilon_3-\varepsilon_1)}.$$

Hence, in this case, the subset $\{\alpha_2\mathbf{v}_2+\mathbf{v}_3+\alpha_4\mathbf{v}_4+\alpha_5\mathbf{v}_5 : \alpha_2,\ \alpha_4,\ \alpha_5\in\mathbb{R}\}\subset\mathscr{G}_1^1$ can be partitioned into the conjugacy classes $[\mathbf{v}_2+\mathbf{v}_3]$, $[-\mathbf{v}_2+\mathbf{v}_3]$, and $[\mathbf{v}_3]$.

Hence the partitioning of $\mathscr{G}_1^{\neq}$ (it follows from the fact that scaling by non-zero real number expands the conjugacy classes of $\mathscr{G}_1^1$ to conjugacy classes of $\mathscr{G}_1^{\neq}$) into the conjugacy classes

$$\begin{array}{ccc} [\mathbf{v}_1+\mathbf{v}_3+\mathbf{v}_5]\,, & [\mathbf{v}_1+\mathbf{v}_3-\mathbf{v}_5]\,, & [\mathbf{v}_1+\mathbf{v}_3]\,, \\ [a\mathbf{v}_1+\mathbf{v}_3]\,, a\notin\{0,1\} & [\mathbf{v}_3+\mathbf{v}_2]\,, & [\mathbf{v}_3-\mathbf{v}_2]\,, \\ & [\mathbf{v}_3] & \end{array}$$

is proved.

Case II. $\phi_1 = 0$. The partitioning of the rest $\mathscr{G}_1^0\,(=\{\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\alpha_4\mathbf{v}_4+\alpha_5\mathbf{v}_5 : \alpha_1,\ \alpha_2,\ \alpha_4,\ \alpha_5\in\mathbb{R}\})$ of $\mathscr{G}$ can be accomplished by coupling the other invariant $\phi_2\,(=\alpha_1)$ given in Eq. (28). Since ϕ_2 is an odd degree homogeneous function of $\alpha_i, i = 1,2,\ldots,5$ we have the following subcases:

Case II.i) $\phi_2 = 1$.
For $\alpha_4 > 0$ and $\alpha_4 < 0$, we choose the representatives respectively as $\mathbf{v}_1 + \mathbf{v}_4$ and $\mathbf{v}_1 - \mathbf{v}_4$. Solving the adjoint transformation equation Eq. (15) with the target coefficients $(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_5)$ as

$$(1, 0, 0, 1, 0) \quad \text{and} \quad (1, 0, 0, -1, 0)$$

one gets the admissible adjoint map $\mathrm{Ad}(e^{\sum_{i=1}^{5} \varepsilon_i \mathbf{v}_i})(\cdot)$ induced by the parameters

$$\varepsilon_2 = \alpha_2 e^{\varepsilon_1},\ \varepsilon_3 = \ln \frac{1}{|\alpha_4|},\ \varepsilon_5 = \frac{1}{2}\alpha_5 \alpha_4^2\, e^{2\varepsilon_1}, \ \text{ where } \ \varepsilon_1,\, \varepsilon_4 \in \mathbb{R}.$$

Hence, for $\alpha_4 \neq 0$, $\mathbf{v}_1 \pm \mathbf{v}_4$ are admissible for generating two inequivalent one-dimensional subalgebras.

The corresponding conjugacy classes $[\mathbf{v}_1 \pm \mathbf{v}_4]$ are accordingly may be found as

$$\alpha_2 = \varepsilon_2\, e^{-\varepsilon_1},\ \alpha_4 = \pm e^{-\varepsilon_3},\ \alpha_5 = 2\varepsilon_5\, e^{2(\varepsilon_3 - \varepsilon_1)}.$$

For $\alpha_4 = 0$, we select the target element as $\mathbf{v}_1$. Hence, the admissible domain of group parameters and the elements in the conjugacy class $[\mathbf{v}_1]$ are given by

$$\varepsilon_2 = \alpha_2 e^{\varepsilon_1},\ \varepsilon_5 = \frac{1}{2}\alpha_5 e^{2(\varepsilon_1 - \varepsilon_3)},\ \varepsilon_1, \varepsilon_3, \varepsilon_4 \in \mathbb{R}.$$
$$\alpha_2 = \varepsilon_2 e^{-\varepsilon_1}\ \alpha_5 = 2\varepsilon_5 e^{2(\varepsilon_3 - \varepsilon_1)}.$$

So, elements in $\{\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_4\mathbf{v}_4 + \alpha_5\mathbf{v}_5 : \alpha_2,\, \alpha_4,\, \alpha_5 \in \mathbb{R}\}$ are partitioned into three conjugacy classes $[\mathbf{v}_1 + \mathbf{v}_4]$, $[\mathbf{v}_1 - \mathbf{v}_4]$ and $[\mathbf{v}_1]$.

Case II.ii) $\phi_2 = 0$.
We have $\alpha_1 = \alpha_3 = 0$. The representatives of conjugacy classes for partitioning of $\{\alpha_2\mathbf{v}_2 + \alpha_4\mathbf{v}_4 + \alpha_5\mathbf{v}_5 : \alpha_2,\, \alpha_4,\, \alpha_5 \in \mathbb{R}\}$ are summarised below.

Table 3: Representatives and adjoint map of other elements in the class.

Case			[Representative]	Adjoint map
$\alpha_2 > 0$	$\alpha_4 > 0$		$[\mathbf{v}_2 + \mathbf{v}_4 + \frac{\alpha_5\alpha_4^2}{\alpha_2^2}\mathbf{v}_5]$	$\varepsilon_1 = \ln\frac{1}{\alpha_2},\ \varepsilon_3 = \ln\frac{1}{\lvert\alpha_4\rvert}$
	$\alpha_4 < 0$		$[\mathbf{v}_2 - \mathbf{v}_4 + \frac{\alpha_5\alpha_4^2}{\alpha_2^2}\mathbf{v}_5]$	
	$\alpha_4 = 0$	$\alpha_5 > 0$	$[\mathbf{v}_2 + \mathbf{v}_5]$	$\varepsilon_1 = \ln\frac{1}{\alpha_2},\ \varepsilon_3 = \ln\frac{\sqrt{\lvert\alpha_5\rvert}}{\alpha_2}$
		$\alpha_5 < 0$	$[\mathbf{v}_2 - \mathbf{v}_5]$	
		$\alpha_5 = 0$	$[\mathbf{v}_2]$	$\varepsilon_1 = \ln\frac{1}{\alpha_2}$
$\alpha_2 < 0$	$\alpha_4 > 0$		$[-\mathbf{v}_2 + \mathbf{v}_4 + \frac{\alpha_5\alpha_4^2}{\alpha_2^2}\mathbf{v}_5]$	$\varepsilon_1 = \ln\left(\frac{1}{\lvert\alpha_2\rvert}\right),\ \varepsilon_3 = \ln\frac{1}{\lvert\alpha_4\rvert}$
	$\alpha_4 < 0$		$[-\mathbf{v}_2 - \mathbf{v}_4 + \frac{\alpha_5\alpha_4^2}{\alpha_2^2}\mathbf{v}_5]$	
	$\alpha_4 = 0$	$\alpha_5 > 0$	$[-\mathbf{v}_2 + \mathbf{v}_5]$	$\varepsilon_1 = \ln\frac{1}{\lvert\alpha_2\rvert},\ \varepsilon_3 = \ln\frac{\sqrt{\lvert\alpha_5\rvert}}{\lvert\alpha_2\rvert}$
		$\alpha_5 < 0$	$[-\mathbf{v}_2 - \mathbf{v}_5]$	
		$\alpha_5 = 0$	$[-\mathbf{v}_2]$	$\varepsilon_1 = \ln\frac{1}{\lvert\alpha_2\rvert}$
$\alpha_2 = 0$	$\alpha_4 > 0$	$\alpha_5 > 0$	$[\mathbf{v}_4 + \mathbf{v}_5]$	$\varepsilon_3 = \ln\frac{1}{\alpha_4},\ \varepsilon_1 = \ln\frac{1}{\alpha_4\sqrt{\lvert\alpha_5\rvert}}$
		$\alpha_5 < 0$	$[\mathbf{v}_4 - \mathbf{v}_5]$	
		$\alpha_5 = 0$	$[\mathbf{v}_4]$	$\varepsilon_3 = \ln\frac{1}{\alpha_4}$
	$\alpha_4 < 0$	$\alpha_5 > 0$	$[-\mathbf{v}_4 + \mathbf{v}_5]$	$\varepsilon_3 = \ln\frac{1}{\lvert\alpha_4\rvert},\ \varepsilon_1 = \ln\frac{1}{\lvert\alpha_4\rvert\sqrt{\lvert\alpha_5\rvert}}$
		$\alpha_5 < 0$	$[-\mathbf{v}_4 - \mathbf{v}_5]$	
		$\alpha_5 = 0$	$[-\mathbf{v}_4]$	$\varepsilon_3 = \ln\frac{1}{\lvert\alpha_4\rvert}$
	$\alpha_4 = 0$	$\alpha_5 > 0$	$[\mathbf{v}_5]$	$\varepsilon_1 = \varepsilon_3 + \ln\frac{1}{\sqrt{\lvert\alpha_5\rvert}}$
		$\alpha_5 > 0$	$[-\mathbf{v}_5]$	

Since scaling of a representative by a non-zero real number gives an element in the conjugacy class, the conjugacy classes partitioning $\{\alpha_2\mathbf{v}_2 + \alpha_4\mathbf{v}_4 + \alpha_5\mathbf{v}_5 : \alpha_2,\, \alpha_4,\, \alpha_5 \in \mathbb{R}\}$ are

$$\begin{array}{ccc} [\mathbf{v}_2 + \mathbf{v}_4 + a\mathbf{v}_5],\ a \in \mathbb{R} & [\mathbf{v}_2 - \mathbf{v}_4 + a\mathbf{v}_5],\ a \in \mathbb{R} & [\mathbf{v}_2 + \mathbf{v}_5], \\ [\mathbf{v}_2 - \mathbf{v}_5], & [\mathbf{v}_4 + \mathbf{v}_5], & [\mathbf{v}_4 - \mathbf{v}_5], \\ [\mathbf{v}_2], & [\mathbf{v}_4], & [\mathbf{v}_5], \end{array}$$

where $\pm\frac{\alpha_5\alpha_4^2}{\alpha_2^2}$ can be treated as a real number a in the real line $\mathbb{R}$.

A combination of all the cases established that the optimal system of one-dimensional subalgebras for $\mathscr{G}$ are elements in the collection (29). ■

4 Group invariant solutions

We use here the elements (IGs) in the optimal system obtained in the previous Sec.3.4 to find group invariant solutions for the system of PDEs in Eq. (3) appearing as the mathematical model for SWWFAVSB. The following investigations are performed in dimensionless variables. To get the solutions in terms of the physical variables, one must properly use the relations in Eqs. (2b) and (2c).

(1) Invariant solution for LGSTs generated by $\mathscr{H}(\mathbf{v_2}+\frac{1}{c}\mathbf{v_4}+a\mathbf{v_5})$, $\mathbf{c}\neq\mathbf{0}$.

The IG is

$$\mathbf{v}\equiv\mathbf{v}_2+\frac{1}{c}\mathbf{v}_4+a\mathbf{v}_5=\frac{\partial}{\partial x}+\frac{1}{c}\frac{\partial}{\partial t}+a\frac{\partial}{\partial\omega}-\alpha a\frac{\partial}{\partial h}. \tag{31}$$

For $u=\theta_1(x,t)$, $\omega=\theta_2(x,t)$ and $h=\theta_3(x)$ to be invariant solution of Eq. (3) for the IG $\mathbf{v}$, the invariance surface condition ([24], pp-331)

$$\left.(\mathbf{v}[u-\theta_1(x,t)],\mathbf{v}[\omega-\theta_2(x,t)],\mathbf{v}[h-\theta_3(x,t)])\right|_{u=\theta_1,\,\omega=\theta_2,\,h=\theta_3}=(0,0,0) \tag{32}$$

gives the following Lagrange auxiliary equations

$$\frac{dx}{1}=\frac{dt}{\frac{1}{c}}=\frac{du}{0}=\frac{d\omega}{a},\ \frac{dx}{1}=\frac{dh}{-\alpha a}.$$

Similarity variables for $\mathbf{v}$ are

$$\zeta=x-ct,\ u=\Phi(\zeta),\ \omega=ax+\Psi(\zeta),\ h=-\alpha ax+\lambda(\zeta). \tag{33}$$

Since $h=h(x)$ is a function of x only, we can use $h_t=0$. So from Eq. (33) we have

$$\lambda'=0.$$

It implies

$$\lambda(\zeta)=d,$$

for some constant d. In terms of the similarity variables given in Eq. (33), the Eq.(3) can be reduced to the system of ODEs for Φ and Ψ

$$(\alpha\Phi-c)\Phi'+\Psi'+a\ =\ 0, \tag{34a}$$
$$(\alpha\Psi+d)\Phi'+(\alpha\Phi-c)\Psi'\ =\ 0. \tag{34b}$$

Eq. (34) can be put into the form

$$\Phi'\ =\ \frac{a(c-\alpha\Phi)}{(\alpha\Phi-c)^2-(\alpha\Psi+d)} \tag{35}$$
$$\Psi'\ =\ \frac{a(\alpha\Psi+d)}{(\alpha\Phi-c)^2-(\alpha\Psi+d)}. \tag{36}$$

Eqs. (34a) and (34b) may be solved to get autonomous and non-autonomous invariants

$$\Lambda_1(\zeta,\Phi,\Psi,\alpha,c,d)\ =\ (\alpha\Phi-c)(\alpha\Psi+d), \tag{37}$$
$$\Lambda_2(\zeta,\Phi,\Psi,\alpha,c,d)\ =\ \frac{1}{\alpha}(\alpha\Phi-c)^2+2\Psi+2a\zeta. \tag{38}$$

Eliminating Ψ from Eqs. (37) and (38), it appears that $(\alpha\Phi-c)$ may be regarded as a real root of the cubic equation

$$z^3+3Hz+G\ =\ 0, \tag{39}$$

with

$$H=\frac{1}{3}(2a\alpha\zeta-2d-\alpha k_2)\quad\text{and}\quad G=2k_1. \tag{40}$$

Here $k_1=\Lambda_1(\zeta,\Phi,\Psi,\alpha,c,d)$ and $k_2=\Lambda_2(\zeta,\Phi,\Psi,\alpha,c,d)$ are independent parameters. Using Cardan's method Eq. (39) can be solved for z as

$$z=\frac{1}{p}(p^2-H),\quad\text{where } p=\left[\tfrac{1}{2}\left\{-G+\sqrt{G^2+4H^3}\right\}\right]^{\frac{1}{3}}. \tag{41}$$

Then from Eqs. (41) and (37), the parameter k_1, k_2, $\mathpzc{a}$, α, $\mathpzc{d}$, c and (independent variable) ζ dependence of the arbitrary functions Φ and Ψ may found as

$$\begin{aligned}
\Phi(\zeta) &= \frac{\left[-k_1+\sqrt{k_1^2+\frac{1}{27}\left\{2\mathpzc{a}\alpha\zeta-2\mathpzc{d}-\alpha k_2\right\}^3}\right]^{\frac{2}{3}}-\frac{1}{3}(2\mathpzc{a}\alpha\zeta-2\mathpzc{d}-\alpha k_2)}{\alpha\left[-k_1+\sqrt{k_1^2+\frac{1}{27}\left\{2\mathpzc{a}\alpha\zeta-2\mathpzc{d}-\alpha k_2\right\}^3}\right]^{\frac{1}{3}}}+\frac{c}{\alpha},\\
\Psi(\zeta) &= \frac{\frac{k_1}{\alpha}\left[-k_1+\sqrt{k_1^2+\frac{1}{27}\left\{2\mathpzc{a}\alpha\zeta-2\mathpzc{d}-\alpha k_2\right\}^3}\right]^{\frac{1}{3}}}{\left[-k_1+\sqrt{k_1^2+\frac{1}{27}\left\{2\mathpzc{a}\alpha\zeta-2\mathpzc{d}-\alpha k_2\right\}^3}\right]^{\frac{2}{3}}-\frac{1}{3}(2\mathpzc{a}\alpha\zeta-2\mathpzc{d}-\alpha k_2)}-\frac{\mathpzc{d}}{\alpha}.
\end{aligned}$$

One may now use Eq. (33) to obtain the explicit space-time and parameter dependence of the physical dimensionless variables u, ω and h as

$$\begin{aligned}
u(x,t) &= \frac{\left[-k_1+\sqrt{k_1^2+\frac{1}{27}\left\{2\mathpzc{a}\alpha\,(x-ct)-2\mathpzc{d}-\alpha k_2\right\}^3}\right]^{\frac{2}{3}}-\frac{1}{3}\{2\mathpzc{a}\alpha\,(x-ct)-2\mathpzc{d}-\alpha k_2\}}{\alpha\left[-k_1+\sqrt{k_1^2+\frac{1}{27}\left\{2\mathpzc{a}\alpha\,(x-ct)-2\mathpzc{d}-\alpha k_2\right\}^3}\right]^{\frac{1}{3}}}+\frac{c}{\alpha},\\
\omega(x,t) &= \frac{\frac{k_1}{\alpha}\left[-k_1+\sqrt{k_1^2+\frac{1}{27}\left\{2\mathpzc{a}\alpha\,(x-ct)-2\mathpzc{d}-\alpha k_2\right\}^3}\right]^{\frac{1}{3}}}{\left[-k_1+\sqrt{k_1^2+\frac{1}{27}\left\{2\mathpzc{a}\alpha\,(x-ct)-2\mathpzc{d}-\alpha k_2\right\}^3}\right]^{\frac{2}{3}}-\frac{1}{3}\{2\mathpzc{a}\alpha\,(x-ct)-2\mathpzc{d}-\alpha k_2\}}\\
&\quad +\mathpzc{a}x-\frac{\mathpzc{d}}{\alpha}\\
h(x) &= \mathpzc{d}-\alpha\mathpzc{a}x,
\end{aligned}\tag{42}$$

with wave velocity c.

Remark 1. It is interesting to observe that the substitution of $\varepsilon_1=0$ and $\varepsilon_3=\ln|c|$, $c\neq 0$ in the adjoint transformation equation Eq. (15) exhibits the subalgebra $\mathscr{H}(\mathbf{v}_2+\frac{1}{c}\mathbf{v}_4+\mathpzc{a}\mathbf{v}_5)$ is conjugate to $\mathscr{H}(\mathbf{v}_2+\mathbf{v}_4+\mathpzc{a}'(a)\mathbf{v}_5)$ for $c>0$ while for $c<0$, $\mathscr{H}(\mathbf{v}_2+\frac{1}{c}\mathbf{v}_4+\mathpzc{a}\mathbf{v}_5)$ is conjugate to $\mathscr{H}(\mathbf{v}_2-\mathbf{v}_4+\mathpzc{a}'(a)\mathbf{v}_5)$, where $\mathpzc{a}'(a)\in\mathbb{R}$, $\forall\, a\in\mathbb{R}$. Then the similarity variables, reduced ODEs, and corresponding GIS of the SNLPDEs in Eq. (3) corresponding to the IGs $\mathbf{v}_2+\mathbf{v}_4+\mathpzc{a}'\mathbf{v}_5$ and $\mathbf{v}_2-\mathbf{v}_4+\mathpzc{a}'\mathbf{v}_5$ in the optimal list (29) can be found respectively by replacing $c=1$ and $c=-1$ in Eqs. (33), (34a), (34b) and Eq. (42).

The solution provided in Eq. (42) is a traveling wave solution for SWWFAVSB Eq. (3), and this solution has yet to be reported before in the literature. One advantage of using Lie symmetry-based reduction using the subalgebra $\mathscr{H}(\mathbf{v}_2+\frac{1}{c}\mathbf{v}_4+\mathpzc{a}\mathbf{v}_5)$ is that the bottom profile $h(x)$ has come out as a part of the solution as $h(x)=\mathpzc{d}-\alpha\mathpzc{a}x$, which provides the form of a uniform slopping bottom topography.

(2) Symmetry reduction for $\mathscr{H}(\mathbf{v_2}\pm\mathbf{v_5})$.

Here the similarity variables corresponding to the IGs $\mathbf{v}_2\pm\mathbf{v}_5$ in optimal class can be found as

$$\zeta=t,\;\; u=\Phi(\zeta),\;\; \omega=\pm x+\Psi(\zeta),\;\; h=\mp\alpha x+\lambda(\zeta),\tag{43}$$

where the functions Φ, Ψ and λ satisfies the system of ODEs

$$\Phi'\pm 1=0,\quad \Psi'=0,\quad \lambda'=0.\tag{44}$$

The solution to these equations is given by

$$\Phi(\zeta)=\mp\zeta+k_1,\quad \Psi(\zeta)=k_2,\quad \lambda(\zeta)=\mathpzc{d}.$$

Use of these solutions in Eq. (43) gives similarity solution of Eq. (3) for the dimensionless physical variables u, ω, h with the dependence of parameters k_1, k_2, $\mathpzc{d}$ as

$$\begin{aligned}
u(x,t) &= \mp t+k_1,\\
\omega(x,t) &= \pm x+k_2,\\
h(x) &= \mp\alpha x+\mathpzc{d}.
\end{aligned}\tag{45}$$

(3) Symmetry reduction for $\mathscr{H}(\mathbf{v_4})$.

One can obtain the similarity variables for $\mathbf{v}_4$ as

$$\zeta = x,\ u = \Phi(\zeta),\ \omega = \Psi(\zeta),\ h = \lambda(\zeta). \tag{46}$$

Use of Eq. (46) in Eq. (3) gives equations for the unknown functions Φ and Ψ as

$$\alpha\Phi\Phi' + \Psi' = 0, \tag{47}$$

$$(\alpha\Psi + \lambda)\Phi' + \Phi(\alpha\Psi' + \lambda') = 0. \tag{48}$$

Eqs. (47) and (48) may be solved to get two autonomous invariants

$$\Lambda_1(\zeta,\,\Phi,\,\Psi,\,\lambda,\,\alpha) = \alpha\Phi^2 + 2\Psi, \tag{49}$$

$$\Lambda_2(\zeta,\,\Phi,\,\Psi,\,\lambda,\,\alpha) = \Phi(\alpha\Psi + \lambda). \tag{50}$$

One can now eliminate Ψ from Eqs. (49) and (50) with taking $k_1 = \Lambda_1(\zeta,\,\Phi,\,\Psi,\,\lambda,\,\alpha)$ and $k_2 = \Lambda_2(\zeta,\,\Phi,\,\Psi,\,\lambda,\,\alpha)$ as independent parameters and obtain a cubic equation for Φ. Solution of this cubic equation followed by the inverse transformation in Eq. (46) provides similarity solution of Eq. (3) involving the parameters k_1, k_2, α and arbitrary function F for the conjugacy class of $\mathbf{v}_4$ as

$$\begin{aligned}
u(x,t) &= \frac{3\left\{-\alpha k_2 + \sqrt{\alpha^2 k_2^2 - \frac{1}{27}(2F(x)+\alpha k_1)^3}\right\}^{\frac{2}{3}} + 2F(x) + \alpha k_1}{3\alpha\left\{-\alpha k_2 + \sqrt{\alpha^2 k_2^2 - \frac{1}{27}(2F(x)+\alpha k_1)^3}\right\}^{\frac{1}{3}}},\\
\omega(x,t) &= \frac{3k_2\left\{-\alpha k_2 + \sqrt{\alpha^2 k_2^2 - \frac{1}{27}(2F(x)+\alpha k_1)^3}\right\}^{\frac{1}{3}}}{3\left\{-\alpha k_2 + \sqrt{\alpha^2 k_2^2 - \frac{1}{27}(2F(x)+\alpha k_1)^3}\right\}^{\frac{2}{3}} + 2F(x) + \alpha k_1} - \frac{F(x)}{\alpha},\\
h(x) &= F(x).
\end{aligned} \tag{51}$$

The solution in Eq. (51) is interesting in the sense that the velocity $u(x,t)$ and the wave amplitude $\omega(x,t)$ are expressed in terms of the arbitrary bottom $h(x) = F(x)$.

Invariant solutions corresponding to $\mathscr{H}(\mathbf{v}_2)$ are only trivial constant solutions. There is no invariant solutions of Eq. (3) for the subalgebras $\mathscr{H}(\mathbf{v}_4+\mathbf{v}_5)$, $\mathscr{H}(\mathbf{v}_4-\mathbf{v}_5)$, and $\mathscr{H}(\mathbf{v}_5)$.

The similarity variables, the reduced system of ODEs, and invariant solutions for admissible one-dimensional subalgebras have been summarised in Table 4.

Table 4: Similarity variables, reduced equations and GIS for some IGs in $\mathscr{OG}_1$ of LGSTs admitted by Eq. (3).

	Subalgebra	Similarity variables	Reduced equations	Invariant solution
1	$\mathscr{H}(\mathbf{v}_1+\mathbf{v}_3+\mathbf{v}_5)$	$\zeta = \frac{x}{t}$, $u = \Phi(\zeta)$, $\omega = \ln x + \Psi(\zeta)$, $h = -\alpha\ln x + \not{d}$.	$\Phi' = \frac{\alpha\Phi-\zeta}{\zeta(\not{d}+\alpha\Psi)-\zeta(\alpha\Phi-\zeta)^2}$, $\Psi' = \frac{-(\alpha\Psi+\not{d})}{\zeta(\not{d}+\alpha\Psi)-\zeta(\alpha\Phi-\zeta)^2}$.	$u(x,t) = \sqrt{-\frac{2}{\alpha}\ln x}$, $\omega(x,t) = \ln x - \frac{\not{d}}{\alpha}$, $h(x) = -\alpha\ln x + \not{d}$, where, $\not{d} \in \mathbb{R}$.
2	$\mathscr{H}(\mathbf{v}_1+\mathbf{v}_3-\mathbf{v}_5)$	$\zeta = \frac{x}{t}$, $u = \Phi(\zeta)$, $\omega = -\ln x + \Psi(\zeta)$, $h = \alpha\ln x + \not{d}$.	$\Phi' = \frac{\zeta-\alpha\Phi}{\zeta(\not{d}+\alpha\Psi)-\zeta(\alpha\Phi-\zeta)^2}$, $\Psi' = \frac{(\alpha\Psi+\not{d})}{\zeta(\not{d}+\alpha\Psi)-\zeta(\alpha\Phi-\zeta)^2}$.	$u(x,t) = \sqrt{\frac{2}{\alpha}\ln x}$, $\omega(x,t) = -\ln x - \frac{\not{d}}{\alpha}$, $h(x) = \alpha\ln x + \not{d}$, where, $\not{d} \in \mathbb{R}$.
3	$\mathscr{H}(\mathbf{v}_1+\mathbf{v}_3)$	$\zeta = \frac{x}{t}$, $u = \Phi(\zeta)$, $\omega = \Psi(\zeta)$, $h = \not{d}$.	$\zeta\Phi' - \Psi' - \alpha\Phi\Phi' = 0$, $\not{d}\Phi' - \zeta\Psi' + \alpha\Psi\Phi' + \alpha\Phi\Psi' = 0$.	$u(x,t) = \frac{2x+c_1t}{3\alpha t}$, $\omega(x,t) = \frac{x^2-2c_1xt+(c_1^2-9\not{d})t^2}{9\alpha t^2}$, $h(x) = \not{d}$, where, $c_1, \not{d} \in \mathbb{R}$.
4	$\mathscr{H}(\mathbf{v}_1+\mathbf{v}_4)$	$\zeta = xe^{-t}$, $u = e^t\Phi(\zeta)$, $\omega = e^{2t}\Psi(\zeta)$, $h = \not{d}x^2$.	$\Phi' = \frac{\Phi(\alpha\Phi-\zeta)-2(\not{d}\zeta\Phi+\Psi)}{(\not{d}\zeta^2+\alpha\Psi)-(\alpha\Phi-\zeta)^2}$, $\Psi' = \frac{2(\alpha\Phi-\zeta)(\not{d}\zeta\Phi+\Psi)-\Phi(\not{d}\zeta^2+\alpha\Psi)}{(\not{d}\zeta^2+\alpha\Psi)-(\alpha\Phi-\zeta)^2}$.	$u(x,t) = -\frac{2x}{\alpha}$, $\omega(x,t) = \frac{c_1\alpha e^{2t}-2x^2}{\alpha}$, $h(x) = 2x^2$, where, $c_1 \in \mathbb{R}$. $u(x,t) = \frac{x}{\alpha}$, $\omega(x,t) = -\frac{x^2}{2\alpha}$, $h(x) = \frac{x^2}{2}$.
5	$\mathscr{H}(\mathbf{v}_1-\mathbf{v}_4)$	$\zeta = xe^{t}$, $u = e^{-t}\Phi(\zeta)$, $\omega = e^{-2t}\Psi(\zeta)$, $h = \not{d}x^2$.	$\Phi' = \frac{\Phi(\alpha\Phi+\zeta)+2(\not{d}\zeta\Phi-\Psi)}{(\alpha\Phi+\zeta)^2-(\alpha\Psi+\not{d}\zeta^2)}$, $\Psi' = \frac{2(\alpha\Phi+\zeta)(\Psi-\not{d}\zeta\Phi)-\Phi(\alpha\Psi+\not{d}\zeta^2)}{(\alpha\Phi+\zeta)^2-(\alpha\Psi+\not{d}\zeta^2)}$.	$u(x,t) = \frac{2x}{\alpha}$, $\omega(x,t) = \frac{c_1\alpha e^{-2t}-2x^2}{\alpha}$, $h(x) = 2x^2$, where $c_1 \in \mathbb{R}$.
6	$\mathscr{H}(\mathbf{v}_4)$	$\zeta = x$, $u = \Phi(\zeta)$, $\omega = \Psi(\zeta)$, $h = \lambda(\zeta)$.	$\alpha\Phi\Phi' + \Psi' = 0$, $(\alpha\Psi+\lambda)\Phi' + \Phi(\alpha\Psi'+\lambda') = 0$.	$u(x,t) = \sqrt{\frac{-2F(x)+\alpha c_1}{\alpha}}$, $\omega(x,t) = F(x)$, $h(x) = \frac{c_2-\alpha F(x)\sqrt{2F(x)-\alpha c_1}}{\sqrt{2F(x)-\alpha c_1}}$, where, $c_1, c_2 \in \mathbb{R}$, and $F(x)$ being arbitrary function of x.

The results presented in Table 4 may be interpreted as the following. The invariance of the solution of Eq. (3) under the subalgebras $\mathscr{H}(\mathbf{v}_1+\mathbf{v}_3\pm\mathbf{v}_5)$ provide the solutions of Eq. (3) having logarithmic bottom $h(x)=\mp\alpha\ln x+d$ where the velocity $u(x,t)$ and amplitude $\omega(x,t)$ do not depend on time t. The wave profile for uniform depth $h(x)=d$ is found by the invariance under $\mathscr{H}(\mathbf{v}_1+\mathbf{v}_3)$. In this case, it may be observed that for fixed x, the velocity $u(x,t)\to 0$ and the wave height $\omega(x,t)\to 0$ as $t\to\infty$. The wave profiles incident upon parabolic bottom $h(x)=dx^2$ for $d\in\left\{2,\frac{1}{2}\right\}$ are obtained by using the invariance under $\mathscr{H}(\mathbf{v}_1\pm\mathbf{v}_4)$. It may be observed here that two different kinds of propagations may be exhibited for $d=2$. The amplitude $\omega(x,t)$ provided in the fourth row blows up to infinity as $t\to\infty$ whereas $\omega(x,t)\to 0$ as $t\to\infty$ in the fifth row. For $d=\frac{1}{2}$, u and ω have no explicit dependence on t.

Table 5 presents the similarity variables and the reduced equations corresponding to IGs in the set of optimal one-dimensional subalgebras for which reduced equations can not be solved exactly.

Table 5: Similarity variables and reduced system of ODEs for some elements in $\mathscr{OG}_1$.

Subalgebra	Similarity variables	Reduced equation
$\mathscr{H}(\mathbf{v}_1)$	$\zeta=t,\ u=x\Phi(\zeta),$ $\omega=x^2\Psi(\zeta),\ h=dx^2.$	$\Phi'=-(\alpha\Phi^2+2\Psi),$ $\Psi'=-3(d\Phi+\alpha\Phi\Psi).$
$\mathscr{H}(\mathbf{v}_3+\mathbf{v}_2)$	$\zeta=\frac{e^x}{t},\ u=e^{-x}\Phi(\zeta),$ $\omega=e^{-2x}\Psi(\zeta),\ h=de^{-2x}.$	$\Phi'=\frac{(\alpha\zeta\Phi-\zeta^2)(\alpha\Phi^2+2\Psi)-3\zeta(\alpha\Phi\Psi+d\Phi)}{(\alpha\zeta\Phi-\zeta^2)^2-\zeta^2(\alpha\Psi+d)},$ $\Psi'=\frac{3\zeta(\alpha\Phi\Psi+d\Phi)(\alpha\Phi-\zeta)-\zeta(\alpha\Psi+d)(\alpha\Phi^2+2\Psi)}{(\alpha\zeta\Phi-\zeta^2)^2-\zeta^2(\alpha\Psi+d)}.$
$\mathscr{H}(\mathbf{v}_3-\mathbf{v}_2)$	$\zeta=\frac{e^{-x}}{t},\ u=e^{x}\Phi(\zeta),$ $\omega=e^{2x}\Psi(\zeta),\ h=de^{2x}.$	$\Phi'=\frac{\zeta(\alpha\Phi+\zeta)(\alpha\Phi^2+2\Psi)-3\zeta\Phi(\alpha\Psi+d)}{\zeta^2\{(\alpha\Phi+\zeta)^2-(\alpha\Psi+d)\}},$ $\Psi'=\frac{3\zeta\Phi(\alpha\Phi+\zeta)(\alpha\Psi+d)-\zeta(\alpha\Psi+d)(\alpha\Phi^2+2\Psi)}{\zeta^2\{(\alpha\Phi+\zeta)^2-(\alpha\Psi+d)\}}.$
$\mathscr{H}(a\mathbf{v}_1+\mathbf{v}_3),\ a\neq 1$	$\zeta=\frac{x}{t^a},\ u=t^{a-1}\Phi(\zeta),$ $\omega=t^{2(a-1)}\Psi(\zeta),\ h=dx^{\frac{2(a-1)}{a}}.$	$\Phi'=\frac{(1-a)\Phi(\alpha\Phi-a\zeta)-\frac{2d(1-a)}{a}\zeta^{\frac{a-2}{a}}\Phi-2(1-a)\Psi}{(\alpha\Phi-a\zeta)^2-\left(\alpha\Psi+d\zeta^{\frac{2(a-1)}{a}}\right)},$ $\Psi'=\frac{\frac{2d(1-a)}{a}\zeta^{\frac{a-2}{a}}\Phi(\alpha\Phi-a\zeta)+2(1-a)\Psi(\alpha\Phi-a\zeta)-(1-a)\Phi\left(\alpha\Psi+d\zeta^{\frac{2(a-1)}{a}}\right)}{(\alpha\Phi-a\zeta)^2-\left(\alpha\Psi+d\zeta^{\frac{2(a-1)}{a}}\right)}.$

5 Conclusion

This work deals with the application of the Lie group theoretic approach for obtaining solutions of SNLPDEs Eq. (3) in $(1+1)$ dimensions appearing in the mathematical modeling of SWWFAVSB. The step-by-step procedure of the Lie group theoretic approach for the SNLPDEs Eq. (3) has been exercised and found that the model admits five parameter LGSTs generated by the infinitesimal generators given in Eq. (9). It is observed that the Lie algebra (of infinitesimal generators), $\mathscr{G}$, is isomorphic to the classical Lie algebra $A_{5,33}^{2,-2}$. For obtaining inequivalent solutions to the SNLPDEs Eq. (3), the invariant functions of inner automorphisms, $\mathrm{Int}(\mathscr{G})$ on $\mathscr{G}$ have been obtained by using three methods. It is observed that invariants of $\mathrm{Int}(\mathscr{G})$ obtained by using an alternative scheme viz., by solving a system of overdetermined equations Eq. (27) for the invariant function ϕ seems more useful in the partitioning of $\mathscr{G}$ into the optimal set of one-dimensional subalgebras than the form of invariants provided in Eq. (21) and Eq. (23) obtained by using Casimir- or Killing polynomials.

A one-dimensional optimal system $\mathscr{OG}_1$ of subalgebras consisting of 18 inequivalent one-dimensional subalgebras of the Lie algebra $\mathscr{G}$ $\left(\equiv A_{5,33}^{2,-2}\right)$ has been derived in Theorem 1. The similarity variables and the reduced system of ODEs in terms of similarity variables for each infinitesimal generator in $\mathscr{OG}_1$ have been provided here. The exact solutions and the invariants of the system for some infinitesimal generators in $\mathscr{OG}_1$ are obtained with various bottom topographies such as the bottom of uniform slope, parabolic bottom, logarithmic bottom, and bottom of uniform depth.

It is observed that the invariance under the one-dimensional subalgebra $\mathscr{H}(\mathbf{v}_2+\frac{1}{c}\mathbf{v}_4+a\mathbf{v}_5)$ provides the traveling wave solution (42) to the SWWFAVSB Eq. (3) for a uniform slopping bottom $h(x)=d-\alpha a x$, and this solution is not reported earlier in the literature. The subalgebra $\mathscr{H}(\mathbf{v}_4)$ provides a solution (51) to the system of Eq. (3) where the velocity u and amplitude ω are time-independent functions of the arbitrary bottom $h(x)$. Possible exact analytical space-time dependent expression of $u(x,t)$ and $\omega(x,t)$ for parabolic bottom $h(x)=dx^2$, $d\in\left\{2,\frac{1}{2}\right\}$ and arbitrary uniform depth $h(x)=d$ are provided in Table 4.

From Table 4 it may be observed, for uniform depth $h(x)=d$, that velocity $u(x,t)\to 0$ and wave amplitude $\omega(x,t)\to 0$ as $t\to\infty$. For $h(x)=\frac{1}{2}x^2$, u and ω have no time dependencies under the invariance of $\mathscr{H}(\mathbf{v}_1+\mathbf{v}_4)$.

Any previously exercised methods do not report the results of our investigations [11, 12, 17].

However, all those symmetries have been lost if the problem has some preassigned initial/boundary conditions which are not invariant under the admitted LGSTs generated by the IGs in the optimal set, $\mathscr{OG}_1$ of one-dimensional subalgebras.

In continuation of the present investigation, its extension to the problems

(i) searching exact/approximate solutions of the system of nonlinear ODEs whose solutions can not be obtained by any available symbolic computer software,

(ii) studies on the evolution of a group invariant solutions obtained here under the application of other IGs in $\mathscr{OG}_1$,

(iii) investigations on the prediction of physical properties of the wave from the qualitative behavior of the solution obtained from the reduced system of nonlinear ODEs,

(iv) extension of symmetry analysis of SNLPDEs appearing in the mathematical studies of tsunami waves in $(2+1)$ dimensions

are in progress and will be reported elsewhere in due course.

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