Solution of Inhomogeneous and Homogeneous Heun's Differential Equations in Nonstandard Analysis

Original Research Article

Abstract

Discussions are presented by Morita and Sato on the problem of obtaining the particular solution of an inhomogeneous differential equation with polynomial coefficients in terms of the Green's function. In succeeding papers, Morita gave discussions of this problem on the basis of nonstandard analysis. It was applied to the hypergeometric, the Hermite, a simple ordinary and a fractional differential equation. In the present paper, this method is applied to the solutions of inhomogeneous and homogeneous Heun's differential equations.

Keywords: Heun's differential equation; nonstandard analysis; particular solution; complementary solution

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1 Introduction

In a series of papers, Morita and Sato [1, 2] and Morita [3, 4, 5] studied the problem of obtaining the particular solutions of differential equations by using the Green's function and nonstandard analysis.

In paper [1], this problem is studied in the framework of distribution theory, where the method is applied to Kummer's and the hypergeometric differential equation. In paper [2], this problem is studied in the framework of nonstandard analysis, where a recipe of solution of the present problem is presented, and it is applied to a simple fractional and a first-order ordinary differential equation. In paper [3], a compact recipe based on nonstandard analysis is obtained by revising the one given in [2], and is applied to Kummer's differential equation.

In [4], we adopt a recipe without the Green's function, and is applied to the hypergeometric differential equation, the differential equations treated in [2] and the Hermite differential equation. In [5], we study the same differential equations as in [4], but the solutions are expressed in terms of the Green's function.

The purpose of the present paper is to give solutions of inhomogeneous and homogeneous Heun's differential equation, by using the method presented in [5].

The presentation follows those in [1, 2, 3], in Introduction and in many descriptions in the following sections.

We use Riemann-Liouville fractional integrals and derivatives, whose definition is given in [6, 7], and also in [3, 4, 5]. The property which we use is presened in Section 1.1. The properties which we use in nonstandard analysis, are presented in Section 1.2, following papers [3, 4, 5], and then contents of the following sections are given in Section **??**,

1.1 Riemann-Liouville fractional integrals and derivatives

We give here some notations to be used. $\mathbb{Z}$ is the set of all integers, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all real numbers and all complex numbers, respectively, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} \mid n > a\}$, $\mathbb{Z}_{<b} = \{n \in \mathbb{Z} \mid n < b\}$ and $\mathbb{Z}_{[a,b]} = \{n \in \mathbb{Z} \mid a \le n \le b\}$ for $a, b \in \mathbb{Z}$ satisfying $a < b$. We also use $\mathbb{R}_{>a} = \{x \in \mathbb{R} \mid x > a\}$ for $a \in \mathbb{R}$, and $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0\}$.

We use the step function $H(t)$ for $t \in \mathbb{R}$, which is equal to 1 if $t > 0$, and to 0 if $t \le 0$, and h_k, which denotes $h_k = 1$ if $k \in \mathbb{Z}_{>-1}$, and $h_k = 0$ if $k \in \mathbb{Z}_{<0}$.

We use the Riemann-Liouville fractional integral and derivative ${}_R D_t^\rho$ for $\rho \in \mathbb{C}$, which is defined in the following remark, that is given in [3, 4, 5].
Remark 1.1. Let $g_\nu(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1} H(t)$ for $\nu \in \mathbb{C}$. Then $g_\nu(t) = 0$ if $\nu \in \mathbb{Z}_{<1}$, and if $\nu \notin \mathbb{Z}_{<1}$,

$$ {}_R D_t^\rho g_\nu(t) = {}_R D_t^\rho \frac{1}{\Gamma(\nu)} t^{\nu-1} H(t) = \frac{1}{\Gamma(\nu-\rho)} t^{\nu-\rho-1} H(t) = g_{\nu-\rho}(t). \quad (1) $$

As a consequence, we have ${}_R D_t^{\nu+n} g_\nu(t) = g_{-n}(t) = 0$ for $n \in \mathbb{Z}_{>-1}$.

In distribution theory [1, 8, 9, 10], we use distribution $\tilde{H}(t)$, which corresponds to function $H(t)$, differential operator D and distribution $\delta(t) = D\tilde{H}(t)$, which is called Dirac's delta function.

1.2 Preliminaries on nonstandard analysis

In nonstandard analysis [11], infinitesimal numbers appear. We denote the set of all infinitesimal real numbers by $\mathbb{R}^0$. We use also its subset $\mathbb{R}^0_{>0}$ given by $\mathbb{R}^0_{>0} = \{\epsilon \in \mathbb{R}^0 \mid \epsilon > 0\}$, which is such that if $\epsilon \in \mathbb{R}^0_{>0}$, there exists $N \in \mathbb{Z}_{>0}$ satisfying $\epsilon < \frac{1}{N}$. Now $\mathbb{R}^0 = \mathbb{R}^0_{>0} \cup \{0\} \cup \mathbb{R}^0_{<0}$, where $\mathbb{R}^0_{<0} = \{\epsilon \in \mathbb{R}^0 \mid \epsilon < 0\}$.

We use $\mathbb{R}^{ns}$, which has subsets $\mathbb{R}$ and $\mathbb{R}^0$. If $x \in \mathbb{R}^{ns}$ and $x \notin \mathbb{R}$, x is expressed as $x_1 + \epsilon$ by $x_1 \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^0$, where x_1 may be $0 \in \mathbb{R}$. Equation $x \simeq y$ for $x \in \mathbb{R}^{ns}$ and $y \in \mathbb{R}^{ns}$, is used, when $x - y \in \mathbb{R}^0$. We denote the set of all infinitesimal complex numbers by $\mathbb{C}^0$, which is the set of complex numbers z which satisfy $|\mathrm{Re}\, z| + |\mathrm{Im}\, z| \in \mathbb{R}^0$. We use $\mathbb{C}^{ns}$, which has subsets $\mathbb{C}$ and $\mathbb{C}^0$. If $z \in \mathbb{C}^{ns}$ and $z \notin \mathbb{C}$, z is expressed as $z_1 + \epsilon$ by $z_1 \in \mathbb{C}$ and $\epsilon \in \mathbb{C}^0$, where z_1 may be $0 \in \mathbb{C}$.

In place of (1), we now use

$$ {}_RD_t^{\rho} g_{\nu+\epsilon}(t) = {}_RD_t^{\rho} \frac{1}{\Gamma(\nu+\epsilon)} t^{\nu-1+\epsilon} H(t) = g_{\nu-\rho+\epsilon}(t) = \frac{1}{\Gamma(\nu-\rho+\epsilon)} t^{\nu-\rho-1+\epsilon} H(t), \tag{2} $$

for all $\rho \in \mathbb{C}$ and $\nu \in \mathbb{C}$, where $\epsilon \in \mathbb{R}^0_{>0}$.

Lemma 1.1. *Let $\rho_1 \in \mathbb{C}$, $\rho_2 \in \mathbb{C}$, $\nu \in \mathbb{C}$, $\epsilon \in \mathbb{R}^0_{>0}$ and $g_{\nu+\epsilon}(t) = \frac{1}{\Gamma(\nu+\epsilon)} t^{\nu+\epsilon-1} H(t)$. Then the index law:*

$$ {}_RD_t^{\rho_1}\, {}_RD_t^{\rho_2} g_{\nu+\epsilon}(t) = {}_RD_t^{\rho_1+\rho_2} g_{\nu+\epsilon}(t) = g_{\nu-\rho_1-\rho_2+\epsilon}(t), \tag{3} $$

always holds.

In nonstandard analysis, in place of $\tilde{H}(t)$ and $\delta(t)$ in distribution theory, $H_\epsilon(t)$ and $\delta_\epsilon(t)$ are used, which are given by

$$ H_\epsilon(t) = {}_RD_t^{-\epsilon} H(t) = g_{1+\epsilon}(t) = \frac{1}{\Gamma(\epsilon+1)} t^{\epsilon} H(t), \quad \delta_\epsilon(t) = \frac{d}{dt} H_\epsilon(t) \tag{4} $$

for $\epsilon \in \mathbb{R}^0_{>0}$. We note that they tend to $H(t)$ and 0, respectively, as $\epsilon \to 0$.

Lemma 1.2. *In the notation in Remark 1.1, $H_\epsilon(t) = g_{1+\epsilon}(t)$, $\delta_\epsilon(t) = g_\epsilon(t)$, and*

$$ {}_RD_t^{\epsilon} H_\epsilon(t) = {}_RD_t^{\epsilon} g_{1+\epsilon}(t) = g_1(t) = H(t), \quad {}_RD_t^{\epsilon} \delta_\epsilon(t) = {}_RD_t^{\epsilon} g_\epsilon(t) = g_0(t) = 0. \tag{5} $$

2 Heun's Differential Equation

Before writing Heun's differential equation, we present a related differential equation given by

$$ \begin{aligned} p({}_RD_t, t)u(t) := \{&(t-t_3)(t-t_1)(t-t_2)\frac{d^2}{dt^2} \\ &+[\gamma_3(t-t_1)(t-t_2) + \gamma_1(t-t_2)(t-t_3) + \gamma_2(t-t_3)(t-t_1)]\frac{d}{dt} \\ &+(\alpha_1\beta_1 t - \alpha_1\beta_1 q_0)\}u(t) + D_0 \cdot {}_RD_t^{-1}u(t) = f(t), \end{aligned} \tag{6} $$

where t_1, t_2, t_3, γ_1, γ_2, γ_3, α_1, β_1, q_0 and D_0 are constants. We express this equation as follows:

$$ \begin{aligned} p({}_RD_t, t)u(t) = [&(A_0 + A_1 t + A_2 t^2 + A_3 t^3)\frac{d^2}{dt^2} + (B_0 + B_1 t + B_2 t^2)\frac{d}{dt} \\ &+(C_0 + C_1 t)]u(t) + D_0 \cdot {}_RD_t^{-1}u(t) = f(t), \end{aligned} \tag{7} $$

where

$$ \begin{aligned} &A_0 = -t_1t_2t_3, \quad A_1 = t_1t_2 + t_2t_3 + t_3t_1, \quad A_2 = -t_1 - t_2 - t_3, \quad A_3 = 1, \\ &B_0 = \gamma_1 t_2 t_3 + \gamma_2 t_3 t_1 + \gamma_3 t_1 t_2, \\ &B_1 = -\gamma_1(t_2+t_3) - \gamma_2(t_3+t_1) - \gamma_3(t_1+t_2), \quad B_2 = \gamma_1 + \gamma_2 + \gamma_3, \\ &C_0 = -\alpha_1\beta_1 q_0, \quad C_1 = \alpha_1\beta_1, \quad D_0 = 0. \end{aligned} \tag{8} $$

Heun's equation is given by

$$
\begin{aligned}
p_{He}(t, {}_RD_t)u(t) := \{ & t(t-1)(t-t_2)\frac{d^2}{dt^2} \\
& +[\gamma_3 t_2 - [\alpha_1+\beta_1+1-\gamma_1+(\gamma_1+\gamma_3)t_2]t + (\alpha_1+\beta_1+1)t^2]\frac{d}{dt} \\
& -\alpha_1\beta_1 q_0 + \alpha_1\beta_1 t\}u(t) = f(t). \qquad (9)
\end{aligned}
$$

This equation is a special one of Equations (6), in which $t_1 = 1$, $t_3 = 0$, $\gamma_2 = \alpha_1 + \beta_1 + 1 - \gamma_1 - \gamma_3$ and $D_0 = 0$. As a consequence, we have the following lemma.

Lemma 2.1. *Heun's equation* (9) *is expressed by the equation which is obtained from Equation* (7)*, by replacing* $p(t, {}_RD_t)$ *by* $p_{He}(t, {}_RD_t)$*, and adopting*

$$
\begin{aligned}
& A_0 = 0, \quad A_1 = t_2, \quad A_2 = -(1+t_2), \quad A_3 = 1, \qquad (10) \\
& B_0 = \gamma_3 t_2, \quad B_2 = \gamma_1+\gamma_2+\gamma_3 = \alpha_1+\beta_1+1, \\
& B_1 = -[\alpha_1+\beta_1+1-\gamma_1+(\gamma_1+\gamma_3)t_2] = -[\gamma_1 t_2 + \gamma_2 + \gamma_3(t_2+1)], \\
& C_0 = -\alpha_1\beta_1 q_0, \quad C_1 = \alpha_1\beta_1, \quad D_0 = 0, \qquad (11)
\end{aligned}
$$

in place of Equation (8)*.*

2.1 Transformed equations of Equations (7) and (9)

We now consider the equation which is satisfied by $\tilde{w}(t) = {}_RD_t^{-\beta}\tilde{u}(t) = {}_RD_t^{-\beta}{}_RD_t^{-\epsilon}u(t) = {}_RD_t^{-\rho}u(t)$, for $\beta \in \mathbb{C}$, $\epsilon \in \mathbb{R}^o_{>0}$ and $\rho = \beta + \epsilon$, when $u(t)$ satifies Equation (7).

Lemma 2.2. *Let* $u(t)$ *be a solution of Equation* (7)*, and* $\tilde{w}(t)$ *be given by* $\tilde{w}(t) = {}_RD_t^{-\rho}u(t)$*. Then* $\tilde{w}(t)$ *satisfies*

$$
\begin{aligned}
p_\rho({}_RD_t, t)\tilde{w}(t) &:= {}_RD_t^{-\rho}p({}_RD_t, t){}_RD_t^{\rho}\tilde{w}(t) \\
&= \{A_0\frac{d^2}{dt^2} + [A_1 t\frac{d^2}{dt^2} + \tilde{B}_0(\rho)\frac{d}{dt}] + [A_2 t^2\frac{d^2}{dt^2} + \tilde{B}_1(\rho)t\frac{d}{dt} + \tilde{C}_0(\rho)] \\
&\quad +[A_3 t^3\frac{d^2}{dt^2} + \tilde{B}_2(\rho)t^2\frac{d}{dt} + \tilde{C}_1(\rho)t] + \tilde{D}_0(\rho)\cdot{}_RD_t^{-1}\}\tilde{w}(t) \\
&= \tilde{f}_\beta(t) := {}_RD_t^{-\rho}f(t), \qquad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
& \tilde{B}_0(\rho) = B_0 - A_1\rho, \quad \tilde{B}_2(\rho) = B_2 - A_3\cdot 3\rho, \quad \tilde{B}_1(\rho) = B_1 - A_2\cdot 2\rho, \\
& \tilde{C}_0(\rho) = C_0 - B_1\rho + A_2\cdot\rho(\rho+1) + A_3\cdot 3\rho(\rho+1), \\
& \tilde{C}_1(\rho) = C_1 - B_2\cdot 2\rho + A_3\cdot 3\rho(\rho+1), \\
& \tilde{D}_0(\rho) = D_0 - C_1\rho + B_2\cdot\rho(\rho+1) - A_3\cdot\rho(\rho+1)(\rho+2). \qquad (13)
\end{aligned}
$$

We call Equation (12) *a transformed equation of Equation* (7)*. When Equation* (8) *with Equation* (13) *is adopted, Equation* (12) *is a transformed equation of Equation* (6)*.*

Proof. Remark 9 in [3] shows that when $\nu \in \mathbb{C}$, $n \in \mathbb{Z}_{>-1}$, $\tilde{u}(t) = \frac{t^{\nu+\epsilon}}{\Gamma(\nu+\epsilon+1)}$ and $\tilde{u}_n(t) = \frac{d^n}{dt^n}\tilde{u}(t)$, we have

$$
\begin{aligned}
{}_RD_t^{-\rho}[t\tilde{u}_n(t)] &= t\cdot {}_RD_t^{-\rho}\tilde{u}_n(t) - \rho\cdot {}_RD_t^{-\rho-1}\tilde{u}_n(t),\\
{}_RD_t^{-\rho}[t^2\tilde{u}_n(t)] &= t\cdot {}_RD_t^{-\rho}[t\tilde{u}_n(t)] - \rho\cdot {}_RD_t^{-\rho-1}[t\tilde{u}_n(t)]\\
&= t^2\cdot {}_RD_t^{-\rho}\tilde{u}_n(t) - 2\rho t\cdot {}_RD_t^{-\rho-1}\tilde{u}_n(t) + \rho(\rho+1)\cdot {}_RD_t^{-\rho-2}\tilde{u}_n(t),\\
{}_RD_t^{-\rho}[t^3\tilde{u}_n(t)] &= t^2\cdot {}_RD_t^{-\rho}[t\tilde{u}_n(t)] - 2\rho t\cdot {}_RD_t^{-\rho-1}[t\tilde{u}_n(t)] + \rho(\rho+1)\cdot {}_RD_t^{-\rho-2}[t\tilde{u}_n(t)]\\
&= t^3\cdot {}_RD_t^{-\rho}\tilde{u}_n(t) - 3\rho t^2\cdot {}_RD_t^{-\rho-1}\tilde{u}_n(t) + 3\rho(\rho+1)t\cdot {}_RD_t^{-\rho-2}\tilde{u}_n(t)\\
&\quad -\rho(\rho+1)(\rho+2)\cdot {}_RD_t^{-\rho-3}\tilde{u}_n(t). \qquad (14)
\end{aligned}
$$

By using these relations in Equation (7), we obtain

$$
\begin{aligned}
p_\rho({}_RD_t,t)\tilde{w}(t) &:= {}_RD_t^{-\rho}p({}_RD_t,t)u(t) = \{(A_0+A_1t+A_2t^2+A_3t^3)\frac{d^2}{dt^2}\\
&\quad +(B_0+B_1t+B_2t^2-A_1\rho-A_2\cdot 2\rho t-A_3\cdot 3\rho t^2)\frac{d}{dt}\\
&\quad +C_0+C_1t-B_1\rho-B_2\cdot 2\rho t+A_2\cdot\rho(\rho+1)+A_3\cdot 3\rho(\rho+1)t\\
&\quad +[D_0-C_1\rho+B_2\cdot\rho(\rho+1)-A_3\cdot\rho(\rho+1)(\rho+2)]{}_RD_t^{-1}\}\tilde{w}(t)\\
&= \tilde{f}_\beta(t) := {}_RD_t^{-\rho}f(t). \qquad (15)
\end{aligned}
$$

This equation is expressed by Equation (12). □

As a corollary of this lemma, we have the following lemma.

Lemma 2.3. *Let $u(t)$ be a solution of Equation* (7)*, and $\tilde{u}(t)$ be given by $\tilde{u}(t) = {}_RD_t^{-\epsilon}u(t)$. Then we obtain the following equation from Equation* (12)*, by replacing ρ by ϵ and $\tilde{w}(t)$ by $\tilde{u}(t)$:*

$$
p_\epsilon({}_RD_t,t)\tilde{u}(t) := {}_RD_t^{-\epsilon}p({}_RD_t,t){}_RD_t^{\epsilon}\tilde{u}(t) = \tilde{f}(t) := {}_RD_t^{-\epsilon}f(t), \qquad (16)
$$

which is a transformed equation of Equation (7)*, when Equations* (8) *and* (13) *are adopted.*

We denote the transformed equations of Heun's equation (9), which correspond to Equations (12) and (16), by Equations (12-He) and (16-He), respectively.

Lemma 2.4. *Lemmas 2.2 and 2.1 show that Equation* (12-He) *is obtained from Equation* (12) *by replacing $p_\rho({}_RD_t,t)$ by $p_{\rho,He}({}_RD_t,t)$, and $p({}_RD_t,t)$ by $p_{He}({}_RD_t,t)$, and using Equations* (10) *and* (11) *in place of Equations* (8)*. In this replacement, Equation* (13) *is replaced by*

$$
\begin{aligned}
\tilde{B}_0(\rho) &= (\gamma_3-\rho)t_2,, \quad \tilde{B}_2(\rho) = \gamma_1+\gamma_2+\gamma_3-3\rho = \alpha_1+\beta_1+1-3\rho,\\
\tilde{B}_1(\rho) &= -[\alpha_1+\beta_1+1-\gamma_1+(\gamma_1+\gamma_3)t_2] + (1+t_2)\cdot 2\rho = B_1+(1+t_2)\cdot 2\rho,\\
\tilde{C}_0(\rho) &= -\alpha_1\beta_1q_0 - B_1\rho + (2-t_2)\cdot\rho(\rho+1),\\
\tilde{C}_1(\rho) &= \alpha_1\beta_1 - (\alpha_1+\beta_1+1)\cdot 2\rho + 3\rho(\rho+1) = (\alpha_1-2\rho)(\beta_1-2\rho)-\rho^2+\rho,\\
\tilde{D}_0(\rho) &= -\alpha_1\beta_1\rho + (\alpha_1+\beta_1+1)\cdot\rho(\rho+1) - \rho(\rho+1)(\rho+2)\\
&= (\alpha_1-\rho-1)(\beta_1-\rho-1)\rho. \qquad (17)
\end{aligned}
$$

Lemmas 2.2 and 2.3 show that Equation (16-He) *is obtained from Equation* (12-He) *by replacing ρ by ϵ, and $\tilde{w}(t)$ by $\tilde{u}(t)$.*

Lemma 2.5. *Lemma 2.4 shows that when we put $\rho = 0$ and replace $\tilde{w}(t)$ by $u(t)$, Equation* (12-He) *is Heun's equation* (9)*.*

Remark 2.1. Lemma 2.4 shows that Equations (12-He) and (16-He) are transformed equations of Heun's equation (9). In Sections 2.2 and 3, we obtain the solution $\tilde{w}(t)$ of Equation (12-He) for the inhomogeneous term $\tilde{f}_\beta(t) = \delta_\epsilon(t) = g_\epsilon(t)$, and then we obtain $\tilde{u}(t)$ and $u(t)$, given by $\tilde{u}(t) = {}_RD_t^\beta \tilde{w}(t)$ and $u(t) = {}_RD_t^{\beta+\epsilon}\tilde{w}(t)$, which are the solutions of Equation (16-He) for the inhomogeneous term given by $\tilde{f}(t) = {}_RD_t^\beta \delta_\epsilon(t) = g_{\epsilon-\beta}(t)$ and, Heun's differential equation (9) for $f(t) = {}_RD_t^\beta \delta_\epsilon(t) = g_{-\beta}(t)$ for $\beta \notin \mathbb{Z}_{>-1}$, respectively.

When $\beta = 0$, $f(t) = 0$ and hence the solution of Heun's equation is a complementary solution, which is studied in Section 4, and we do not consider the case of $\beta = n \in \mathbb{Z}_{>0}$, for which $f(t) = g_{-n}(t) = 0$.

2.2 Solutions of Heun's differential equation

We now use $\tilde{w}(t)$ and $\tilde{f}_\beta(t)$ expressed by

$$\tilde{w}(t) = \sum_{k=0}^{\infty} p_k \frac{1}{\Gamma(\alpha+k+1)} t^{\alpha+k} H(t) = \sum_{k=0}^{\infty} p_k g_{\alpha+k+1}(t), \tag{18}$$

$$\tilde{f}_\beta(t) = \sum_{k=0}^{\infty} c_k \frac{1}{\Gamma(\epsilon+k)} t^{\epsilon+k-1} H(t), \tag{19}$$

where $p_0 \neq 0$, $\alpha = \nu + \epsilon$ or $\alpha = \nu$, $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<0}$ and $\epsilon \in \mathbb{R}^0_{>0}$. We then prepare the following equations:

$$\begin{aligned} \frac{d}{dt}\tilde{w}(t) &= \sum_{k=0}^{\infty} p_k g_{\alpha+k}(t), \quad t\frac{d^2}{dt^2}\tilde{w}(t) = \sum_{k=0}^{\infty} p_k(\alpha+k-1) g_{\alpha+k}(t), \\ \tilde{w}(t) &= \sum_{k=1}^{\infty} p_{k-1}\cdot g_{\alpha+k}(t), \quad t^n\frac{d^n}{dt^n}\tilde{w}(t) = \sum_{k=1}^{\infty} p_{k-1}(\alpha+k-n)_n \cdot g_{\alpha+k}(t), \quad n \in \mathbb{Z}_{[1,2]}, \\ t^n\frac{d^{n-1}}{dt^{n-1}}\tilde{w}(t) &= \sum_{k=2}^{\infty} p_{k-2}(\alpha+k-n)_n\cdot g_{\alpha+k}(t), \quad n \in \mathbb{Z}_{[1,3]}; \quad {}_RD_t^{-1}\tilde{w}(t) = \sum_{k=2}^{\infty} p_{k-2}\cdot g_{\alpha+k}(t). \end{aligned} \tag{20}$$

By using Equation (20), $\tilde{f}_\beta(t)$ given by Equation (19), and Equations (10) and (17), Equation (12-He) is expressed as follows:

$$\begin{aligned} p_{\rho,He}({}_RD_t, t)\tilde{w}(t) &:= {}_RD_t^{-\rho} p_{He}({}_RD_t, t){}_RD_t^\rho \tilde{w}(t) \\ &= \sum_{k=0}^{\infty}\{p_k[A_1(\alpha+k-1)+\tilde{B}_0(\rho)] - h_{k-1}p_{k-1}Q_k(\alpha,\rho) \\ &\quad + h_{k-2}p_{k-2}R_k(\alpha,\rho)\}\frac{1}{\Gamma(\alpha+k)} t^{\alpha+k-1}H(t) = \tilde{f}_\beta(t), \end{aligned} \tag{21}$$

where $h_{k-l} = 1$ if $k - l \in \mathbb{Z}_{>-1}$, $h_{k-l} = 0$ if $k - l \in \mathbb{Z}_{<0}$, and

$$A_1(\alpha+k-1) + \tilde{B}_0(\rho) = t_2(\alpha+k-1+\gamma_3-\rho), \tag{22}$$

$$Q_k(\alpha,\rho) = -[A_2(\alpha+k-2)+\tilde{B}_1(\rho)](\alpha+k-1) - \tilde{C}_0(\rho), \quad k \in \mathbb{Z}_{>0}, \tag{23}$$

$$R_k(\alpha,\rho) = [[A_3(\alpha+k-3)+\tilde{B}_2(\rho)](\alpha+k-2)+\tilde{C}_1(\rho)](\alpha+k-1)+\tilde{D}_0(\rho), \quad k \in \mathbb{Z}_{>1}. \tag{24}$$

Lemma 2.6. *Let* $\tilde{f}_\beta(t)$ *be given by Equation* (19), $A_1(\alpha+k-1)+\tilde{B}_0(\rho)$, $Q_k(\alpha,\rho)$ *and* $R_k(\alpha,\rho)$ *be* *given by Equations* (22), (23) *and* (24), *and* p_k *and* α *be so determined that*

$$p_0 t_2(\alpha-1+\gamma_3-\rho)\frac{1}{\Gamma(\alpha)}t^{\alpha-1}H(t)=c_0\frac{1}{\Gamma(\epsilon)}t^{\epsilon-1}H(t), \tag{25}$$

$$p_1 t_2(\alpha+\gamma_3-\rho)-p_0Q_1(\alpha,\rho)]\frac{1}{\Gamma(\alpha+1)}t^{\alpha}H(t)=c_1\frac{1}{\Gamma(\epsilon+1)}t^{\epsilon}H(t),$$

$$p_k t_2(\alpha+k-1+\gamma_3-\rho)-p_{k-1}Q_k(\alpha,\rho)+p_{k-2}R_k(\alpha,\rho)]\frac{1}{\Gamma(\alpha+k)}t^{\alpha+k-1}H(t)$$
$$=c_k\frac{1}{\Gamma(\epsilon+k)}t^{\epsilon+k-1}H(t), \quad k\in\mathbb{Z}_{>1}. \tag{26}$$

Then $\tilde{w}(t)$ *given by Equation* (18) *is a solution of Equation* (21).

Lemma 2.7. *When* $c_0=1$, *Equation* (25) *is satisfied by* $\alpha=\epsilon$ *and* $p_0=\frac{1}{t_2(\epsilon-1+\gamma_3-\rho)}$.

Lemma 2.8. *When* $c_0=0$ *or* $\epsilon=0$, *the righthand side of Equation* (25) *is* 0. *In this case, Equation* (25) *is satisfied by* $\alpha=0$ *or* $\alpha=1-\gamma_3+\rho$, *and by any value of* p_0.

Lemma 2.9. *When* $c_k=0$ *for* $k\in\mathbb{Z}_{>0}$, *we use* $\tilde{p}_k$ *in place of* $\frac{p_k}{p_0}$ *for* $k\in\mathbb{Z}_{>-1}$, *Then the equations in Equation* (26) *are expressed as* $\tilde{p}_0=1$ *and*

$$\tilde{p}_k=\frac{1}{t_2(k-1+\alpha+\gamma_3-\rho)}[\tilde{p}_{k-1}Q_k(\alpha,\rho)-h_{k-2}\tilde{p}_{k-2}R_k(\alpha,\rho)], \quad k\in\mathbb{Z}_{>0}. \tag{27}$$

We also use the coefficients P_k *in place of* $\tilde{p}_k$. *They are defined by* $\tilde{p}_0=P_0=1$ *and*

$$\tilde{p}_k=\frac{1}{t_2^k(\alpha+\gamma_3-\rho)_k}P_k, \quad \text{i.e.} \quad P_k=t_2^k(\alpha+\gamma_3-\rho)_k\tilde{p}_k, \quad k\in\mathbb{Z}_{>-1}. \tag{28}$$

Now in place of Equation (27), *we have* $P_0=1$ *and*

$$P_k=P_{k-1}Q_k(\alpha,\rho)-t_2(k-2+\alpha+\gamma_3-\rho)h_{k-2}P_{k-2}R_k(\alpha,\rho), \quad k\in\mathbb{Z}_{>0}. \tag{29}$$

3 Particular Solutions

In the present section, we consider the solution $\tilde{w}(t)$ of Equation (21) in the form of Equation (18), assuming that $\tilde{f}_\beta(t)=\delta_\epsilon(t)$, $c_0=1$, $c_k=0$ for $k\in\mathbb{Z}_{>0}$ and $\alpha=\epsilon$, in Equations (25) and (26).

Theorem 3.1. (i) *In the above condition, Lemmas 2.7 and 2.9 show that the coefficints* p_0 *and* $\tilde{p}_k=\frac{p_k}{p_k}$ *for* $k\in\mathbb{Z}_{>-1}$ *are given by*

$$p_0=\frac{1}{t_2(-1+\gamma_3-\beta)}, \quad \tilde{p}_0=1, \tag{30}$$

$$\tilde{p}_k=\frac{1}{t_2(k-1+\gamma_3-\beta)}[\tilde{p}_{k-1}Q_k(\epsilon,\rho)-h_{k-2}\tilde{p}_{k-2}R_k(\epsilon,\rho)], \quad k\in\mathbb{Z}_{>0}, \tag{31}$$

and the solution of Equation (21) *is expressed by*

$$\tilde{w}(t)=\sum_{k=0}^{\infty}p_k\frac{1}{\Gamma(\epsilon+k+1)}t^{\epsilon+k}H(t)=p_0\sum_{k=0}^{\infty}\tilde{p}_k\frac{1}{\Gamma(\epsilon+k+1)}t^{\epsilon+k}H(t). \tag{32}$$

Remark 2.1 shows that by using Equation (32), *we obtain the solution* $\tilde{u}(t)={}_RD_t^\beta\tilde{w}(t)$ *of Equation* (16-He) *for* $\tilde{f}(t)={}_RD_t^\beta\delta_\epsilon(t)$ *and* $\beta\neq\mathbb{Z}_{>0}$, *as follows:*

$$\tilde{u}(t)=p_0\sum_{k=0}^{\infty}\tilde{p}_k\frac{1}{\Gamma(\epsilon+k+1-\beta)}t^{\epsilon+k-\beta}H(t)$$
$$=p_0\frac{1}{\Gamma(\epsilon+1-\beta)}\sum_{k=0}^{\infty}\tilde{p}_k\frac{1}{(\epsilon+1-\beta)_k}t^{\epsilon+k-\beta}H(t), \tag{33}$$

and $u(t) = {}_R D_t^{\epsilon} \tilde{u}(t)$ is obtained from Equation (33).

(ii) *We note that if we replace $Q_k(\epsilon, \rho)$ and $R_k(\epsilon, \rho)$ in Equation* (31) *by $Q_k(0, \beta)$ and $R_k(0, \beta)$, respectively, so that $\tilde{p}_0 = 1$, and*

$$\tilde{p}_k \simeq \frac{1}{t_2(k-1+\gamma_3-\beta)}[\tilde{p}_{k-1}Q_k(0,\beta) - h_{k-2}\tilde{p}_{k-2}R_k(0,\beta)], \quad k \in \mathbb{Z}_{>0}, \tag{34}$$

$\tilde{w}(t)$ and $\tilde{u}(t)$ given by Equations (32) *and* (33)*, respectively, are deviated by a contribution of $O(\epsilon)$, which can be neglected, and hence we can adopt it.*

By Equations (23), (24) *and* (17)*, $Q_k(0, \beta)$ and $R_k(0, \beta)$ are given by*

$$\begin{aligned} Q_1(0,\beta) &= \alpha_1\beta_1 q_0 + B_1\beta + (2-t_2)\cdot\beta(\beta+1), \\ Q_k(0,\beta) &= [(1+t_2)(k-2-2\beta) - B_1](k-1) + \alpha_1\beta_1 q_0 + B_1\beta + (2-t_2)\cdot\beta(\beta+1), \\ R_k(0,\beta) &= [(k-2+\alpha_1-\frac{3}{2}\beta)(k-2+\beta_1-\frac{3}{2}\beta) - (\alpha_1+\beta_1)\frac{1}{2}\beta + \frac{3}{4}\beta^2 + \beta](k-1) \\ &\quad -[(\alpha_1-\beta-1)(\beta_1-\beta-1)-\beta-1]\beta, \quad k \in \mathbb{Z}_{>1}. \end{aligned} \tag{35}$$

Remark 3.1. Following Lemma 2.4 in [5], we denote the solution given by Equation (32), by $G_{Heun,\beta,\epsilon}(t,0)$. When we put $\epsilon = 0$ in this solution, the obtained $G_{Heun,\beta,0}(t,0) = {}_R D_t^{\epsilon} G_{Heun,\beta,\epsilon}(t,0)$ is a complementary solution of Equation (21) for $\epsilon = 0$.

The solutions $\tilde{u}(t)$ and $u(t)$ are expressed by ${}_R D_t^{\beta} G_{Heun,\beta,\epsilon}(t,0)$ and ${}_R D_t^{\beta+\epsilon} G_{Heun,\beta,\epsilon}(t,0)$, respectively. When $\beta = 0$, these solutions are expressed by $G_{Heun,\epsilon}(t,0)$ and $G_{Heun,0}(t,0) = {}_R D_t^{\epsilon} G_{Heun,\epsilon}(t,0)$, respectively.

Corollary 3.1. (i) *When $\beta = 0$, $\tilde{u}(t)$ given by Equation* (33)*, in which $\beta = 0$, is a particular solution of Equation* (16-He) *for $\tilde{f}(t) = \delta_\epsilon(t)$. In this case, in place of Equation* (31)*, we have the equations which are obtained from those in it by replacing β by 0 and ρ by ϵ.*

(ii) *Followig Theorem 3.1(ii), when $\beta = 0$, we may use $Q_k(0)$ and $R_k(0)$ in place of $Q_k(0,\beta)$ and $R_k(0,\beta)$ in Equation* (34)*, so that $\tilde{p}_0 = 1$ and*

$$\tilde{p}_k \simeq \frac{1}{t_2(k-1+\gamma_3)}[\tilde{p}_{k-1}Q_k(0) - h_{k-2}\tilde{p}_{k-2}R_k(0)], \quad k \in \mathbb{Z}_{>0}, \tag{36}$$

where $Q_k(0)$ and $R_k(0)$ are given by

$$\begin{aligned} Q_1(0) &:= Q_1(0,0) = \alpha_1\beta_1 q_0, \\ Q_k(0) &:= Q_k(0,0) = [(1+t_2)(k-2) - B_1](k-1) + \alpha_1\beta_1 q_0, \quad k \in \mathbb{Z}_{>0}, \\ R_k(0) &:= R_k(0,0) = (k-2+\alpha_1)(k-2+\beta_1)(k-1), \quad k \in \mathbb{Z}_{>1}. \end{aligned} \tag{37}$$

Remark 3.2. In Remark 3.1, the solution $\tilde{w}(t)$ which appears in Theorems 3.1 is called $G_{Heun,\beta,\epsilon}(t,0)$.

3.1 Use of coefficients P_k

Theorem 3.2. (i) *In Theorem 3.1(i), we have the particular solution of Equation* (16-He)*, given by Equation* (33)*. We now define P_k by Equation* (28) *for $\alpha = \epsilon$, that is*

$$\tilde{p}_k = \frac{1}{t_2^k(\gamma_3-\beta)_k} P_k, \quad \text{i.e.} \quad P_k = t_2^k(\gamma_3-\beta)_k \tilde{p}_k, \quad k \in \mathbb{Z}_{>-1}. \tag{38}$$

By using Equation (29) *for $\alpha = \epsilon$, we obtain $P_0 = 1$ and*

$$P_k = P_{k-1}Q_k(\epsilon,\rho) - t_2(k-2+\gamma_3-\beta)h_{k-2}P_{k-2}R_k(\epsilon,\rho), \quad k \in \mathbb{Z}_{>0}. \tag{39}$$

and then the particular solution of Equation (16-He)*, given by Equation* (33)*, is expressed by*

$$\tilde{u}(t) = p_0 \sum_{k=0}^{\infty} P_k \frac{1}{(\gamma_3-\beta)_k \Gamma(k+1-\beta+\epsilon) t_2^k} t^{k-\beta+\epsilon} H(t). \tag{40}$$

and $u(t) = {}_R D_t^{\epsilon} \tilde{u}(t)$ *is obtained from Equation* (40)*.*

(ii) *In Theorem 3.1(ii), it is proposed to use Equation* (34) *in place of Equation* (31)*. We now propose to use the following eqution in place of Equation* (39)*:*

$$P_k \simeq P_{k-1} Q_k(0,\beta) - t_2(k-2+\gamma_3-\beta) h_{k-2} P_{k-2} R_k(0,\beta), \quad k \in \mathbb{Z}_{>0}. \tag{41}$$

Corollary 3.2. (i) *When* $\beta = 0$*,* $\tilde{u}(t)$ *given by Equation* (40) *is a particular solution of Equation* (16-He) *for* $\tilde{f}(t) = \delta_\epsilon(t)$*, where Equation* (39) *for* $\beta = 0$ *is used.*

(ii) *Corresponding to Theorem 3.2(ii), when* $\beta = 0$*, we propose to use Equation* (41)*, by replacing* $Q_k(0,\beta)$ *and* $R_k(0,\beta)$ *by* $Q_k(0)$ *and* $R_k(0)$*, respectively, where* $Q_k(0)$ *and* $R_k(0)$ *are given in Equation* (37)*.*

3.2 Use of coefficients a_k

Theorem 3.3. (i) *In Theorem 3.1, we have a particular solution* $\tilde{u}(t)$ *of Equation* (16-He) *for the inhomogeneous term* $\tilde{f}(t) = {}_R D_t^{\beta} \delta_\epsilon(t) = g_{\epsilon-\beta}(t)$*. We now define* a_k *by*

$$\tilde{p}_k = (\epsilon-\beta+1)_k \cdot a_k, \quad \text{i.e.} \quad a_k = \frac{1}{(\epsilon-\beta+1)_k} \tilde{p}_k, \quad k \in \mathbb{Z}_{>-1}, \tag{42}$$

and then we see that the solution $\tilde{u}(t)$ *of Equation* (16-He)*, given by Equation* (33)*, is expressed by*

$$\tilde{u}(t) = p_0 \sum_{k=0}^{\infty} \tilde{p}_k \frac{1}{\Gamma(\epsilon-\beta+k+1)} t^{\epsilon-\beta+k} H(t) = p_0 \frac{1}{\Gamma(\epsilon-\beta+1)} \sum_{k=0}^{\infty} a_k t^{\epsilon-\beta+k} H(t), \tag{43}$$

where a_k *satisfy* $a_0 = 1$ *and*

$$\begin{aligned} a_k = {} & \tfrac{1}{(\epsilon-\beta+k)} \tfrac{1}{t_2(k-1+\gamma_3-\beta)} [a_{k-1} Q_k(\epsilon,\rho) \\ & - \tfrac{1}{\epsilon-\beta+k-1} h_{k-2} a_{k-2} R_k(\epsilon,\rho)], \qquad k \in \mathbb{Z}_{>0}. \end{aligned} \tag{44}$$

Remark 2.1 shows that $u(t) = {}_R D_t^{\epsilon} \tilde{u}(t)$ *is obtained from Equation* (43)*.*

(ii) *Following Theorems 3.1(ii) and 3.2(ii), we now propose to use the following equtions in place of Equation* (44)*:*

$$\begin{aligned} a_k \simeq {} & \frac{1}{(\epsilon-\beta+k)} \frac{1}{t_2(k-1+\gamma_3-\beta)} [a_{k-1} Q_k(0,\beta) \\ & - \frac{1}{\epsilon-\beta+k-1} h_{k-2} a_{k-2} R_k(0,\beta)], \quad k \in \mathbb{Z}_{>0}. \end{aligned} \tag{45}$$

Proof. By using the first equation of Equation (42) in Equation (31), we obtain

$$\begin{aligned} (\epsilon-\beta+1)_k a_k = {} & \frac{1}{t_2(k-1+\gamma_3-\beta)} [(\epsilon-\beta+1)_{k-1} a_{k-1} Q_k(\epsilon,\rho) \\ & -(\epsilon-\beta+1)_{k-2} h_{k-2} a_{k-2} R_k(\epsilon,\rho)], \end{aligned} \tag{46}$$

This gives Equation (44). □

Corollary 3.3. (i) *When $\beta = 0$, $\tilde{u}(t)$ given by Equation* (43) *for $\beta = 0$, is a particular solution of Equation* (16-He) *for $\tilde{f}(t) = \delta_\epsilon(t)$, where Equation* (44) *for $\beta = 0$ is used.*

(ii) *Corresponding to Theorem 3.3(ii), when $\beta = 0$, we propose to use Equation* (45) *for $\beta = 0$, by replacing $Q_k(0, \beta)$ and $R_k(0, \beta)$ by $Q_k(0)$ and $R_k(0)$, respectively,*
Remark 3.3. In Remark 3.1, the solutions $\tilde{u}(t)$ and $u(t)$ which appear in Theorems 3.1, 3.2 and 3.3 are called ${}_RD_t^{\beta} G_{Heun,\beta,\epsilon}(t, 0)$ and ${}_RD_t^{\beta+\epsilon} G_{Heun,\beta,\epsilon}(t, 0)$, respectively.
Remark 3.4. In Remark 3.1, $\tilde{u}(t)$ which appear in Corollaries 3.1, 3.2 and 3.3, are called $G_{Heun,\epsilon}(t, 0)$.

4 Complementary Solutions

In the present section, we apply the results in Section 2.2, to the cases of $\beta = 0$ and $f(t) = \tilde{f}_\beta(t) = 0$. Lemma 2.8 shows two choices. We first consider the case of $\alpha = \epsilon = 0$.
Remark 4.1. In Corollaries 3.1(i), 3.2(i) and 3.3(i), the solutions $\tilde{u}(t)$ for the cases of $\beta = 0$, $f(t) = \tilde{f}_\beta(t) = \delta_\epsilon(t)$ and $\alpha = \epsilon$ are given. The solutions $u(t)$ in the present section, are obtained from them by $u(t) = {}_RD_t^{\epsilon}\tilde{u}(t)$ or by replacing $\tilde{u}(t)$ by $u(t)$, ϵ by 0, and a value of p_0 by an arbitrary number.
Theorem 4.1. *In the case stated above, Lemmas 2.8 and 2.9 show that by using Equations* (19) *and* (27) *for $\alpha = \epsilon = 0$ and $\rho = 0$, a complementary solution $u(t)$ of Equation* (9) *is given by*

$$u(t) = \sum_{k=0}^{\infty} p_k \frac{1}{k!} t^k H(t) = p_0 \sum_{k=0}^{\infty} \tilde{p}_k \frac{1}{k!} t^k H(t), \tag{47}$$

where p_0 is any number, $p_k = p_0 \tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$, and $\tilde{p}_k$ for $k \in \mathbb{Z}_{>-1}$ satisfy $\tilde{p}_0 = 1$ and

$$\tilde{p}_k = \frac{1}{t_2(k - 1 + \gamma_3)} [\tilde{p}_{k-1} Q_k(0) - h_{k-2} \tilde{p}_{k-2} R_k(0)], \quad k \in \mathbb{Z}_{>0}, \tag{48}$$

where $Q_k(0)$ and $R_k(0)$ are given in Equation (37)*.*

Note here that Equation (48) *is obtained from Equation* (36)*, by replacing $\simeq$ by $=$.*
Theorem 4.2. *Lemmas 2.8 and 2.9 show that by using Equations* (28) *and* (29) *for $\alpha = \epsilon = 0$ and $\rho = 0$, P_k is defined by $\tilde{p}_k = \frac{1}{t_2^k (\gamma_3)_k} P_k$, and the solution $u(t)$ of Equation* (9)*, given by Equation* (47)*, is expressed as follows:*

$$u(t) = p_0 \sum_{k=0}^{\infty} P_k \frac{1}{(\gamma_3)_k k!} (\frac{t}{t_2})^k H(t). \tag{49}$$

Here p_0 is any number, and P_k for $k \in \mathbb{Z}_{>-1}$ are given by $P_0 = 1$ and

$$P_k = P_{k-1} Q_k(0) - t_2(k - 2 + \gamma_3) h_{k-2} P_{k-2} R_k(0), \quad k \in \mathbb{Z}_{>0}. \tag{50}$$

Theorem 4.3. *The complementary solution of Heun's differential equation* (9)*, which is given by Equation* (47)*, is also expressed as follows:*

$$u(t) = p_0 \sum_{k=0}^{\infty} a_k t^k, \tag{51}$$

where p_0 is any number, and a_k are related with $\tilde{p}_k$ by

$$\tilde{p}_k = a_k k!, \quad \text{i.e.} \quad a_k = \tilde{p}_k \frac{1}{k!}, \quad k \in \mathbb{Z}_{>1}. \tag{52}$$

Then we cofirm that a_k satisfy $a_0 = 1$ and

$$\begin{aligned} a_k = &\frac{1}{(\gamma_3 + k - 1)kt_2}[a_{k-1}Q_k(0) - \frac{1}{k-1}h_{k-2}a_{k-2}R_k(0)] \\ = &\frac{1}{(\gamma_3 + k - 1)kt_2}\{[a_{k-1}[(1+t_2)(k-2+\gamma_3) + \gamma_2 + \gamma_1 t_2](k-1) + \alpha_1\beta_1 q_0] \\ &-h_{k-2}a_{k-2}(k-2+\alpha_1)(k-2+\beta_1)\}, \quad k \in \mathbb{Z}_{>0}. \end{aligned} \tag{53}$$

Proof. By using the first equation of Equation (52) in Equation (48), we obtain

$$k! \cdot a_k = \frac{1}{t_2(\gamma_3 + k - 1)}[(k-1)! \cdot a_{k-1}Q_k(0) - (k-2)! \cdot h_{k-2}a_{k-2}R_k(0)].$$

This gives the first equality in Equation (53). □

This result is given in Section 3.3 in [12] and in Section 8.2 in [13].

Remark 4.2. Remark 4.1 states that when p_0 is given by Equation (30) for $\beta = 0$, the solutions $u(t)$ in Equations (47), (48) and (51) are obtained from the solutions $\tilde{u}(t)$ given in Corollaries 3.1(i), 3.2(i) and 3.3(i). In Remark 3.1, the solutions $u(t)$ are called $G_{Heun,0}(t, 0)$.

4.1 Complementary solution, II

In Theorems 4.1∼4.3, we studied the case of $\tilde{f}(t) = 0$, $\beta = 0$ and $\alpha = \epsilon = 0$ in Lemma 2.8. We now study the case of $\alpha = 1 - \gamma_3$ and $\epsilon = 0$ in place of $\alpha = \epsilon = 0$.

Theorem 4.4. *Lemmas 2.8 and 2.9 show that by using Equations* (19) *and* (27) *for $\alpha = 1 - \gamma_3$ and $\rho = 0$, we obtain the complementary solution of Equation* (9)*, given by*

$$\begin{aligned} u(t) = &p_0 \sum_{k=0}^{\infty} \tilde{p}_k \frac{1}{\Gamma(2 - \gamma_3 + k)} t^{1-\gamma_3+k} H(t) \\ = &\frac{1}{\Gamma(2-\gamma_3)} t^{1-\gamma_3} p_0 \sum_{k=0}^{\infty} \tilde{p}_k \frac{1}{(2-\gamma_3)_k} t^k H(t), \end{aligned} \tag{54}$$

where p_0 is any number, $\tilde{p}_0 = 1$ and

$$\tilde{p}_k = \frac{1}{t_2 k}[\tilde{p}_{k-1}Q_k(1-\gamma_3) - h_{k-2}\tilde{p}_{k-2}R_k(1-\gamma_3)], \quad k \in \mathbb{Z}_{>0}, \tag{55}$$

$$\begin{aligned} Q_k(1-\gamma_3) := Q_k(1-\gamma_3, 0) = &[(1+t_2)(k-1-\gamma_3) - B_1](k-\gamma_3) + \alpha_1\beta_1 q_0, \quad k \in \mathbb{Z}_{>0}, \\ R_k(1-\gamma_3) := R_k(1-\gamma_3, 0) = &[[(k-2-\gamma_3) + \alpha_1 + \beta_1 + 1](k-1-\gamma_3) + \alpha_1\beta_1](k-\gamma_3) \\ = &(k-1-\gamma_3+\alpha_1)(k-1-\gamma_3+\beta_1)(k-\gamma_3), \quad k \in \mathbb{Z}_{>1}. \end{aligned} \tag{56}$$

Theorem 4.5. *In Theorem 4.4, p_0 is any number, and $\tilde{p}_k$ satisfiy Equation* (55)*. By using Equation* (28) *for $\alpha = 1 - \gamma_3$ and $\rho = 0$, we define P_k by*

$$\tilde{p}_k = \frac{1}{t_2^k k!} P_k, \quad \text{i.e.} \quad P_k = t_2^k k! \cdot \tilde{p}_k, \quad k \in \mathbb{Z}_{>-1}. \tag{57}$$

Then P_k satisfy $P_0 = 1$, and

$$P_k = P_{k-1}Q_k(1-\gamma_3) - t_2(k-1)h_{k-2}P_{k-2}R_k(1-\gamma_3), \quad k \in \mathbb{Z}_{>0}. \tag{58}$$

By using Equation (57) *in Equation* (54)*, the complementary solution of Equation* (9) *is expressed by*

$$u(t) = p_0 \sum_{k=0}^{\infty} P_k \frac{1}{t_2^k k! \cdot \Gamma(1-\gamma_3+k+1)} t^{1-\gamma_3+k} H(t). \tag{59}$$

Proof. Using the first equation of Equation (57) in Equation (55), we obtain

$$\frac{1}{t_2^k k!}P_k = \frac{1}{t_2 k}[\frac{1}{t_2^{k-1}(k-1)!}P_{k-1}Q_k(1-\gamma_3) - \frac{1}{t_2^{k-2}(k-2)!}h_{k-2}P_{k-2}R_k(1-\gamma_3)], \tag{60}$$

which gives Equation (58). □

Theorem 4.6. *The complementary solution of Heun's equation, which is given by Equation* (54)*, is also expressed as follows:*

$$u(t) = p_0 \frac{1}{\Gamma(2-\gamma_3)} t^{1-\gamma_3} \sum_{k=0}^{\infty} a_k t^k H(t), \tag{61}$$

where p_0 *is any number, and* a_k *are defined by*

$$\tilde{p}_k = (2-\gamma_3)_k a_k, \quad \text{i.e.} \quad a_k = \frac{1}{(2-\gamma_3)_k}\tilde{p}_k, \quad k \in \mathbb{Z}_{>-1}. \tag{62}$$

By using the first equation of Equation (62) *in Equation* (55)*, we obtain* $a_0 = \tilde{p}_0 = 1$*, and*

$$\begin{aligned} a_k &= \frac{1}{(2-\gamma_3)_k} p_k = \frac{1}{(2-\gamma_3)_k t_2 k}[\tilde{p}_{k-1}Q_k(1-\gamma_3) - h_{k-2}\tilde{p}_{k-2}R_k(1-\gamma_3)] \\ &= \frac{1}{(2-\gamma_3+k-1)t_2 k}[a_{k-1}Q_k(1-\gamma_3) - \frac{1}{2-\gamma_3+k-2}h_{k-2}a_{k-2}R_k(1-\gamma_3)] \\ &= \frac{1}{t_2 k(1-\gamma_3+k)}\{a_{k-1}[[(1+t_2)(k-1)+\gamma_2+\gamma_1 t_2](k-\gamma_3)+\alpha_1\beta_1 q_0] \\ &\quad -h_{k-2}a_{k-2}(k-1-\gamma_3+\alpha_1)(k-1-\gamma_3+\beta_1)\}, \quad k \in \mathbb{Z}_{>0}. \end{aligned} \tag{63}$$

5 Conclusion

In a preceding paper [5] of the present author, the particular solutions of Kummer's and the hypergeometric differential equation are obtained for the inhomogeneous term given by $f(t) = g_{-\beta}(t) = \frac{1}{\Gamma(-\beta)}t^{-1-\beta}$ for $\beta \in \mathbb{C}\backslash\mathbb{Z}_{>-1}$. When the desired solution of Kummer's equation is $u(t)$, we construct a transformed differntial equation of Kummer's equation, which is satisfied $\tilde{u}(t) = {}_R D_t^{-\epsilon} u(t)$, and obtain its solution $\tilde{u}(t)$ and the desired solution by $u(t) = {}_R D_t^{\epsilon} \tilde{u}(t)$.

In Section 3, we present the solution of the same problem for the case of Heun's equation. The solutions obtained are given in three formats.

In Section 4, we obtain two complementary solutions of Heun's equation. They are expressed in three formats. The complementary solution in one format is in agreement with a solution presented in the past, given in Section 3.3 in [12] and in Section 8.2 in [13].

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