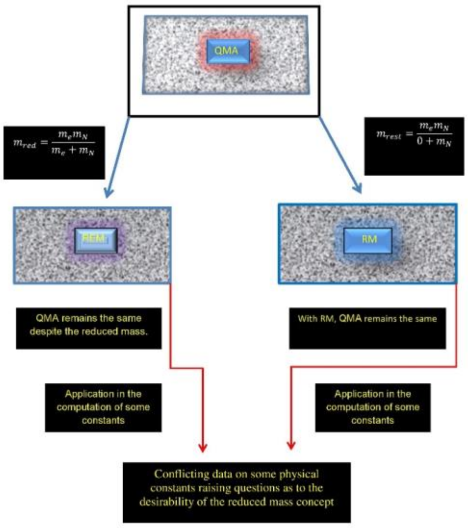
Desirability of Reduced mass Concept, Conflicting CODATA Values of some Constants: Variants of Fine structure constant and Rydberg constant Equations

Abstract. The use of the reduced mass concept as opposed to the rest mass seems to give conflicting results. The objectives were to derive variants of the equation of the Rydberg constant (R_∞), fine structure constant, and wave number that relate with the Rydberg constant. The derived R_∞ equations are variants of each other and of those known in the literature; this is also applicable to the fine structure constant (α). The values of R_∞ range between 10973731.6 and 10973733.89 /m for the rest mass case, while for the reduced mass case, the range is between 10966253.06 and 10967784.63 /m; regarding 'α', the derived equations gave results that are the same. The use of rest mass and reduced mass gave conflicting values for physical constants. Other variants of the equation for R_∞ may be determined in the future.

Keywords: Average ionisation energy, fine structure constant equation variants, reduced mass, rest mass, Rydberg constant equation variants, wave number.

Classification numbers: 32.10.Bi; 32.10.Dk

Graphical abstract



A scheme, which summarises the issues examined and discussed in the aftermath of the theoretical study; all abbreviations, QMA, REM, and RM, are also a reflection of the homogeneity and heterogeneity of matter even if any of them is a proton, recalling quarks in this context. QMA, REM, and RM denote quantity of matter, reduced mass, and rest mass respectively; m_{red}, m_{rest}, m_N, and m_e are the reduced mass, rest mass, mass of a nucleus and mass of an electron respectively for algebraic description.

1. Introduction

"This section is best presented taking into account anonymous views regarding the triviality of the desirability of the reduced mass concept, ignoring all other developments in the article; this is unethical considering the fact that the fine structure constant, for instance, which attained a consistent value, has continued to be subject to experimental reevaluation in order to achieve uncertainty on a pico-scale or less. I provide a summary section that outlines the foundation of this position".

The desirability of the use of reduced mass concept as opposed to the rest mass seems to give conflicting results. "Reduced mass is a term for the appropriate mass corresponding to a linear combination of coordinates involving two or more individual particles (in the present context the particles are atoms)" [1]; the reduced mass is that of a system in motion under mutual forces when one of two particles is observed from the position of the other. [2]. In deriving the equation of reduced mass concept, the equations of motion of two mutually interacting bodies is reduced to a single equation describing the motion of one body in a reference frame centred in the other body (Britannica.com). Barbosa [3] who sees the reduced mass as a quantity studied in a two-particle system problem in mechanics have attempted to explain the origin of the reduced mass concept. According to the author's note [3], the reduced mass of a two-particle system has its origin in the combination of noninertial nature of an observer attached to particle 1 leading to an inertial force (F_{in}); the force ($F_{1(2)}$) which drives the observer is intimately related to the force applied to particle 2 ($F_{2(1)}$) through the Newtonian 3rd law of motion as they arise from the mutual interaction. It is pertinent that the core physicists should delve into the definitions and come with simpler interpretation without losing meaning in a future research. The reduced mass is known to cause the stark broadening of the spectral lines of the hydrogen atoms; it has also been implicated in the deformation of the central part of the line shape of the spectrum [3]. Meanwhile, the Rydberg constant has a role in spectroscopy and is related to the fine structure constant. Hence the desirability of the reduced mass concept, the conflicting CODATA [4, 5] values of some constants, and the variants of the fine structure constant and Rydberg constant equations have to be examined or investigated. The goal is to reveal a possible conflict in some fundamental physical constants by embarking on the following objectives: 1) Derive variants of the equation of Rydberg constant and compare the resulting value with literature; 2) derive variants of the equation relating wave number with Rydberg constant at the atomic level; 3) show that there could be multiple variants of the equation of fine structure constant. Considering the fact that the

Rydberg constant has applications in scientific research and engineering, need to point out ways that may create different values of the constant due to the use of either rest mass or reduced mass may serve as motivation in this study. This is not to the exclusion of the fine structure constant.

2. Theory

2.1 The ratio of the product of the atomic radius and the momentum of the orbital electron to the energy level as a function of the composite fundamental constant

In this section, the relationship between the ratio of the product of the momentum and the ionisation energy-dependent radius of the atom to the energy level and a composite fundamental physical constant are derived. In line with pedagogical principle, one begins from the known to the unknown. The Rydberg constant (R_∞) is given as [5]:

$$R_\infty = \alpha^2 m_e c / 2\,h, \tag{1}$$

where α, m_e *c,* and *h* are the fine structure constant, rest mass of an electron, velocity of light in a vacuum, and Planck constant respectively. The 'α' is given as [5]:

.
$$\alpha = e^2 / 2h\varepsilon_0 c, \tag{2}$$

where ε_0 and *e* are the permittivity in free space and the charge of an electron. Substitute Eq. (2) into Eq. (1) to give:

$$R_\infty = m_e e^4 / 8\,h^3 \varepsilon_0^2 c, \tag{3}$$

Equation (3) is definitely not a novelty, but with Eqs (1) and (2), it serves an immediate reference.

. It has been shown that the effective nuclear charge (Z_{eff}) is inversely proportional to α, directly proportional to the principal quantum number (n_i), and directly proportional to the square root of the average ionisation energy (E_i) [6]. The subscript, *i* means that any variable (principal quantum number, average ionisation energy *etc*.) referred to could be for any energy level; thus,

$$Z_{eff} = n_i \left(\frac{2E_i}{m_e}\right)^{½} / \alpha c \quad , \tag{4}$$

Therefore,

$$c = n_i\,(2E_i/m_e)^{½} / Z_{eff}\,\alpha, \tag{5}$$

Substitute Eq. (5) into Eq. (3) to give:

$$R_\infty = \left(Z_{eff}\, m_e e^4 \alpha / 8\, h^3 \varepsilon_0^2\, n_i\right)\left(\frac{m_e}{2E_i}\right)^{½}, \tag{6}$$

Solving for E_i in Eq. (6) gives:

$$E_i = m_e\left(Z_{eff}\, m_e e^4 \alpha / 8\, h^3 \varepsilon_0^2\, n_i\, R_\infty\right)^2/2 \tag{7}$$

To foreclose error, the value of average ionisation energy, E_i for hydrogen is calculated using 2016 CODATA values of the fundamental constants to give 2.179872321 exp. (−18) J. Substitute Eq. (2) into Eq. (7) to get

$$E_i = m_e^3\left(Z_{eff}\, e^6/16\, h^4 \varepsilon_0^3\, n_i\, c\, R_\infty\right)^2/2, \tag{8}$$

Solving for Z_{eff} in Eq. (8) gives:

$$Z_{eff} = \left(\frac{512E_i}{m_e^3}\right)^{½} \varepsilon_0^3 h^4\, n_i R_\infty c/e^6, \tag{9}$$

In order to validate Eq. (9) the value of Z_{eff} for hydrogen is calculated to give: 1.000000001. Given Bohr's equation for average ionisation energy, such as: $E_i = Z_{eff}^2 e^4 m_e/8\varepsilon_0^2\, n_i^2 h^2$, then, substitute Eq. (9) into Bohr's equation for ionisation energy and equate the result to the Coulomb equation to give:

$$\frac{512E_i}{m_e^3}(\varepsilon_0^3 h^4 n_i R_\infty c)^2 e^4 m_e/8\varepsilon_0^2 n_i^2 h^2 e^{12} = \frac{Z_{eff} e^2}{8\,\pi\,\varepsilon_0 a_i}, \tag{10a}$$

where a_i is the average ionisation energy-dependent radius of any atom other than hydrogen. Simplification and solving for Z_{eff} gives:

$$Z_{eff} = \frac{512E_i \varepsilon_0^5 h^6 R_\infty^2 c^2 a_i \pi}{m_e^2\, e^{10}}, \tag{10b}$$

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To be simplistic, Eq. (10b) can be factorised to yield an equation of two factors such as: $Z_{eff} = x.y$ where x is equal to $\varepsilon_0^2 h^4/m_e^2\, e^4$ and y is equal to $512E_i\varepsilon_0^3 h^2 R_\infty^2 c^2 a_i \pi/m_e^2\, e^6$; the product of x and y does not change the value of Z_{eff}. Thus given the equation of Bohr's radius (for hydrogen atom to be specific), which when squared and multiplied by π^2 should give the same result as if x was multiplied by π^2 and also divided by π^2. Thus, this is such that x is equal to $\pi^2\left(\frac{\varepsilon_0^2 h^4}{\pi^2 m_e^2\, e^4}\right)$. Therefore, with Bohr's equation for the radius of hydrogen where n_i and Z_{eff} are equal to one, $\pi^2(\varepsilon_0^2 h^4/\pi^2\, m_e^2\, e^4)$ should be ≡ $a_0^2\pi^2$ where a_0 is the Bohr's radius for hydrogen; Therefore, Eq. (10b) becomes:

$$Z_{eff} = 512E_i a_i \left(\frac{\varepsilon_0 \pi}{e^2}\right)^3 (hR_\infty c\, a_0)^2, \tag{11}$$

Since the atomic radius is inversely proportional to the average ionisation energy-a kinetic energy- on the basis of Coulomb's law, one can multiply both sides of Eq. (11) by a_i and convert E_i to its relation to momentum (p) to give:

$$a_i Z_{\mathrm{eff}} = 512 p^2 a_i^2 \left(\frac{\varepsilon_0 \pi}{e^2}\right)^3 (hR_\infty c\, a_0)^2/2\, m_e, \tag{12}$$

In Eq. (12), $a_i Z_{eff} = n_i^2 h^2 \varepsilon_0/\pi e^2 m_e = a_0 n_i^2$; thus,

$$p^2 a_i^2 = \frac{m_e a_0 n_i^2}{2^8}\left(\frac{e^2}{\varepsilon_0 \pi}\right)^3 \left(\frac{1}{hR_\infty c\, a_0}\right)^2, \tag{13}$$

$$\frac{pa_i}{n_i} = \frac{1}{16\, h\, c\, R_\infty}\left(\frac{m_e}{a_0}\frac{e^6}{\varepsilon_0^3 \pi^3}\right)^{½}, \tag{14}$$

Given the equation: $(8E_H a_0)^3 = (e^2/\pi\varepsilon_0)^3$ where E_H specifically denotes the average ionisation energy of hydrogen, one can rewrite, Eq. (14) for the purpose of calculational simplicity, as follows:

$$pa_i = n_i a_0 \,(2E_{\mathrm{H}}^3 m_e)^{½}/hR_\infty c, \tag{15}$$

If $a_i = a_0$ and $n_i = 1$, then, hydrogen is referred to:

$$R_\infty = \,(2E_{\mathrm{H}}^3 m_e)^{½}/h\, c\, p, \tag{16a}$$

Equation (16a) can be simplified to obtain equation similar to a literature version [7]:

$$R_\infty = E_{\mathrm{H}}/h\, c, \tag{16b}$$

There are two clear corollaries:

$$R_\infty = n_i a_0 \,(2E_{\mathrm{H}}^3 m_e)^{½}/h\, c\, pa_i \quad , \tag{17a}$$

Equation (17a) can be simplified further if one solves mathematically for p (*i.e.* $(2m_e E_i)^{½}$**)** and $a_i \left(= \frac{e^2 n_i}{8\,\pi\varepsilon_0 (E_{\mathrm{H}} E_i)^{½}}\right)$ which is defined elsewhere [8] is known; after rearrangement the results is:

$$R_\infty = \frac{8\,\pi\,\varepsilon_0 a_0 E_{\mathrm{H}}^2}{h\, c\, e^2}, \tag{17b}$$

To preclude doubt, Eq. (17b) = 1.09781463 exp. (+7)/m if there is no deliberate mistake. Like Eq. (2), one can derive a similar equation for effective nuclear charge as follows: First recall that [6]:

$$\frac{1}{\alpha} = \left(\frac{m_e c^2}{2E_{\mathrm{H}}}\right)^{½}, \tag{18}$$

Also, E_H can be expresses as [6-8]:

$$E_{\mathrm{H}} = \; n_i^2 E_i / Z_{\mathrm{eff}}^2, \tag{19}$$

Substitute Eq. (19) into Eq. (18) to give after rearrangement:

$$m_e c^2 = \frac{2}{\alpha^2} \frac{n_i^2 E_i}{Z_{\mathrm{eff}}^2} \quad , \tag{20}$$

Solving for Z_{eff} gives [6]:

$$Z_{eff} = \frac{n_i}{\alpha c} \left(\frac{2E_i}{m_e} \right)^{½}, \tag{21}$$

Substitute Eq. (2) into Eq. (21) to give:

$$Z_{eff} = \frac{2 n_i v h \varepsilon_0}{e^2}, \tag{22}$$

where v is the velocity of the electron and n_i may be ≥ 1; this is determined as the result of $(2E_i/m_e)^{½}$, where $E_i = m_e v^2/2$. Thus, given the reciprocal of Eq. (2), one sees the similarity in the mathematical "structure" as follows:

$$1/\alpha = \frac{2\, n_0\, c\, h\, \varepsilon_0}{e^2} \tag{23}$$

where $n_0 = 1$. Note that absence of n_0 does not change anything; rather wherever it appears is for the purpose of completeness. The obvious corollary is that:

$$\frac{Z_{eff}}{n_i\, v} = \frac{1}{n_0\, c\, \alpha} = \frac{2h\, \varepsilon_0}{e^2}, \tag{24}$$

$$Z_{eff} = \frac{n_i\, v}{n_0\, c\, \alpha}, \tag{25}$$

Equation (25) can be verified not just only with hydrogen but with any other hydrogenic ion such as ($\mathrm{H_e^{1+}}$); if so, given the average ionisation energy of the latter, the value of Z_{eff} is: 1.999999998. Equation (14) can be rewritten to give two equations.

$$v = \frac{n_i}{16\, a_i\, h\, c\, R_\infty} \left(\frac{e^6}{m_e\, a_0\, \varepsilon_0^3\, \pi^3} \right)^{½}, \tag{26}$$

The values of a_i can range between a_0 and $a_0\, n_i^2/Z_{eff}$ where Z_{eff} is between 1 and < 137 and n_i can range between 1 and 7 or more. For easy reference, one can rewrite the equation stated earlier with equation number below. Thus, as in the literature [8], the radius of any atom is expressed as:

$$a_i = \frac{e^2 n_i}{8\, \pi \varepsilon_0 (E_{\mathrm{H}} E_i)^{½}}, \tag{27}$$

2.2 Fine structure constant–based equations for the determination of the radius of any atom

Note that if hydrogen is referred to E_H is = E_i and n_i = 1; if it is multi-proton hydrogenic ion, E_H is ≠ E_i, though n_i remains = 1. One can then re-express a_i in terms of v as follows: Given that E_i is = to E_H Z_{eff}^2 / n_i^2, Eq. (27) becomes:

$$a_i = \frac{e^2 n_i^2}{8\pi\varepsilon_0 E_H Z_{eff}}, \quad (28a)$$

Then substituting Eq. (25) into Eq. (28a) gives:

$$a_i = \frac{e^2 n_i n_0 c \alpha}{8\pi\varepsilon_0 E_H v}, \quad (28b)$$

Taking v as $(2Z_{eff} e^2/8\pi\varepsilon_0 a_i m_e)^{½}$ and substitute into Eq. (28b) to give:

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$$a_i = \frac{e^2 n_i n_0 c \alpha (8\pi\varepsilon_0 a_i m_e)^{½}}{8\pi\varepsilon_0 E_H (2e^2 Z_{eff})^{½}}$$

Simplification gives:

$$a_i = \frac{e^2 n_0^2 c^2 \alpha^2 m_e n_i^2}{16 E_H^2 \pi\varepsilon_0 Z_{eff}}, \quad (29)$$

Re-substitution of Eq. (25) into Eq. (29) gives:

$$a_i = \frac{e^2 n_0^3 c^3 \alpha^3 m_e n_i}{16 E_H^2 \pi\varepsilon_0 v}, \quad (30)$$

One can repeat the operation leading to Eq. (29) to give:

$$a_i = \frac{(c\alpha n_0)^6 m_e^3 n_i^2 e^2}{64 E_H^4 \pi\varepsilon_0 Z_{eff}}, \quad (31)$$

Ultimately, one can further re-substitute Eq. (25) into Eq. (31) as in Eq. (29) to give:

$$a_i = \frac{(c\alpha n_0)^7 m_e^3 n_i e^2}{64 E_H^4 \pi\varepsilon_0 v}, \quad (32)$$

Clearly, it can be seen that none of the equations is final because there is no end to substitutions. Experimental information is needed in order to proceed with Eqs (28a), (28b), (29), (30), (31), and (32) all of which give the same result.

2.3 Deductions with respect to fine structure constant

Further deductions can be made with respect to the redefinition of the equations of fine structure constant. Beginning from Eqs (28b), (30), and (32), there are possibilities of reproducing Eq. (18) in terms of the fine structure constant if all the fundamental constants and experimental variables, the average ionisation energies, are substituted into the equations. The resulting fine structure constant is approximately 7.2935 exp. (−3). In all cases the equation is: $\alpha = \frac{1}{c\, n_0}\left(\frac{2E_{\mathrm{H}}}{m_e}\right)^{½}$ where n_0 is always equal to one. The equation became a reality after substituting Eq. (27) and $(2E_i/m_e)^{½}$ into the respective equations and solving for α. Rederiving the equation based on Eqs (29) and (31) requires substituting Bohr's equation. This is despite information about ionisation energies, a deterministic variable first discovered by a Russian scientist, yet most western scientists but not limited to them, have been vehement in criticising Bohr's equation ($a_i = n_i^2 h^2 \varepsilon_0 / \pi m_e e^2 Z_{eff}$): It appears no motivation is seen in the Russian scientist, yet there is no better alternative that could be seen as a motivation for rejecting the possibility that ionisation energies that are experimentally generated anywhere in the global space can be used to determine the effective nuclear charge. How effective nuclear charge can be calculated based on ionisation energies and enabling equations are available in the literature [6]. With Eqs (29) and (31), the new equations after the substitution of Bohr's equation for any atom given above into each equation and solving for the fine structure constant are respectively:

$$\alpha = \frac{4\, h\, \varepsilon_0\, E_{\mathrm{H}}}{e^2\, m_e\, c}, \tag{33}$$

$$\alpha = \frac{2}{c\, n_0}\left(\frac{E_{\mathrm{H}}^2\, h\, \varepsilon_0}{m_e^2\, e^2}\right)^{⅓} \quad , \tag{34}$$

If Bohr's equation for ionisation energy is substituted into Eqs (33) and (34), the expected result is very similar to the values obtainable from the equation given as: $\alpha = e^2/2\varepsilon_0 hc$. The latter can be found in any standard advanced text books. It must be made known that other variants of the equation can be derived if the series beyond Eq. (32) is continued. The value of the fine structure constant expected from Eqs (33) and (34) is approximately 7.2945 exp. (−3). Equation (34) is a major innovation as far as the variants of the equation of fine structure constant is concerned.

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2.4 Deductions with respect to wave number

The core atomic physicists, Bohr, Rydberg-Ritz *etc* have concern for wave number though such concern is restricted to hydrogenic atoms and ions only. Moreover, it is all about emission spectrum. It is pointless being in doubt because, a_i and p can be separately calculated as along as the average ionisation energy and energy level are known; this is usually so for most elements. Here, deductively or as a corollary, one can relate the wave number of multi-electron atoms to the Rydberg constant. In this regard, refer to Eq. (26) where the momentum p is equal to $m_e v$ which, in turn, is equal to h/λ, the well-known de Broglie equation where the reciprocal of the wave length ($1/\lambda$) is equal to the wave number (∇). Therefore, given Eq. (15), derivation is as follows:

$$\frac{h\,a_i}{\lambda} = n_i a_0\,(2E_{\mathrm{H}}^3 m_e)^{½}/hR_\infty c, \tag{35}$$

"No beating the gun", just step-by-step, until one obtains the following:

$$\frac{a_i}{n_i\,\lambda} = a_0\,(2E_{\mathrm{H}}^3 m_e)^{½}/h^2 R_\infty c, \tag{36}$$

$$\nabla = n_i a_0\,(2E_{\mathrm{H}}^3 m_e)^{½}/h^2 R_\infty c\, a_i, \tag{37}$$

For the purpose of emphasis, two equations attributed to Bohr are: 1) $E_{\mathrm{i}} = Z_{eff}^2\, e^4\, m_e \,/\, 8\varepsilon_0^2 n_i^2 h^2$ and 2) $a_i = n_i^2 h^2 \varepsilon_0 / \pi\, m_e e^2 Z_{eff}$. Substituting the latter into the former and solving for E_{i} gives:

$$E_i = n_i^2 h^2 \,/\, 8\,\pi^2\, m_e\, a_i^2, \tag{38}$$

Based on Eq. (38), the momentum (h/λ) of the electron is

$$p = n_i h/2\pi a_{\mathrm{i}}, \tag{39}$$

Consequently, the wave number is:

$$\nabla = n_i/2\,\pi\, a_i, \tag{40}$$

With Eqs (37) and (40) the following can be derived. The equations contain n_i/a_i. Therefore,

$$\frac{1}{2\,\pi} = \frac{\mathrm{a}_0\,(2E_{\mathrm{H}}^3 m_e)^{½}}{R_\infty h^2\, c}, \tag{41a}$$

$$R_\infty = \frac{\mathrm{a}_0\,\pi\,(8E_{\mathrm{H}}^3 m_e)^{½}}{h^2\, c}, \tag{41b}$$

For the avoidance of doubt precluding any mistake the value of R_∞ obtainable from Eq. (41b) is: 1.09781174 exp. (+7)/m.

3. Results and discussion

Worthy of note in this research is the observation that the physical variables, the wave number, average ionisation energies, and the square of the effective nuclear charge vary with the square of nuclear charge in the same way (Figure 1). The square of the nuclear charge is chosen

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because it best correlates with the physical variable determined based on the average ionisation energy.

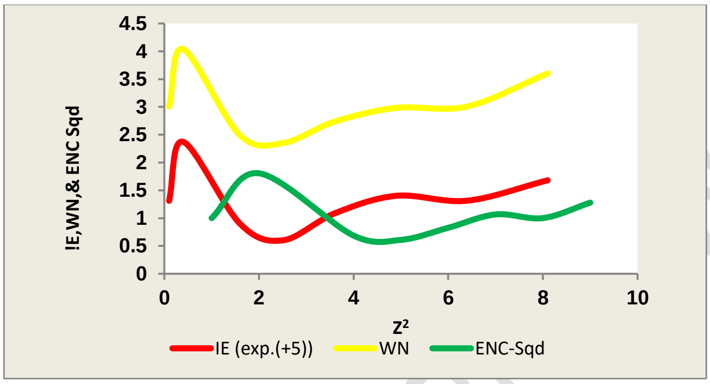

Fig. 1 The plot illustrating the trend of all related physical variables with the square of atomic number. The legends, IE, WN, and ENC-Sqd represent the average ionisation energy, wave number, effective nuclear charge squared. The IE, WN, and ENC-Sqd values were seperately plotted versus the square of atomic number/10. Division by ten was to enhance resolution of the curves despite sum points of contact or intersections.

The values of the Rydberg constant calculated in this research ranges between 10973731.6-10973733.89/m for the rest mass case while, for the reduced mass case, the range is between 10966253.06 and 1096784.63/m (Table 1) which compares with the CODATA [4] value of 10973731.568/ m. This is against the backdrop of higher values of the average ionisation energy of hydrogen and Bohr's radius calculated using rest mass and reduced mass respectively. Variants of the equations for the fine structure constant (α), Eq. (33) and Eq. (34), in particular, are not known in the literature. The values are exactly the same for the reduced mass case, 7.294715912 exp. (−3) and 7.297352577 exp. (−3), for the rest mass case. The α achieved with accuracy of 81 PPT is equal to 7.297352563 exp. (−3) [9] and the CODATA [5] value is 7.2973525664 exp. (−3) (Table 1). The rest mass value is higher than CODATA and Morel *et al.* [9] values.

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Table 1. Calculated Rydberg constant (R_∞) and fine structure constant (α) based on rest mass and reduced mass of the electron.

Equations (parameter)		Based on R-M	Based on Red-M
$R_\infty = 8\,\pi\,\varepsilon_0 a_0 E_{\mathrm{H}}^2/h\,c\,e^2$	(I)	10973731.6	10966253.06
$R_\infty = E_{\mathrm{H}}/h\,c$ (Here & in the lit.)	(II)	10973731.6	10967784.63
$R_\infty = \alpha^2 m_e\,c/2\,h$ (CODATA-I)	(III)	10973731.57	10967758.834
$R_\infty = e^4 m_e/8\,\varepsilon_0^2\,ch^3$ (CODATA-II)	(IV)	10973733.89	10967760.69
α/exp. (−3): Eq. (33)		7.297352577	7.294715912
α/exp. (−3): Eq. (34)		7.297352577	7.294715912

Equation (I) was derived in this research and Eq. (II), was a rederivation in this research, which reproduced the literature (lit.) version. Equation (III) (CODATA-I) and Eq. (IV) (CODATA-II) are CODATA records and derived equation respectively, in which fine structure constant equation was substituted into Eq. (III) to give Eq. (IV). The ionisation energy-based radii of atoms used can be found in the literature [10]: α is the fine structure constant; R-M and Red-M are the rest mass and reduced mass respectively.

The values of the wave number calculated based on Eq. (40) are not influenced by the mass of the electron if the ionisation based radii were calculated using experimentally determined average ionisation energies. An exception may apply to hydrogen if either the rest mass or reduced mass were applied for the calculation of Bohr's radius. The calculations based on Eq. (37) are influenced by the use of either rest mass or reduced mass. The values ranges between 2.735 and 4.036 exp.(+9)/m for Eq. (37); with Eq. (40) it is between 3.006 and 4.038 exp. (+9)/m (Table 2).

Table 2. Calculated wave numbers based on reduced mass and rest mass of the electron.

Elements	H	He	Be	B	C	N	O	F
∇* (Eq. (37)) (exp. (9)/m)	3.006	4.038	2.490	2.348	2.736	2.983	3.007	3.403
∇(Eq. (37)) (exp. (9)/m)	3.005	4.036	2.495	2.347	2.735	2.982	3.006	3.401
∇* (Eq.(40)	3.006	4.038	2.490	2.348	2.736	2.983	3.007	3.402
∇(Eq.(40)	3.008	-	-	-	-	-	-	-

∇* is the wave number affected by the use of rest mass. ∇ is the wave number affected by the use of reduced mass. Values are approximations to the 3rd decimal places.

Various equations were derived, notable of which is the equation unknown in the literature for the calculation of the Rydberg constant (R_∞) and fine structure constant (α) as well. With the various equations, it has been possible to show that the use of reduced mass as opposed to rest mass leads to lower values of physical constants compared to CODATA values. Higher values of R_∞ were obtained from all equations in which Bohr's radius and the average ionisation energy of hydrogen were calculated using the rest mass of the electron. These revelations are made possible by the application of the derived equations, which are therefore analysed and discussed as follows: Given Eq. (17a), it could be seen that R_∞ is inversely proportional to the product of the momentum, *p,* of the orbital electron and its distance from the nucleus, a_i, for multi-electron atoms, whereas it is inversely proportional to the *p* (Eq. (16a)) for hydrogen and hydrogenic ions as well. Incidentally, R_∞ is also directly proportional to the square of the average ionisation energy (E_H) of hydrogen atom (Eq. 17b). Also, Eq. (37) shows that R_∞ is inversely proportional to the product of the wave number and the ionisation energy-dependent atomic radius and directly proportional to the square root of the 3rd power of the E_H for all multi-electron atoms; it is strictly directly proportional to the square root of the 3rd power of the E_H for the hydrogen atom only. With this apparent analysis, one can now x-ray the physical implications.

It is not certain whether or not the concept of reduced mass is applied in the spectroscopic determination of ionisation energies. If not, all the equations where R_∞ is E_H-dependent should give the same results. On the contrary, if theoretically calculated hydrogenic average ionisation energy is the case, the application of either the rest mass or the reduced mass in Bohr's equation for the equation of average ionisation energy would obviously give different results. Recall that Bohr's equation for the radius is inversely proportional to the mass of the electron; using rest mass gives a lower value of the radius (≈5.289643 exp. (−11) m), while the reduced mass (9.104424487 exp. (−31) kg) calculated by deploying all 2012 CODATA [4] values of relevant physical constants gives 5.292523949 exp. (−11) m, which is slightly higher than the 2012 CODATA value of 5.2917721092 exp. (−11) m [4]. If these values are substituted into the Coulomb equation for hydrogen atoms, different values of the kinetic energies (E_H values) are obtainable. In a similar way, if the rest mass and the reduced mass are separately substituted into Bohr's equation ($E_\mathrm{i} = Z_{eff}^2\, e^4\, m_e \,/\, 8\varepsilon_0^2\, n^2 h^2$) for ionisation energy, different results are expected. One is higher than the other (2.179872176 exp.

(−18) and 2.178690844 exp. (−18) J). If these values are substituted again into the Coulomb equation for the hydrogen atom, the radii above, the first and the second are expected. Yet all calculations of R_∞ involving "physical constants" where rest mass featured gave values that were higher than those where reduced mass was the case. Besides, the results from the former (the rest mass case) are very similar to the CODATA value. Therefore, it seems the rest mass played a role in the CODATA value of R_∞. Then the question is: what is the desirability of the use of the reduced mass concept as opposed to the rest mass, which seems to give a conflicting result? In any case, if E_H is experimentally generated and gives a consistent result, then R_∞ may be regarded as a fundamental physical constant, considering the fact that the rest mass (an expression of the quantity of matter in any specific object) is also constant.

With a clear presence of the ratio of average ionisation energy to any kind of electron mass in all the variants of the equation of fine structure constant (Eqs (33) and (34)), the results of calculation should be electron mass of any kind invariant. A recent experimental study showed the value (7.297352563 exp. (−3)) of the fine structure constant with an accuracy of 81 PPT [9], which is lower than the calculated value in this study and the CODATA value, giving the impression that one or more of the fundamental physical terms in the equation of the fine structure constant may be less accurate. However, one may wish to know if a lower value from any future experiment results in higher accuracy. This has become a pertinent question because the CODATA value or any other unknown in the course of this research seems not to be a standard reference for comparison. Be it as it may, the fine structure constant is directly proportional to E_H (Eq. (33)), and it is directly proportional to the cube root of the square of E_H (Eq. (34)).

Equation (36), once again, implies that the ratio of the ionisation energy-dependent radius of any atom to the product of the wave length and principal quantum number is equal to the ratio of composite fundamental constants. Ultimately, the wave number for any multi-electron atom is now related to the Rydberg constant. Note that Eq. (37) is generalisable to all atoms, both hydrogenic and nonhydrogenic atoms and ions. The only experimental information needed, for instance, is the average ionisation energy coupled with the enabling equation, Eq. (27), for instance. In order to evaluate Eq. (37), the first members of the groups in the first period of the period table were used as examples whose first average ionisation energies are known. If there is any way by which a_i can be determined either experimentally or theoretically, it is not just the average ionisation energy that can be calculated according to the conventional Bohr's equation but the wave number as well. The wave number is directly proportional to the square root of the cube of E_H and the principal quantum number and inversely proportional to the radius, as shown in Eqs (37) and (40).

There seems to be more interest in fine structure constant than Rydberg constant despite the fact that the latter is related to the former in the equation given as follows [4, 5]: $R_\infty = \alpha^2 m_e c /2h$. The α has been studied for various reasons including reevaluation of the standard model [11, 12]. Some studies are also devoted to measurements in other to achieve better results in terms of higher precision [13-14]; what looks like theoretical method entailed the application of quartic equations to which is related the much talked about (but yet to be fully elucidated) golden ratio and a-not-too clear classical "harmonic proportions" [16, 17]. In this research, however, variants of the equation of α were derived giving results that are always not more than 1/137. These are in addition to the derived equation relating effective nuclear charge to the α [6] and the equations that confirm the universality of R_∞; this is so because such equation can be used to either determine it or be used to determine nuclear property such as mass radius of any nucleon [7]. Recent mention of R_∞ is in connection with Rydberg atom explored in the determination of the angle-of-arrival of an incident radio-frequency (RF) wave or signal [18]; this seems to be an application of Rydberg concept in technology besides spectroscopic facilities. The 'α' has application in the derivation of the equation of the mass radius of the nucleon and larger subatomic particles [19].

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It is interesting to note that the physical variables exhibit a similar trend with an increasing value of the square of the atomic number. Recall Fig. 1 in this regard. It is akin to the chemical periodicity of elements. Different atoms are characterised by different average ionisation energies, which is another name for the average kinetic energy needed to expel an electron from an uncertain position in its ground state to infinity. It is not unlikely that an electron may, under unnatural conditions (or such natural conditions within the stars, the sun being the nearest of such), move to unspecified lower energy with the release of an amount of energy equal to its ionisation energy at its ground-state energy level. Therefore, wave number may not be restricted to the spectra of different elements; any elementary particle in motion has a de Broglie wave length and, as such, a wave number.

Finally, in the light of the concern shown in the reduced mass concept is the following: The analysis of results from three commonly used theoretical models for field ionisation, Keldysh, Perelomov-Popov-Terent'ev (PPT), and Ammosov-Delone-Krainov (ADK), yielded an unexpected result [20]. This arose from the observation that the inclusion of the ponderomotive potential and the Stark shift decreased the K transition rate under the influence of the Stark shift at an unexpectedly low rate, against theoretical prediction [20]. As a pointer to the desirability of reduced mass, one may wish to know if the reduced mass has no effect on the stark shift, which otherwise could have enhanced its effect on the K transition rate.

4. Summary

The reduced mass is equal to 9.10442528 exp. (−31) kg while the rest mass is 9.1093837015 kg.

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Rydberg constant is equal to 10973731.56816; Bohr's radius (a_0) is equal to 5.29177210903 exp. (−11) m. This is calculated by exploring reduced mass. These are CODATA values. Exploring rest mass yields a_0 equal to 5.28964303 exp. (−11) m.

Pertinent Rydberg equations are:

$$R_\infty = \alpha^2 m_e c/2h \tag{i}$$

Exploring rest mass gives 10972267.4/m (2021 CODATA value = 10973731.568 160/m)

Exploring rest mass gives 10966295/m; therefore use of rest mass gives value similar to CODATA value.

$$R_\infty = e^4 m_e/8\varepsilon_0^2 h^3 c \tag{ii}$$

Exploring rest mass gives 10973697.5/m; this value is much closer to 2021 CODATA value.

Exploring reduced mass gives 10967724.3/m

$$R_\infty = E_H/hc \text{ (iii)}$$

Exploring average ionization energy calculated using Bohr's radius calculated using rest mass gives 10972267.4/m; this the same as value obtained with Eq. (i).

Exploring average ionization energy calculated using Bohr's radius calculated using reduced mass gives 10969316.8/m.

Rydberg frequency (R_f) $= \alpha^2 m_e c^2 / 2h$ (or cR_∞) (iv)

Exploring rest mass gives 3.2898419602508 *exp*. (+15) Hz (2021 CODATA value = 3.289 841 960 2508 *exp*. (+15) Hz).

Exploring reduced mass gives 3.28805123 *exp*. (+15) Hz

Hartree energy (*HE*) $= \alpha^2 m_e c^2$ (v)

Exploring rest mass gives 4.35974472 *exp*. (−18) J (2021 CODATA value = 4.3597447222071 *exp*. (−18) J).

Exploring rest mass gives 4.35858147 *exp*. (−18) J

$$a_0 = n^2 h^2 \varepsilon_0 / \pi m_e e^2 Z \quad \text{(vi)}$$

Average ionization energy$= e^2 / 8\pi\varepsilon_0\, a_0$ (vii)

Exploring rest mass for calculating a_0 gives 5.28964303 exp. (−11) m. Substitution into Eq. (vii) gives 2.17987243 *exp*. (−28) J. Exploring reduced mass for calculating a_0 gives 5.29177210903 *exp*. (−11) m (2021 CODATA value). Substitute into Eq. (vii) gives 2.17899539 *exp*. (−28) J. Therefore, anyone "with a legal mind and pride for the right reason" should be certain that the values determined for each sort of fundamental physical constant differ from one another. By taking a broad view, the esteemed authors of CODATA values could appreciate why there was agreement on the need to investigate rest or reduced mass when determining physical constants. *There is absolutely no need to deploy spyware since data are available for verification.*

5. Conclusion

The use of rest mass and reduced mass gives conflicting values for physical constants. The question as to the desirability of the reduced mass concept is valid considering the observed value calculated in this research based on rest mass is very similar to the CODATA value. Variants of the fine structure constant, radii of atoms, and Rydberg constant equations were derivable. Other variants of the equations for the fine structure constant and, consequently, the Rydberg constant may be determined in the future.

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