Short Research Article

Ultrafilter in Graph Theory: Relationship to Tree-decomposition

Abstract: The investigation of width parameters in both graph theory and algebraic contexts has attracted considerable attention. Among these parameters, tree width has emerged as a crucial metric. In this concise paper, we introduce a natural definition of Ultrafilters on graphs and demonstrate their equivalence to graph Tangles, establishing a dual relationship with tree width. Our goal is to extend this research in the future by delving deeper into the properties of Ultrafilters on graphs, particularly focusing on uncovering unique characteristics that manifest in various graph classes.
Keyword: filter, ultrafilter, tangle, tree-decomposition, path-decomposition, tree-width, Bramble

1. Introduction

In recent years, there has been a notable surge in research interest concerning the exploration of width parameters in the realms of both graph theory and algebraic contexts [1-9, 11-22, 28-34]. Width parameters pertain to metrics derived from tree-like structures, commonly referred to as graph decompositions. Among these metrics, one of notable significance is the tree width, which has been demonstrated to serve as a pivotal metric in delineating the intricacy of diverse mathematical entities, encompassing graphs and matroids (refer to [35-45]).

The concept of "Tangle," defined by Robertson et al., is known to be dual to the width parameter known as "tree width" on graphs. Therefore, research on Tangle, which are intimately connected with width parameters, is of paramount importance.

The notion of a "Filter" is well-established within the realms of topology and algebra. Simply put, a filter can be interpreted as a collection of sets containing a specific element, and it serves as a useful tool for discussing convergence properties in mathematics. In the domain of Boolean algebra, maximal filters are referred to as "ultrafilters." Due to their versatile nature, Ultrafilters hold substantial and wide-ranging significance, finding applications in a multitude of fields including topology, algebra, logic, set theory, lattice theory, matroid theory, graph theory, combinatorics, measure theory, model theory, and functional analysis. Consequently, research on Ultrafilters and related studies thrives, reflecting their utmost importance in various domains (as supported by references [23-27, 46-58, 83-89]).

In this paper, we establish a natural definition of Ultrafilters on graphs and demonstrate their equivalence to graph Tangles. It is truly remarkable how two seemingly unrelated concepts can reveal profound connections with just a few additional conditions. Our goal is to extend this research in the future by delving deeper into the characteristics of Ultrafilters on graphs, particularly exploring the unique features that emerge in various graph classes.

2. Definitions and Notations in this paper

This section provides mathematical definitions of each concept.

A graph $G$is a mathematical structure composed of nodes (vertices) connected by edges, representing relationships or connections. $V(G)$ represents the set of vertices (nodes) in a graph G, $E(G)$ represents the set of edges in the same graph G, and $G=(V,E)$ signifies that G is a graph defined by a pair of sets, V for vertices and E for edges.A subgraph is a subset of a graph consisting of selected vertices and edges. Additionally, in this paper, we utilize the natural number k.

2.1 Filters on Boolean Algebras

We provide an explanation of Filters in Boolean Algebras. Boolean Algebras are mathematical structures used for operations, like AND, OR, NOT, commonly used in logic and computer science.
The definition of a filter in a Boolean algebra $(X, \cup, \cap)$ is given below.As mentioned in the introduction, Filters and Ultrafilters are fundamental concepts in mathematics. The complement of an filter in a Boolean algebra $(X, \cup, \cap)$ is referred to as an ideal in a Boolean algebra $(X, \cup, \cap)$.

Definition 1: In a Boolean algebra $(X, \cup, \cap)$, a set family $F \subseteq 2^X$ satisfying the following conditions is called a filter on the carrier set X.
(FB1) $A, B \in F \Rightarrow A \cap B \in F$,
(FB2) $A \in F, A \subseteq B \subseteq X \Rightarrow B \in F$,
(FB3) $\emptyset$ is not belong to F.

In a Boolean algebras $(X, \cup, \cap)$, A maximal filter is called an ultrafilter and satisfies the following axiom (FB4):
(FB4) $\forall A \subseteq X$, *either* $A \in F$ *or* $X / A \in F$.

2.2 *G*-Tangle on the graph

We provide an explanation of *G*-Tangle on the graph.
In the context of graph theory, a tangle in a graph G is a way to describe how the vertices can be separated into distinct groups based on certain conditions. Let us first define a "separation."
A separation of a graph G is a pair of subgraphs (A, B) that satisfy the conditions:
- $V(A) \cup V(B) = V(G)$, where $V(X)$ denotes the vertex set of X.
- $V(A) \cap V(B)$ is non-empty but minimal, meaning there are no subsets of A and B that can further be called a separation.The order of a separation (A, B) is defined to be $|V(A) \cap V(B)|$, the number of vertices that A and B share.
The definition of a tangle on the graph is provided below. Tangles are well-known for their deep connection with tree-decompositions.

Definition 2 [11]: Let G be a graph. A G-tangle of order k is a family T of separations of G satisfying the following conditions.
(T0) The order of all separations $(A, B) \in T$ is less than k.
(T1) For all separations (A, B) of G of order less than k, either $(A, B) \in T$ or $(B, A) \in T$.
(T2) If $(A_1, B_1),(A_2, B_2),(A_3, B_3) \in T$ then $A_1 \cup A_2 \cup A_3 \neq G$.
(T3) $V(A) \neq V(G)$ for all $(A, B) \in T$

In the field of graph width parameters, duality theorems are frequently discussed. It is known that Tangle and Tree-width have the following duality relationship. In this paper, we consider about duality relationship like following theorem.
Theorem 1[11]: If there exists a G-Tangle of order $k - 1$, then the tree-width is at least k.

The definition of path tangle, which has a deep connection with path-decompositions, are outlined below. A path-decomposition is a tree structure that restricts tree-decompositions to a path-like structure.
Definition 3: Let G be a graph. A G-path-tangle of order k is a family T of separations of G satisfying the following conditions.
(T0) The order of all separations $(A, B) \in T$ is less than k.
(T1) For all separations (A, B) of G of order less than k, either $(A, B) \in T$ or $(B, A) \in T$.
(LT2) If $(A_1, B_1),(A_2, B_2)$, $|V(A_3)| = 1$ then $A_1 \cup A_2 \cup A_3 \neq G$.
(T3) $V(A) \neq V(G)$ for all $(A, B) \in T$

Path-width has been the subject of numerous studies and holds similar significance in research due to its practical applications in the real world (ex. [59-65]).

From Theorem 1, which establishes the duality between tree-width and tangle, we can derive the following.

Theorem 2: Let G be a graph. If there exists a G-path tangle of order k - 1, then the -width is at least k.

Additionally, in reference [35], the theorems known as "path-width" and "blockage" have been established and proven.

Theorem 3[35]: Let G be a graph. There is a blockage of order k if and only if the path-width of G is at least k.

3 Ultrafilter on the graph:Obstruction to tree-decomposition

We provide an explanation of G-Ultrafilter on the graph.

The definition of a G-Ultrafilter on the graph is given below. We naturally extend the definition from Boolean algebras to graphs. The complement of a graph ultrafilter is referred to as a maximal ideal on a graph.

Definition 4: Let G be a graph. A G-Ultrafilter of order k is a family F of separations of G satisfying the following conditions.

(F0) The order of all separations $(A, B) \in F$ is less than k.

(F1) For all separations (A, B) of G of order less than k, either $(A, B) \in F$or $(B, A) \in F$.

(F2) $(A_1, B_1) \in F, A_1 \subseteq A_2$, (A_2, B_2) of G of order less than k $\Rightarrow (A_2, B_2) \in$F,

(F3)$(A_1, B_1) \in$F, $(A_2, B_2) \in$F, $(A_1 \cap A_2, B_1 \cup B_2)$ of G of order less than k

$\Rightarrow (A_1 \cap A_2, B_1 \cup B_2) \in$F,

(F4) *If* $V(A) = V(G)$, *then* $(A, B) \in F$.

Proving the Main Theorem of this paper, which establishes the equivalence between Ultrafilters on graphs and Tangles.

Theorem 4. Let G be a graph. T is a G-Tangle of separations of order k in graph iff$F = \{(A,B) \mid (B,A) \in T\}$ is an G-Ultrafilter of separations of order k in graph.

Proof. To prove this Theorem, we need to establish a bidirectional implication: that if T is a G-tangle of order k, then F defined by $F = \{(A,B) \mid (B,A) \in T\}$is a G-ultrafilter of order k, and vice versa. We will prove both directions separately.

First, we consider about forward Direction.

We show that F satisfies (F0). Let (A, B) be any separation in F. Then by definition of F, we have (B, A) in T. Since T is a G-tangle of order k, we know that the order of (B, A) is less than k. Hence, the order of (A, B) is also less than k. Thus, F satisfies condition (F0).

We show that$F$ satisfies (F1). Let (A, B) be any separation of G of order less than k. Then either (A, B) or (B, A) is in T. This means either (A, B) or (B, A) is in F, satisfying the condition (F1).

We show that F satisfies (F2). To establish that F satisfies axiom (F2) , we need to show that if (A_1, B_1) belongs to F , $A_1 \subseteq A_2$, and (A_2, B_2) is a separation of G of order less than k , then (A_2, B_2) belongs to F.

Given the set $T = \{(B, A) \mid (A, B) \in F\}$defines a G-tangle of order k, we know that (B_1, A_1) belongs to T. Now, since$A_1 \subseteq A_2$, we have $B_2 \subseteq B_1$. By the property (T2) of tangles, we can infer that (B_2, A_2) belongs to$T$, because otherwise $B_1 \cup B_2$would cover the whole graph, which contradicts axiom (T2).

Hence, (B_2, A_2)belongs to T, which means (A_2, B_2) belongs to F, establishing that F satisfies condition (F2).

We show that F satisfies (F3). Assume to the contrary that F does not satisfy (F3). This means that there exist separations (A_1, B_1) and (A_2, B_2)in Fsuch that the separation $(A_1 \cap A_2, B_1 \cup B_2)$ is not in F.

If $(A_1 \cap A_2, B_1 \cup B_2)$ is not in F , then, by the definition of F as $F = \{(A,B) \mid (B,A) \in T\}$, it follows that the separation $(B_1 \cup B_2, A_1 \cap A_2)$ is not in Teither.

Remember that T is a G-tangle, so for every separation of order less than k, either one of its orientations is in T. Since $(B_1 \cup B_2, A_1 \cap A_2)$ is not in T, the only possibility left is that its opposite orientation, $(A_1 \cap A_2, B_1 \cup B_2)$, must be in T. This, however, contradicts our assumption.

Furthermore, because (B_1, A_1) and (B_2, A_2) are both in T, by axiom (T2), the union of any three separations from Tshould not cover the entire graph. Yet, if we take (B_1, A_1), (B_2, A_2), and $(B_1 \cup B_2, A_1 \cap A_2)$, their union covers the entire graph, G. This contradicts (T2) of the definition of a G-tangle.So axiom (F3) holds. Axiom (F4) obviously holds.

Next, we consider about backward direction.
We show that$T$ satisfies axiom (T0). For any separation (B, A)in T, we have (A, B)in F. By condition (F0), the order of (A, B) is less than k. Hence, the order of $(B, A)$is also less than k. Thus, T satisfies the condition (T0).
We show that$T$ satisfies axiom (T1). To prove this, let (A, B) be any separation of G of order less thank. By axiom (F1), we know that either (A, B) or (B, A) belongs to F. Hence, it follows that either (B, A) or (A, B) must be in$T$, verifying the condition (T1).
We show that$T$ satisfies (T2). Suppose we have(B_1, A_1), (B_2, A_2), (B_3, A_3) in T. We aim to prove that $B_1 \cup B_2 \cup B_3 \neq G$. By definition of T, we know (A_1, B_1), (A_2, B_2), (A_3, B_3) are in F.
Now, if we assume that $B_1 \cup B_2 \cup B_3 = G$, then it would imply that $A_1 \cap A_2 \cap A_3 = \emptyset$, which contradicts (F3) as it would lead to a separation of order less than k not being in F. Hence, $B_1 \cup B_2 \cup B_3 \neq G$, proving axiom (T2).
We show thatT satisfiesaxiom (T3). To prove T satisfies axiom (T3), let's consider a separation (B, A)in T. It implies that (A, B) is in$F$. By condition (F4)of F, $V(B) \neq V(G)$, confirming that T satisfies condition (T3) . This proof is completed.

From Theorem 1 and Theorem 4, the following can be deduced:
Theorem 5: Let G be a graph. If there exists a G-Ultrafilter of order $k - 1$, then the tree-width is at least k.

4. Path ultrafilter on the graph: Obstruction to path-decomposition
By imposing additional constraints on filters on graphs, it becomes possible to establish a dual relationship with path-width. The definition for this is as follows. The complement of a graph path ultrafilter is referred to as a maximal path ideal on a graph.

Definition 5: Let G be a graph. A G-Path Ultrafilter of order k is a family F of separations of G satisfying the following conditions.
(F0) The order of all separations $(A, B) \in F$ is less than k.
(F1) For all separations (A, B) of G of order less than k, either $(A, B) \in F$or $(B, A) \in F$.
(F2) $(A_1, B_1) \in F$, $A_1 \subseteq A_2$, (A_2, B_2) of G of order less than k $\Rightarrow (A_2, B_2) \in F$,
(F3) $(A_1, B_1) \in F$, $|V(A_2)| = |V(G)| - 1$, $(A_1 \cap A_2, B_1 \cup B_2)$ of G of order less than k
$\Rightarrow (A_1 \cap A_2, B_1 \cup B_2) \in F$,
(F4) *If* $V(A) = V(G)$, *then* $(A, B) \in F$.

Theorem 6: Let G be a graph. If there exists a G-Path Ultrafilter of order $k - 1$, then the path-width is at least k.

5. Bramble: Closely related to Tree-decomposition
Bramble is a concept closely related to graph-width parameters such as tree-decomposition, and like other concepts, it has been the subject of various research efforts (ex.[67-82]).

We are providing an explanation of G-bramble with reference to [82]. Consider a graph G, where k is a positive integer. Two subgraphs A and B touch if either their vertex sets have a non-empty intersection $(V(A) \cap V(B) \neq \emptyset)$, or there exists an edge $e \in E(G)$ that connects a vertex from A to a vertex from B.
A set $X \subseteq V(G)$ is said to "cover" a subgraph $B \subseteq G$ if their intersection is non-empty $(X \cap V(B) \neq \emptyset)$.$X$ is considered to "cover" a family B of subgraphs of G if it covers all subgraphs $B \in B$.

Now, we define a *G*-bramble of *G* as a family *B* consisting of connected subgraphs of *G* in which any two subgraphs within the family touch. The size of the bramble *B* is denoted simply as |B|. The order of the bramble, denoted as the parameter *k*, is the smallest integer *k* such that there exists a set *X* with $|X| = k$ that covers the entire family *B.*

There is a relationship between *G*-Bramble and Tree-width.
Theorem 7[66]: Let *G* be a graph.A graph *G* has a *G*-bramble of order *k* if and only if it has tree-width at least $k – 1$.

There is a relationship between *G*-Bramble and *G*-Ultrafilter.
Theorem 8. Let *G* be a graph*. T* is a*G*-Ultrafilter of separations of order *k* in graph, then *T* is a *G*-Bramble of separations of order *k* in graph.
Proof: Let *T* be a *G*-Ultrafilter of separations of order *k* in graph. To show *T* is a *G*-Bramble, we need to show:
1) Elements of *T* are connected subgraphs of *G.*
2) Any two subgraphs within the family *T* touch.

1) Elements of *T* are connected subgraphs of *G*:
By the definition of a separation in a graph, each element of *T* consists of a pair of subgraphs *(A, B).*Both*A* and *B* are subgraphs of *G* and hence, elements of *T a*re subgraphs of *G.*

2) Any two subgraphs within the family *T* touch:
Let's take any two separations (A_1, B_1) and (A_2, B_2) from *T*.Given the nature of a *G*-Ultrafilter, either(A_1, B_1) or (B_1, A_1) belongs to T, and either(A_2, B_2)or(B_2, A_2) belongs to*T.*
Based on the properties of the*G*-Ultrafilter, if A_1 intersects withA_2 orB_2, or if B_1 intersects with A_2 or B_2, then they touch. Using the axioms (F0) and (F1), it is clear that such an intersection will always exist, ensuring that the two separations touch.
Thus, based on these points, *T* satisfies the requirements of a *G*-Bramble. Hence, if*T* is a *G*-Ultrafilter of separations of order *k i*n a graph, then*T* is a *G*-Bramble of separations of order *k* in the graph.This completes the proof.

6.Future tasks: Ultraproduct on graphs and Finite intersection property on graphs
We investigate the characteristics of ultrafilters on graphs across various graph classes. For example, we anticipate advancements in the study of ultrafilters on specialized graphs such as fuzzy graphs[90-92], neutrosophic graphs[93-95], and plithogenic graphs[96-100].

Additionally, we not only introduce the concept of ultraproducts on graphs and analyze their properties but also explore topics such as the Ultrafilter Axiom of Choice and the finite intersection property within the framework of ultrafilters on graphs. Ultraproducts are a well-established mathematical concept that has been extensively researched across multiple domains [83–89].

Moreover, we are considering the exploration of additional properties and potential applications of ultrafilters on graphs in future studies.

Data Availability
This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Conflict of Interest Statement
The author declares no conflicts of interest.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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