A Duality Principle and Concerning Convex Dual Formulation for a Model in Micro-Magnetism

Abstract

This article develops a duality principle applicable to originally non-convex primal variational formulations. More specifically, we establish a convex (in fact concave) dual variational formulation for a model in micro-magnetism. The results are obtained through basic tools of functional analysis, calculus of variations, duality and optimization theory in infinite dimensional spaces. We emphasize such a convex dual formulation obtained may be applied to a large class of similar models in the calculus of variations, including models in elasticity and phase transition.

Keywords: Duality principle, model in micro-magnetism, concave dual formulation.

MSC: 49N15

1 Introduction

This article develops a duality principle applicable to a large class of models in the calculus of variations. Specifically in this text, in a first step, we present applications to a model in micro-magnetism.

We emphasize the results on duality theory here addressed and developed are inspired mainly in the approaches of J.J.Telega, W.R. Bielski and co-workers presented in the articles [1, 2, 3, 4]. Other main reference is the article by Toland, [5].

Details on theoretical results in micro-magnetism may be found in [6].

Moreover, details on the Sobolev spaces involved may be found in [7].

Similar results and models are addressed in [8, 9, 10, 11, 12].

Basic results on convex analysis are addressed in [13]. Other similar results and approaches may be found in [14, 15, 16].

Now we start to describe the primal variational formulation for the model in micro-magnetism in question.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V_1 \times V_2 \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} J(m,f) &= \frac{\alpha}{2}\int_\Omega |\nabla m|^2\,dx + \varphi(m) - \int_\Omega H\cdot m\,dx \\ &+\frac{1}{2}\int_{\mathbb{R}^3} |f|^2\,dx \end{aligned} \qquad (1)$$

where $\alpha > 0$, $m = (m_1, m_2, m_3) \in V_1 = W^{1,2}(\Omega;\mathbb{R}^3)$, represents a magnetization field concerning a ferromagnetic sample Ω and

$$f \in V_2 = L^2(\mathbb{R}^3;\mathbb{R}^3)$$

is a vectorial field on $\mathbb{R}^3$.

Moreover $H : \Omega \rightarrow \mathbb{R}^3$ is a known external magnetic field.

Such a functional is considered subject to the following constraints

$$|m| = \sqrt{m_1^2 + m_2^2 + m_3^2} = 1, \text{ in } \Omega,$$

$$\text{Curl } f = \mathbf{0}, \text{ in } \mathbb{R}^3,$$

$$\text{div } (-f + m\chi_\Omega) = 0, \text{ in } \mathbb{R}^3.$$

Here we have denoted

$$\chi_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases} \qquad (2)$$

Moreover, for a functional $\varphi : V_1 \rightarrow \mathbb{R}$, we assume the following multi-well format

$$\varphi(m) = \int_\Omega \min_{k\in\{1,\cdots,M\}} g_k(m)\,dx,$$

where $g_k : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a convex and twice differentiable function $\forall k \in \{1,\cdots,M\}$ for a fixed $M \in \mathbb{N}$.

Here we also denote

$$B = \left\{ t = \{t_k\} \text{ measurable } : 0 \leq t_k \leq 1, \ \forall k \in \{1,\cdots,M\} \text{ and } \sum_{k=1}^M t_k = 1, \text{ in } \Omega \right\}.$$

With such statements and definitions in mind, we define the functional

$$J_1 : V_1 \times V_2 \times B \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\},$$

by

$$\begin{aligned} J_1(m,f,t) &= \frac{\alpha}{2}\int_\Omega |\nabla m|^2\,dx + \int_\Omega \sum_{k=1}^M t_k g_k(m)\,dx \\ &-\int_\Omega H\cdot m\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |f|^2\,dx \\ &+Ind_0(m) + Ind_1(m,f) + Ind_2(f), \end{aligned} \qquad (3)$$

where

$$Ind_0(m) = \begin{cases} 0, & \text{if } \sum_{k=1}^3 m_k^2 = 1, \text{ in } \Omega, \\ +\infty, & \text{otherwise,} \end{cases} \tag{4}$$

$$Ind_1(m,f) = \begin{cases} 0, & \text{if } \operatorname{div}(-f + m\chi_\Omega) = 0, \text{ in } \mathbb{R}^3, \\ +\infty, & \text{otherwise,} \end{cases} \tag{5}$$

$$Ind_2(f) = \begin{cases} 0, & \text{if } \text{Curl } f = \mathbf{0}, \text{ in } \mathbb{R}^3, \\ +\infty, & \text{otherwise,} \end{cases} \tag{6}$$

2 The main duality principle

Denoting $Y = Y^* = L^2(\Omega)$ and $Y_1 = Y_1^* = L^2(\Omega;\mathbb{R}^3)$, we start by defining the functionals $F_1 : V_1 \times B \to \mathbb{R}$ and $F_2 : V_2 \to \mathbb{R}$ by

$$\begin{aligned} F_1(m,t) &= \frac{\alpha}{2}\int_\Omega |\nabla m|^2\,dx + \int_\Omega \sum_{k=1}^M t_k g_k(m)\,dx \\ &\quad - \int_\Omega H\cdot m\,dx, \end{aligned} \tag{7}$$

$$F_2(f) = \frac{1}{2}\int_{\mathbb{R}^3} |f|^2\,dx.$$

Thus, already including the Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3$ for the respective concerning constraints, we obtain

$$\begin{aligned} J_1(m,f,t) &= F_1(m,t) + F_2(f) \\ &\quad + Ind_0(m) + Ind_1(m,f) + Ind_2(f) \\ &\geq F_1(m,t) + F_2(f) \\ &\quad + \left\langle \sum_{k=1}^3 m_k^2 - 1, \lambda_3 \right\rangle_{L^2} \\ &\quad + \langle \operatorname{div}(-f + m\chi_\Omega), \lambda_2\rangle_{L^2(\mathbb{R}^3)} + \langle \text{ Curl } f, \lambda_1\rangle_{L^2(\mathbb{R}^3)}, \\ &\geq \inf_{m\in V_1}\left\{ F_1(m,t) + \left\langle \sum_{k=1}^3 m_k^2 - 1, \lambda_3 \right\rangle_{L^2} - \langle m, \nabla\lambda_2\rangle_{L^2} \right\} \\ &\quad + \inf_{f\in V_2}\left\{ \langle f, \nabla\lambda_2 + \text{ Curl } \lambda_1\rangle_{L^2(\mathbb{R}^3)} + F_2(f) \right\} \\ &= -F_1^*(\lambda_2,\lambda_3,t) - F_2^*(\lambda_2,\lambda_1), \end{aligned} \tag{8}$$

$\forall \lambda = (\lambda_1,\lambda_2,\lambda_3) \in A^*,\ t \in B$, where

$$A^* = \{\lambda \in Y_1^* \times Y^* \times Y^* \ :\ -\alpha\nabla^2 + \lambda_3 > \mathbf{0} \text{ and } \lambda_2 = 0, \text{ on } \partial\Omega\}.$$

Here we have defined $F_1^* : Y^* \times Y^* \times B \to \mathbb{R}$ and $F_2^* : Y^* \times Y_1^* \to \mathbb{R}$ by

$$F_1^*(\lambda_2,\lambda_3,t) = \sup_{m\in V_1}\left\{ -F_1(m,t) - \left\langle \sum_{k=1}^3 m_k^2 - 1, \lambda_3 \right\rangle_{L^2} + \langle m, \nabla\lambda_2\rangle_{L^2} \right\},$$

and

$$\begin{aligned} F_2^*(\lambda_2,\lambda_1) &= \sup_{f\in V_2}\left\{-\langle f,\nabla\lambda_2+\text{ Curl }\lambda_1\rangle_{L^2(\mathbb{R}^3)}-F_2(f)\right\} \\ &= \frac{1}{2}\int_{\mathbb{R}^3}|\nabla\lambda_2+\text{ Curl }\lambda_1|^2\,dx. \end{aligned} \tag{9}$$

Define now $J^*: A^*\times B\to\mathbb{R}$ by

$$J^*(\lambda,t)=J^*(\lambda_1,\lambda_2,\lambda_3,t)=-F_1^*(\lambda_2,\lambda_3,t)-F^*(\lambda_2,\lambda_1).$$

Observe that, from the previous lines, we have obtained

$$\begin{aligned} &\inf_{(m,f)\in V_1\times V_2}\{J(m,f)+Ind_0(m)+Ind_1(m,f)+Ind_2(f)\} \\ = &\inf_{(m,f,t)\in V_1\times V_2\times B} J_1(m,f,t) \\ \geq &\inf_{t\in B}\left\{\sup_{\lambda\in A^*}J^*(\lambda,t)\right\} \\ \geq &\sup_{\lambda\in A^*}\left\{\inf_{t\in B}J^*(\lambda,t)\right\}. \end{aligned} \tag{10}$$

We recall the constraints

$$0\leq t_k\leq 1,\text{ in }\Omega,\ \forall k\in\{1,\cdots,M\}$$

are equivalent to

$$t_k^2-t_k\leq 0,\text{ in }\Omega,\forall k\in\{1,\cdots,M\}.$$

Thus, already including the related Lagrange multipliers, we define the Lagrangian L by

$$L(\lambda,t,\tilde{\lambda})=J^*(\lambda,t)+\sum_{K=1}^M\left\langle t_k^2-t_k,\left(\tilde{\lambda}_k^0\right)^2\right\rangle_{L^2}+\left\langle\sum_{k=1}^M t_k-1,\tilde{\lambda}_1\right\rangle_{L^2}.$$

Let $(\hat{\lambda},t_0,\tilde{\lambda})\in A^*\times B\times L^2(\Omega;\mathbb{R}^{M+1})$ be such that

$$\delta L(\hat{\lambda},t_0,\tilde{\lambda})=\mathbf{0}.$$

Observe that, defining

$$H_1(m,\hat{\lambda}_2,\hat{\lambda}_3,t_0)=-F_1(m,t_0)-\left\langle\sum_{k=1}^3 m_k^2-1,\hat{\lambda}_3\right\rangle_{L^2}+\langle m,\nabla\hat{\lambda}_2\rangle_{L^2},$$

we have that

$$F_1^*(\hat{\lambda}_2,\hat{\lambda}_3,t_0)=\sup_{m\in V_1}\{H_1(m,\hat{\lambda}_3,\hat{\lambda}_2,t_0)\}=H_1(\hat{m},\hat{\lambda}_2,\hat{\lambda}_3,t_0),$$

where $\hat{m}\in V_1$ is such that

$$\frac{\partial H_1(\hat{m},\hat{\lambda}_2,\hat{\lambda}_3,t_0)}{\partial m_k}=\mathbf{0},$$

$\forall k \in \{1,2,3\}$.

Observe also that symbolically, we may write

$$\frac{\partial F_1^*(\hat{\lambda}_2,\hat{\lambda}_3,t_0)}{\partial \lambda_2} = -\text{ div }\left(\frac{\partial F_1^*(\hat{\lambda}_2,\hat{\lambda}_3,t_0)}{\partial(\nabla \lambda_2)}\right).$$

On the other hand, also symbolically, we have

$$\begin{aligned}\frac{\partial F_1^*(\hat{\lambda}_2,\hat{\lambda}_3,t_0)}{\partial(\nabla \lambda_2)} &= \frac{\partial H_1(\hat{m},\hat{\lambda}_3,\hat{\lambda}_2,t_0)}{\partial(\nabla \lambda_2)} \\ &\quad + \frac{\partial H_1(\hat{m},\hat{\lambda}_3,\hat{\lambda}_2,t)}{\partial m}\frac{\partial \hat{m}}{\partial(\nabla \lambda_2)} \\ &= \frac{\partial H_1(\hat{m},\hat{\lambda}_3,\hat{\lambda}_2,t_0)}{\partial(\nabla \lambda_2)} \\ &= \hat{m}. \end{aligned} \tag{11}$$

In summary, we have got

$$\frac{\partial F_1^*(\hat{\lambda}_2,\hat{\lambda}_3,t_0)}{\partial \lambda_2} = -\text{ div }\hat{m}.$$

Similarly, we may obtain

$$\frac{\partial F_1^*(\hat{\lambda}_2,\hat{\lambda}_3,t_0)}{\partial \lambda_3} = -\left(\sum_{k=1}^3 \hat{m}_k^2 - 1\right).$$

From such a results and the variation of L in λ_3 we obtain the following extremal equation

$$-\frac{\partial F_1^*(\hat{\lambda}_2,\hat{\lambda}_3,t_0)}{\partial \lambda_3} = \left(\sum_{k=1}^3 \hat{m}_k^2 - 1\right) = 0, \text{ in } \Omega.$$

From the extremal equation

$$\frac{\partial L(\hat{\lambda},t_0,\tilde{\lambda})}{\partial \lambda_2} = \mathbf{0}$$

we obtain

$$-\frac{\partial F_1^*(\hat{\lambda}_2,\hat{\lambda}_3,t_0)}{\partial \lambda_2} - \frac{\partial F_2^*(\hat{\lambda}_2,\hat{\lambda}_1)}{\partial \lambda_2} = \mathbf{0},$$

so that

$$\text{div }(\hat{m}\chi_\Omega) + \text{ div }(\nabla\hat{\lambda}_2 + \text{ Curl }\hat{\lambda}_1) = 0, \text{ in } \mathbb{R}^3.$$

Thus, denoting

$$\hat{f} = -\nabla\hat{\lambda}_2 - \text{ Curl }\hat{\lambda}_1,$$

we have got

$$\text{div }(\hat{m}\chi_\Omega) - \text{ div }\hat{f} = 0, \text{ in } \mathbb{R}^3.$$

From the variation of L in λ_1, we may obtain

$$\frac{\partial F_2^*(\hat{\lambda}_2,\hat{\lambda}_1)}{\partial \lambda_1} = \mathbf{0}$$

so that

$$\text{Curl }(\nabla\hat{\lambda}_2 - \text{ Curl }\hat{\lambda}_1) = \mathbf{0},$$

that is,

$$\text{Curl }\hat{f} = \mathbf{0}, \text{ in } \mathbb{R}^3.$$

From such results, we have obtained

$$Ind_0(\hat{m}) = 0,\ Ind_1(\hat{m}, \hat{f}) = 0 \text{ and } Ind_2(\hat{f}) = 0.$$

Moreover, from the standard Legendre transform properties, we may also obtain

$$F_1^*(\hat{\lambda}_2, \hat{\lambda}_3, t_0) = -F_1(\hat{m}, t_0) - \left\langle \sum_{k=1}^{3} \hat{m}_k^2 - 1, \hat{\lambda}_3 \right\rangle_{L^2} + \langle \hat{m}, \nabla\hat{\lambda}_2\rangle_{L^2},$$

and

$$F_2^*(\hat{\lambda}_2, \hat{\lambda}_1) = -F_2(\hat{f}) - \langle \hat{f}, \nabla\hat{\lambda}_2 + \text{ Curl }\hat{\lambda}_1\rangle_{L^2(\mathbb{R}^3)}.$$

Combining these last two equations, we have obtained

$$\begin{aligned} J^*(\hat{\lambda}, t_0) &= -F_1^*(\hat{\lambda}_2, \hat{\lambda}_3, t_0) - F_2^*(\hat{\lambda}_2, \hat{\lambda}_1) \\ &= F_1(\hat{m}, t_0) + F_2(\hat{f}) \\ &= F_1(\hat{m}, t_0) + F_2(\hat{f}) \\ &\quad + Ind_0(\hat{m}) + Ind_1(\hat{m}, \hat{f}) + Ind_2(\hat{f}) \\ &= J_1(\hat{m}, \hat{f}, t_0). \end{aligned} \tag{12}$$

Suppose $t_0 \in B$ is such that

$$J^*(\hat{\lambda}, t_0) = \inf_{t \in B} J^*(\hat{\lambda}, t).$$

From an evident concavity of J^* in λ, we have

$$J^*(\hat{\lambda}, t_0) = \sup_{\lambda \in A^*} J^*(\lambda, t_0).$$

From such results and a Standard Saddle Point Theorem, we may infer that

$$\begin{aligned} J^*(\hat{\lambda}, t_0) &= \sup_{\lambda \in A^*} \left\{ \inf_{t \in B} J^*(\lambda, t) \right\} \\ &= \inf_{t \in B} \left\{ \sup_{\lambda \in A^*} J^*(\lambda, t) \right\}. \end{aligned} \tag{13}$$

Joining the pieces, we have got

$$\begin{aligned}
J(\hat{m},\hat{f}) &= J(\hat{m},\hat{f})+Ind_0(\hat{m})+Ind_1(\hat{m},\hat{f})+Ind_2(\hat{f}) \\
&= \inf_{(m,f)\in V_1\times V_2}\{J(m,f)+Ind_0(m)+Ind_1(m,f)+Ind_2(f)\} \\
&= J_1(\hat{m},\hat{f},t_0) \\
&= \inf_{(m,f,t)\in V_1\times V_2\times B} J_1(m,f,t) \\
&= J^*(\hat{\lambda},t_0) \\
&= \sup_{\lambda\in A^*}\left\{\inf_{t\in B} J^*(\lambda,t)\right\} \\
&= \inf_{t\in B}\left\{\sup_{\lambda\in A^*} J^*(\lambda,t)\right\}. \qquad (14)
\end{aligned}$$

The objective of this section is complete.

3 Conclusion

In this article, we have developed a duality principle and a related convex dual variational formulation for a model in micro-magnetism.

We highlight the results here obtained are applicable to a large class of models in the calculus of variations, including some plate and shell non-linear theories, models in superconductivity and phase transition, among many others.

In a near future research we intend to apply such results to some of these mentioned related models.

1. **Conflict of interest declaration: The author declares no conflict of interest concerning this article.**

References

[1] W.R. Bielski, A. Galka, J.J. Telega, *The Complementary Energy Principle and Duality for Geometrically Nonlinear Elastic Shells. I. Simple case of moderate rotations around a tangent to the middle surface.* Bulletin of the Polish Academy of Sciences, Technical Sciences, Vol. 38, No. 7-9, 1988.

[2] W.R. Bielski and J.J. Telega,*A Contribution to Contact Problems for a Class of Solids and Structures,* Arch. Mech., 37, 4-5, pp. 303-320, Warszawa 1985.

[3] J.J. Telega, *On the complementary energy principle in non-linear elasticity. Part I: Von Karman plates and three dimensional solids*, C.R. Acad. Sci. Paris, Serie II, 308, 1193-1198; Part II: Linear elastic solid and non-convex boundary condition. Minimax approach, ibid, pp. 1313-1317 (1989)

[4] A.Galka and J.J.Telega, *Duality and the complementary energy principle for a class of geometrically non-linear structures. Part I. Five parameter shell model; Part II. Anomalous dual variational priciples for compressed elastic beams*, Arch. Mech. 47 (1995) 677-698, 699-724.

[5] J.F. Toland, *A duality principle for non-convex optimisation and the calculus of variations*, Arch. Rat. Mech. Anal., **71**, No. 1 (1979), 41-61.

[6] H. Kronmuller and M. Fahnle, Micromagnetism and Microstructure of Ferromagnetic Solids, Cambridge University Press, 2003.

[7] R.A. Adams and J.F. Fournier, Sobolev Spaces, 2nd edn. (Elsevier, New York, 2003).

[8] F. Botelho, Functional Analysis and Applied Optimization in Banach Spaces, Springer Switzerland, 2014.

[9] F.S. Botelho, Functional Analysis, Calculus of Variations and Numerical Methods for Models in Physics and Engineering, CRC Taylor and Francis, Florida, 2020.

[10] F.S. Botelho, *Variational Convex Analysis*, Ph.D. thesis, Virginia Tech, Blacksburg, VA -USA, (2009).

[11] F. Botelho, Topics on Functional Analysis, Calculus of Variations and Duality, Academic Publications, Sofia, (2011).

[12] F. Botelho, *Existence of solution for the Ginzburg-Landau system, a related optimal control problem and its computation by the generalized method of lines*, Applied Mathematics and Computation, 218, 11976-11989, (2012).

[13] R.T. Rockafellar, Convex Analysis, Princeton Univ. Press, (1970).

[14] F.S. Botelho, *Dual Variational Formulations for a Large Class of Non-Convex Models in the Calculus of Variations*, Mathematics 2023, 11(1), 63; https://doi.org/10.3390/math11010063 - 24 Dec 2022

[15] H. Attouch, G. Buttazzo and G. Michaille, *Variational Analysis in Sobolev and BV Spaces*, MPS-SIAM Series in Optimization, Philadelphia, 2006.

[16] F.S. Botelho, Duality Principles and Numerical Procedures for a Large Class of Non-Convex Models in the Calculus of Variations. Preprints 2023, 2023020051. https://doi.org/10.20944/preprints202302.0051.v95