Original Research Article

Generalized stability of a general quintic functional equation

Abstract. The general quintic functional equation is a generalization of many functional equations such as Jensen, general quadratic, general cubic, and general quartic functional equations. In this paper, we investigate the generalized stability of the general quintic functional equation.

AMS Subject Classification: 39B82; 39B52.

Key Words: stability of a functional equation; general quintic functional equation; general quintic mapping.

1 Introduction

In this paper, let V, X, and Y be a real vector space, a real normed space, and a real Banach space, respectively. The result of the stability of additive functional equation obtained by Hyers [2] as an answer to an question of group isomorphism raised by Ulam [9] in 1940 became the starting point for stability of functional equations, and many mathematicians followed him to study the stability of various types of functional equations (see [1, 8] for more generalized results).

Consider the general quintic functional equation

$$\sum_{i=0}^{6} \binom{6}{i} (-1)^{6-i} f(x+iy) = 0 \tag{1.1}$$

for all $x, y \in V$. If $f : V \to Y$ is a solution mapping of the functional equation (1.1), then we call the mapping f a general quintic mapping. The result obtained by Y. H. Lee for the Hyers-Ulam-Rassas stability of the function equation (1.1) is shown in the following theorem.

Theorem 1.1 (Theorem 2 in [6]) *Let $p \neq 1, 2, 3, 4, 5$ be a fixed nonnegative real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\left\| \sum_{i=0}^{6} {}_6C_i (-1)^{6-i} f(x+iy) \right\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.2}$$

for all $x, y \in X$. *Then there exists a general quintic mapping* F *with* $F(0) = 0$ *and a constant* $K(p)$ *such that*

$$\|f(x) - f(0) - F(x)\| \leq K(p)\theta\|x\|^p$$

for all $x \in X$.

The hyperstability of the functional equation (1.1) obtained by S. S. Jin and Y. H. Lee is as follows.

Theorem 1.2 (Theorem 2.4 in [4]) *Let* $p < 0$ *be a real number. Suppose that* $f : X \to Y$ *is a mapping satisfying the inequality* (1.2) *for all* $x, y \in X \setminus \{0\}$. *Then* f *satisfies the functional* (1.1).

Y.H. Lee and S. M. Jung [7] obtained partial results of the generalized stability of the functional equation (1.1) using the fixed point method. On the other hand, S. S. Jin and Y. H. Lee [3] used the method of P. Găvruta in [1] to obtain partial stability results of (1.1), too.

In this paper, we will show concise results that have improved the existing results on the stability of the general quintic functional equation in the spirit of P. Găvruta through a clearer proof. In particular, we will extend the range of partial results of the generalized stability of the functional equation (1.1) obtained by Y.H. Lee and S. M. Jung [7] and S. S. Jin and Y. H. Lee [3] to general results.

2 Stability of a general quintic functional equation

Throughout this paper, for a given mapping $f : V \to Y$, we use the following abbreviations:

$$\begin{aligned}
\tilde{f}(x) :=& f(x) - f(0),\\
f_1(x) :=& \frac{1}{5040}\big(\tilde{f}(16x) - 60\tilde{f}(8x) + 1120\tilde{f}(4x) - 7680\tilde{f}(2x) + 16384\tilde{f}(x)\big),\\
f_2(x) :=& -\frac{1}{2688}\big(\tilde{f}(16x) - 58\tilde{f}(8x) + 1008\tilde{f}(4x) - 5888\tilde{f}(2x) + 8192\tilde{f}(x)\big),\\
f_3(x) :=& \frac{1}{4608}\big(\tilde{f}(16x) - 54\tilde{f}(8x) + 808\tilde{f}(4x) - 3456\tilde{f}(2x) + 4096\tilde{f}(x)\big),\\
f_4(x) :=& -\frac{1}{21504}\big(\tilde{f}(16x) - 46\tilde{f}(8x) + 504\tilde{f}(4x) - 1856\tilde{f}(2x) + 2048\tilde{f}(x)\big),\\
f_5(x) :=& \frac{1}{322560}\big(\tilde{f}(16x) - 30\tilde{f}(8x) + 280\tilde{f}(4x) - 960\tilde{f}(2x) + 1024\tilde{f}(x)\big),\\
\underset{y}{\overset{6}{\Delta}} f(x) :=& \sum_{i=0}^{6}\binom{6}{i}(-1)^{6-i}f(x+iy),\\
\Gamma f(x) :=& f(32x) - 62f(16x) + 1240f(8x) - 9920f(4x) + 31744f(2x) - 32768f(x)
\end{aligned}$$

for all $x, y \in V$. By laborious computation we can get some useful equalities in the following lemma.

Lemma 2.1 *For a given mapping $f : V \to Y$, the equalities*

$$
\begin{aligned}
&\overset{6}{\underset{y}{\Delta}} \tilde{f}(x) = \overset{6}{\underset{y}{\Delta}} f(x),\\
&\Gamma\tilde{f}(x) = \overset{6}{\underset{4x}{\Delta}} f(8x) + 6\overset{6}{\underset{4x}{\Delta}} f(4x) + 21\overset{6}{\underset{-4x}{\Delta}} f(24x) + 56\overset{6}{\underset{2x}{\Delta}} f(8x) + 336\overset{6}{\underset{2x}{\Delta}} f(6x)\\
&\qquad + 904\overset{6}{\underset{2x}{\Delta}} f(4x) + 1504\overset{6}{\underset{2x}{\Delta}} f(2x) + 1680\overset{6}{\underset{-2x}{\Delta}} f(12x) + 896\overset{6}{\underset{x}{\Delta}} f(4x)\\
&\qquad + 5376\overset{6}{\underset{x}{\Delta}} f(3x) + 13056\overset{6}{\underset{x}{\Delta}} f(2x) + 15616\overset{6}{\underset{x}{\Delta}} f(x) + 8064\overset{6}{\underset{-x}{\Delta}} f(6x), \qquad (2.1)\\
&\tilde{f}_1(x) - \frac{\tilde{f}_1(2x)}{2} = -\frac{\Gamma\tilde{f}(x)}{10080}, \qquad \tilde{f}_1(x) - 2\tilde{f}_1\left(\frac{x}{2}\right) = \frac{1}{5040}\Gamma\tilde{f}\left(\frac{x}{2}\right), \qquad (2.2)\\
&\tilde{f}_2(x) - \frac{\tilde{f}_2(2x)}{4} = \frac{\Gamma\tilde{f}(x)}{10752}, \qquad \tilde{f}_2(x) - 4\tilde{f}\left(\frac{x}{2}\right) = -\frac{1}{2688}\Gamma\tilde{f}\left(\frac{x}{2}\right), \qquad (2.3)\\
&\tilde{f}_3(x) - \frac{\tilde{f}_3(2x)}{8} = -\frac{\Gamma\tilde{f}(x)}{36864}, \qquad \tilde{f}_3(x) - 8\tilde{f}_3\left(\frac{x}{2}\right) = \frac{1}{4608}\Gamma\tilde{f}\left(\frac{x}{2}\right), \qquad (2.4)\\
&\tilde{f}_4(x) - \frac{\tilde{f}_4(2x)}{16} = \frac{\Gamma\tilde{f}(x)}{344064}, \qquad \tilde{f}_4(x) - 16\tilde{f}_4\left(\frac{x}{2}\right) = -\frac{1}{21504}\Gamma\tilde{f}\left(\frac{x}{2}\right), \qquad (2.5)\\
&\tilde{f}_5(x) - \frac{\tilde{f}_5(2x)}{32} = -\frac{\Gamma\tilde{f}(x)}{10321920}, \qquad \tilde{f}_5(x) - 32\tilde{f}_5\left(\frac{x}{2}\right) = \frac{1}{322560}\Gamma\tilde{f}\left(\frac{x}{2}\right), \qquad (2.6)\\
&f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x) \qquad (2.7)
\end{aligned}
$$

hold for all $x, y \in V$.

Lemma 2.2 *If $f : V \to Y$ satisfies the functional equation $\overset{6}{\underset{y}{\Delta}} f(x) = 0$ for all $x, y \in V$, then the mappings $\tilde{f}_1, \ldots, \tilde{f}_5 : V \to Y$ satisfies*

$$\tilde{f}_k(2x) = 2^k\tilde{f}_k(x) \qquad (2.8)$$

for all $x \in V$ and each $k \in \{1, 2, 3, 4, 5\}$.

Proof. If $f : V \to Y$ satisfies the functional equation $\overset{6}{\underset{y}{\Delta}} f(x) = 0$ for all $x, y \in V$, then $f : V \to Y$ satisfies the functional equation $\Gamma\tilde{f}(x) = 0$ from (2.1). Therefore, the equality (2.8) follows from the equalities (2.2), (2.3), (2.4), (2.5), and (2.6). □

According to Corollary 6 in [5], we obtain following Lemma.

Lemma 2.3 *For a given mapping $f : V \to Y$, if there exist a mapping $F : V \to Y$ and a* *function* $\phi : V \setminus \{0\} \to [0, \infty)$ *that satisfy*

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{1}{2^i} \phi(2^i x) < \infty \quad or \tag{2.9}$$

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{1}{2^{(\ell+1)i}} \phi(2^i x) + \sum_{i=0}^{\infty} 2^{\ell i} \phi\left(\frac{1}{2^i} x\right) < \infty \quad or \tag{2.10}$$

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} 2^{5i} \phi\left(\frac{1}{2^i} x\right) < \infty \tag{2.11}$$

for all $x \in V \setminus \{0\}$ *and for some* $\ell \in \{1, 2, 3, 4\}$, *where* $F(x) = \sum_{k=1}^{5} F_k(x)$ *and every* F_k *has the property* (2.8), *then the mapping* F *is uniquely determined.*

Lemma 2.4 *If a mapping* $f : V \to Y$ *satisfies the functional equation* $\overset{6}{\underset{y}{\Delta}} f(x) = 0$ *for all* $x, y \in V \backslash \{0\}$, *then it is a general quintic mapping.*

Proof. It is clear that $\overset{6}{\underset{0}{\Delta}} f(x) = 0$ for all $x \in V$ and

$$\overset{6}{\underset{y}{\Delta}} f(0) = \overset{6}{\underset{-y}{\Delta}} f(6y) = 0$$

for all $y \in V \backslash \{0\}$. So $\overset{6}{\underset{y}{\Delta}} f(x) = 0$ for all $x, y \in V$ as desired. □

Now we show the generalized stability theorem of (1.1).

Theorem 2.5 *Let* $\varphi : (V \backslash \{0\})^2 \to [0, \infty)$ *be a function satisfying one of the following conditions*

$$\sum_{i=0}^{\infty} 2^{-i} \varphi(2^i x, 2^i y) < \infty, \tag{2.12}$$

$$\sum_{i=0}^{\infty} 4^{-i} \varphi(2^i x, 2^i y) < \infty \quad and \quad \sum_{i=0}^{\infty} 2^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.13}$$

$$\sum_{i=0}^{\infty} 8^{-i} \varphi(2^i x, 2^i y) < \infty \quad and \quad \sum_{i=0}^{\infty} 4^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.14}$$

$$\sum_{i=0}^{\infty} 16^{-i} \varphi(2^i x, 2^i y) < \infty \quad and \quad \sum_{i=0}^{\infty} 8^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.15}$$

$$\sum_{i=0}^{\infty} 32^{-i} \varphi(2^i x, 2^i y) < \infty \quad and \quad \sum_{i=0}^{\infty} 16^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.16}$$

$$\sum_{i=0}^{\infty} 32^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{2.17}$$

for all $x, y \in V\backslash\{0\}$. Suppose that $f : V \to Y$ is a mapping such that

$$\left\| \underset{y}{\overset{6}{\Delta}} f(x) \right\| \leq \varphi(x,y) \tag{2.18}$$

for all $x, y \in V\backslash\{0\}$. Then there exists a unique general quintic mapping F such that

$$\|\tilde{f}(x) - F(x)\| \leq \frac{1}{10080} \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{2^i}, \tag{2.19}$$

$$\|\tilde{f}(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{2^i}{5040} \Phi\left(\frac{x}{2^{i+1}}\right) + \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{10752 \cdot 4^i}, \tag{2.20}$$

$$\|\tilde{f}(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{4^i}{2688} \Phi\left(\frac{x}{2^{i+1}}\right) + \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{36864 \cdot 8^i}, \tag{2.21}$$

$$\|\tilde{f}(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{8^i}{4608} \Phi\left(\frac{x}{2^{i+1}}\right) + \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{344064 \cdot 16^i}, \tag{2.22}$$

$$\|\tilde{f}(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{16^i}{21504} \Phi\left(\frac{x}{2^{i+1}}\right) + \sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{10321920 \cdot 32^i}, \tag{2.23}$$

$$\|\tilde{f}(x) - F(x)\| \leq \sum_{i=4}^{\infty} \frac{32^i}{322560} \Phi\left(\frac{x}{2^{i+1}}\right) \tag{2.24}$$

for all $x \in V\backslash\{0\}$ if φ satisfies (2.12), (2.13), (2.14), (2.15), (2.16), or (2.17), respectively, where $\Phi : V\backslash\{0\} \to [0, \infty)$ is the function defined by

$$\begin{aligned}\Phi(x) :=& \varphi(8x, 4x) + 6\varphi(4x, 4x) + 21\varphi(24x, -4x) + 56\varphi(8x, 2x) + 336\varphi(6x, 2x) \\ &+ 904\varphi(4x, 2x) + 1504\varphi(2x, 2x) + 1680\varphi(12x, -2x) + 896\varphi(4x, x) \\ &+ 5376\varphi(3x, x) + 13056\varphi(2x, x) + 15616\varphi(x, x) + 8064\varphi(6x, -x).\end{aligned}$$

Proof. Notice that, from(2.1) and (2.18), we have

$$\begin{aligned}\|\Gamma\tilde{f}(x)\| =& \Big\| \underset{4x}{\overset{6}{\Delta}} f(8x) + 6 \underset{4x}{\overset{6}{\Delta}} f(4x) + 21 \underset{-4x}{\overset{6}{\Delta}} f(24x) + 56 \underset{2x}{\overset{6}{\Delta}} f(8x) + 336 \underset{2x}{\overset{6}{\Delta}} f(6x) \\ &+ 904 \underset{2x}{\overset{6}{\Delta}} f(4x) + 1504 \underset{2x}{\overset{6}{\Delta}} f(2x) + 1680 \underset{-2x}{\overset{6}{\Delta}} f(12x) + 896 \underset{x}{\overset{6}{\Delta}} f(4x) \\ &+ 5376 \underset{x}{\overset{6}{\Delta}} f(3x) + 13056 \underset{x}{\overset{6}{\Delta}} f(2x) + 15616 \underset{x}{\overset{6}{\Delta}} f(x) + 8064 \underset{-x}{\overset{6}{\Delta}} f(6x) \Big\| \\ \leq& \Phi(x)\end{aligned} \tag{2.25}$$

for all $x \in V$. We prove the theorem in two steps.

Step 1. Let $k \in \{1, 2, 3, 4, 5\}$ and $\delta \in \{-1, 1\}$, and let φ satisfy

$$\sum_{n=0}^{\infty} \frac{\varphi(2^{\delta n} x, 2^{\delta n} y)}{2^{\delta k n}} < \infty \tag{2.26}$$

for all $x, y \in V\backslash\{0\}$. Together with

$$\frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta kn}} - \frac{\tilde{f}_k(2^{\delta(n+m)}x)}{2^{\delta k(n+m)}} = \sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_k(2^{\delta i}x)}{2^{\delta ki}} - \frac{\tilde{f}_k(2^{\delta(i+1)}x)}{2^{\delta k(i+1)}}\right)$$

and (2.2), (2.3), (2.4), (2.5), (2.6), (2.25), we have the inequalities

$$\left\|\frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m}x)}{2^{n+m}}\right\| \le \frac{1}{10080}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{2^i}\right\| \le \frac{1}{10080}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{2^i},$$

$$\left\|2^n\tilde{f}_1\left(\frac{x}{2^n}\right) - 2^{n+m}\tilde{f}_1\left(\frac{x}{2^{n+m}}\right)\right\| \le \frac{1}{5040}\sum_{i=n}^{n+m-1}\left\|2^i\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \le \frac{1}{5040}\sum_{i=n}^{n+m-1}2^i\Phi\left(\frac{x}{2^{i+1}}\right),$$

$$\left\|\frac{\tilde{f}_2(2^n x)}{4^n} - \frac{\tilde{f}_2(2^{n+m}x)}{4^{n+m}}\right\| \le \frac{1}{10752}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{4^i}\right\| \le \frac{1}{10752}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{4^i},$$

$$\left\|4^n\tilde{f}_2\left(\frac{x}{2^n}\right) - 4^{n+m}\tilde{f}_2\left(\frac{x}{2^{n+m}}\right)\right\| \le \frac{1}{2688}\sum_{i=n}^{n+m-1}\left\|4^i\ \Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \le \frac{1}{2688}\sum_{i=n}^{n+m-1}4^i\Phi\left(\frac{x}{2^{i+1}}\right),$$

$$\left\|\frac{\tilde{f}_3(2^n x)}{8^n} - \frac{\tilde{f}_3(2^{n+m}x)}{8^{n+m}}\right\| \le \frac{1}{36864}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{8^i}\right\| \le \frac{1}{36864}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{8^i},$$

$$\left\|8^n\tilde{f}_3\left(\frac{x}{2^n}\right) - 8^{n+m}\tilde{f}_3\left(\frac{x}{2^{n+m}}\right)\right\| \le \frac{1}{4608}\sum_{i=n}^{n+m-1}\left\|8^i\ \Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \le \frac{1}{4608}\sum_{i=n}^{n+m-1}8^i\Phi\left(\frac{x}{2^{i+1}}\right),$$

$$\left\|\frac{\tilde{f}_4(2^n x)}{16^n} - \frac{\tilde{f}_4(2^{n+m}x)}{16^{n+m}}\right\| \le \frac{1}{344064}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{16^i}\right\| \le \frac{1}{344064}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{16^i},$$

$$\left\|16^n\tilde{f}_3\left(\frac{x}{2^n}\right) - 16^{n+m}\tilde{f}_3\left(\frac{x}{2^{n+m}}\right)\right\| \le \frac{1}{21504}\sum_{i=n}^{n+m-1}\left\|16^i\ \Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \le \frac{1}{21504}\sum_{i=n}^{n+m-1}16^i\Phi\left(\frac{x}{2^{i+1}}\right),$$

$$\left\|\frac{\tilde{f}_5(2^n x)}{32^n} - \frac{\tilde{f}_5(2^{n+m}x)}{32^{n+m}}\right\| \le \frac{1}{10321920}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{32^i}\right\| \le \frac{1}{10321920}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{32^i},$$

and

$$\left\| 32^n \tilde{f}_5\left(\frac{x}{2^n}\right) - 32^{n+m}\tilde{f}_5\left(\frac{x}{2^{n+m}}\right)\right\| \leq \frac{1}{322560}\sum_{i=n}^{n+m-1}\left\|32^i\, \Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\|$$
$$\leq \frac{1}{322560}\sum_{i=n}^{n+m-1} 32^i\Phi\left(\frac{x}{2^{i+1}}\right)$$

for all $x \in V\backslash\{0\}$ and $n, m \in \mathbb{N}\cup\{0\}$. It leads us to prove that $\left\{\frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta k n}}\right\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$ if φ satisfies (2.26). Moreover, since Y is complete and $\tilde{f}_k(0) = 0$, the sequence converges for all $x \in V$. It follows that we can define a mapping $F_{\delta k} : V \to Y$ by

$$F_{\delta k}(x) := \lim_{n\to\infty}\frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta k n}} \tag{2.27}$$

for all $x \in V$ if φ satisfies (2.26). Now we observe that the equality

$$\begin{aligned}\underset{y}{\overset{6}{\Delta}} F_{\delta k}(x) =& F_{\delta k}(x+6y) - 6F_{\delta k}(x+5y) + 15F_{\delta k}(x+4y) - 20F_{\delta k}(x+3y)\\ &+ 15F_{\delta k}(x+2y) - 6F_{\delta k}(x+y) + F_{\delta k}(x)\\ =& \lim_{n\to\infty}\Bigg(\frac{\tilde{f}_k(2^{\delta n}(x+6y))}{2^{\delta n}} - 6\frac{\tilde{f}_k(2^{\delta n}(x+5y))}{2^{\delta n}} + 15\frac{\tilde{f}_k(2^{\delta n}(x+4y))}{2^{\delta n}}\\ &- 20\frac{\tilde{f}_k(2^{\delta n}(x+3y))}{2^{\delta n}} + 15\frac{\tilde{f}_k(2^{\delta n}(x+2y))}{2^{\delta n}} - 6\frac{\tilde{f}_k(2^{\delta n}(x+y))}{2^{\delta n}} + \frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta n}}\Bigg)\end{aligned}$$

holds for all $x, y \in V\backslash\{0\}$. Together with the definition of $\tilde{f}_1$, if φ satisfies (2.26) for $k = 1$, then we have

$$\begin{aligned}\left\|\underset{y}{\overset{6}{\Delta}} F_{\delta 1}(x)\right\| =& \lim_{n\to\infty}\Bigg\|\frac{1}{5040}\left(\frac{\tilde{f}(2^{\delta n+4}(x+6y))}{2^{\delta n}} - 6\frac{\tilde{f}(2^{\delta n+4}(x+5y))}{2^{\delta n}} + \cdots + \frac{\tilde{f}(2^{\delta n+4}x)}{2^{\delta n}}\right)\\ &- \frac{60}{5040}\left(\frac{\tilde{f}(2^{\delta n+3}(x+6y))}{2^{\delta n}} - 6\frac{\tilde{f}(2^{\delta n+3}(x+5y))}{2^{\delta n}} + \cdots + \frac{\tilde{f}(2^{\delta n+3}x)}{2^{\delta n}}\right)\\ &+ \frac{1120}{5040}\left(\frac{\tilde{f}(2^{\delta n+2}(x+6y))}{2^{\delta n}} - 6\frac{\tilde{f}(2^{\delta n+2}(x+5y))}{2^{\delta n}} + \cdots + \frac{\tilde{f}(2^{\delta n+2}x)}{2^{\delta n}}\right)\\ &- \frac{7680}{5040}\left(\frac{\tilde{f}(2^{\delta n+1}(x+6y))}{2^{\delta n}} - 6\frac{\tilde{f}(2^{\delta n+1}(x+5y))}{2^{\delta n}} + \cdots + \frac{\tilde{f}(2^{\delta n+1}x)}{2^{\delta n}}\right)\\ &+ \frac{16384}{5040}\left(\frac{\tilde{f}(2^{\delta n}(x+6y))}{2^{\delta n}} - 6\frac{\tilde{f}(2^{\delta n}(x+5y))}{2^{\delta n}} + \cdots + \frac{\tilde{f}(2^{\delta n}x)}{2^{\delta n}}\right)\Bigg\|\\ =& \lim_{n\to\infty}\Bigg\|\frac{16384\Delta^6_{2^{\delta n}y}f(2^{\delta n}x)}{5040\cdot 2^{\delta n}} - \frac{7680\Delta^6_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{5040\cdot 2^{\delta n}} + \frac{1120\Delta^6_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{5040\cdot 2^{\delta n}}\\ &- \frac{60\Delta^6_{2^{\delta n+3}y}f(2^{\delta n+3}x)}{5040\cdot 2^{\delta n}} + \frac{\Delta^6_{2^{\delta n+4}y}f(2^{\delta n+4}x)}{5040\cdot 2^{\delta n}}\Bigg\|\\ \leq& \lim_{n\to\infty}\Bigg(\frac{1024\varphi(2^{\delta n}x, 2^{\delta n}y)}{315\cdot 2^{\delta n}} + \frac{480\varphi(2^{\delta n+1}x, 2^{\delta n+1}y)}{315\cdot 2^{\delta n}} + \frac{2\varphi(2^{\delta n+2}x, 2^{\delta n+2}y)}{9\cdot 2^{\delta n}}\end{aligned}$$

$$+\frac{\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{84\cdot 2^{\delta n}}+\frac{\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{5040\cdot 2^{\delta n}}\Bigg) = 0$$

for all $x,y\in V\backslash\{0\}$. In a similar way, by the definition of $\tilde{f}_k$, if φ satisfies (2.26) for $k=2,3,4,5$, respectively, then we get

$$\begin{aligned}\left\|\overset{6}{\underset{y}{\Delta}}F_{\delta 2}(x)\right\| &= \lim_{n\to\infty}\Bigg\|\frac{\Delta^6_{2^{\delta n+4}y}f(2^{\delta n+4}x)}{-2688\cdot 4^{\delta n}}+\frac{58\Delta^6_{2^{\delta n+3}y}f(2^{\delta n+3}x)}{2688\cdot 4^{\delta n}}-\frac{1088\Delta^6_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{2688\cdot 4^{\delta n}}\\&\quad+\frac{5888\Delta^6_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{2688\cdot 4^{\delta n}}-\frac{8192\,\Delta^6_{2^{\delta n}y}f(2^{\delta n}x)}{2688\cdot 4^{\delta n}}\Bigg\|\\&\le \lim_{n\to\infty}\Bigg(\frac{\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{2688\cdot 4^{\delta n}}+\frac{58\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{2688\cdot 4^{\delta n}}+\frac{1088\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{2688\cdot 4^{\delta n}}\\&\quad+\frac{5888\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{2688\cdot 4^{\delta n}}+\frac{8192\varphi(2^{\delta n}x,2^{\delta n}y)}{2688\cdot 4^{\delta n}}\Bigg)\\&= 0,\end{aligned}$$

$$\begin{aligned}\left\|\overset{6}{\underset{y}{\Delta}}F_{\delta 3}(x)\right\| &= \lim_{n\to\infty}\Bigg\|\frac{\Delta^6_{2^{\delta n+4}y}f(2^{\delta n+4}x)}{4608\cdot 4^{\delta n}}-\frac{54\Delta^6_{2^{\delta n+3}y}f(2^{\delta n+3}x)}{4608\cdot 4^{\delta n}}+\frac{808\Delta^6_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{4608\cdot 4^{\delta n}}\\&\quad-\frac{3456\Delta^6_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{4608\cdot 4^{\delta n}}+\frac{4096\,\Delta^6_{2^{\delta n}y}f(2^{\delta n}x)}{4608\cdot 4^{\delta n}}\Bigg\|\\&\le \lim_{n\to\infty}\Bigg(\frac{\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{4608\cdot 4^{\delta n}}+\frac{54\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{4608\cdot 4^{\delta n}}+\frac{808\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{4608\cdot 4^{\delta n}}\\&\quad+\frac{3456\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{4608\cdot 4^{\delta n}}+\frac{4096\varphi(2^{\delta n}x,2^{\delta n}y)}{4608\cdot 4^{\delta n}}\Bigg)\\&= 0,\end{aligned}$$

$$\begin{aligned}\left\|\overset{6}{\underset{y}{\Delta}}F_{\delta 4}(x)\right\| &= \lim_{n\to\infty}\Bigg\|\frac{\Delta^6_{2^{\delta n+4}y}f(2^{\delta n+4}x)}{-21504\cdot 8^{\delta n}}+\frac{46\Delta^6_{2^{\delta n+3}y}f(2^{\delta n+3}x)}{21504\cdot 8^{\delta n}}-\frac{504\Delta^6_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{21504\cdot 8^{\delta n}}\\&\quad+\frac{1856\Delta^6_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{21504\cdot 8^{\delta n}}-\frac{2048\,\Delta^6_{2^{\delta n}y}f(2^{\delta n}x)}{21504\cdot 8^{\delta n}}\Bigg\|\\&\le \lim_{n\to\infty}\Bigg(\frac{\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{21504\cdot 8^{\delta n}}+\frac{46\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{21504\cdot 8^{\delta n}}+\frac{504\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{21504\cdot 8^{\delta n}}\\&\quad+\frac{1856\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{21504\cdot 8^{\delta n}}+\frac{2048\varphi(2^{\delta n}x,2^{\delta n}y)}{21504\cdot 8^{\delta n}}\Bigg)\\&= 0,\end{aligned}$$

$$\begin{aligned}\left\|\overset{6}{\underset{y}{\Delta}} F_{\delta 5}(x)\right\| &= \lim_{n\to\infty}\left\|\frac{\Delta^6_{2^{\delta n+4}y}f(2^{\delta n+4}x)}{322560\cdot 8^{\delta n}} - \frac{30\Delta^6_{2^{\delta n+3}y}f(2^{\delta n+3}x)}{322560\cdot 8^{\delta n}} + \frac{280\Delta^6_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{322560\cdot 8^{\delta n}}\right.\\ &\qquad \left. - \frac{960\Delta^6_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{322560\cdot 8^{\delta n}} + \frac{1024\,\Delta^6_{2^{\delta n}y}f(2^{\delta n}x)}{322560\cdot 8^{\delta n}}\right\|\\ &\le \lim_{n\to\infty}\left(\frac{\varphi(2^{\delta n+4}x,2^{\delta n+4}y)}{322560\cdot 8^{\delta n}} + \frac{30\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{322560\cdot 8^{\delta n}} + \frac{280\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{322560\cdot 8^{\delta n}}\right.\\ &\qquad \left. + \frac{960\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{322560\cdot 8^{\delta n}} + \frac{1024\varphi(2^{\delta n}x,2^{\delta n}y)}{322560\cdot 8^{\delta n}}\right)\\ &= 0\end{aligned}$$

for all $x, y \in V\backslash\{0\}$. And then, since $\Delta^6_y F_{\delta k}(x) = 0$ for all $x, y \in V\backslash\{0\}$, the mapping $F_{\delta k}$ is a general quintic mapping for all $k \in \{1,2,3,4,5\}$ and $\delta \in \{+1,-1\}$ by Lemma 2.4.

Step 2. Now we define the desired general quintic mapping F for all cases.

(1) Let φ satisfy the condition (2.12), then F_1, F_2, F_3, F_4, and F_5 are defined by (2.27). We put a general quintic mapping $F : V \to Y$ by

$$F(x) := F_1(x) + F_2(x) + F_3(x) + F_4(x) + F_5(x)$$

for all $x \in V$. Observe that. by (2.2), (2.3), (2.4), (2.5), and (2.6), we have

$$\begin{aligned}&\left\|\tilde f(x) - \sum_{k=1}^{5}\frac{\tilde f_k(2^n x)}{2^{kn}}\right\| \le \sum_{i=0}^{n-1}\left\|\sum_{k=1}^{5}\left(\frac{\tilde f_k(2^i x)}{2^{ki}} - \frac{\tilde f_k(2^{i+1}x)}{2^{k(i+1)}}\right)\right\|\\ &= \sum_{i=0}^{n-1}\left(\frac{1}{10080\cdot 2^i} - \frac{1}{10752\cdot 4^i} + \frac{1}{36864\cdot 8^i} - \frac{1}{344064\cdot 16^i} + \frac{1}{10321920\cdot 32^i}\right)\|\Gamma\tilde f(2^i x)\|\\ &\le \sum_{i=0}^{n-1}\left\|\frac{\Gamma\tilde f(2^i x)}{10080\cdot 2^i}\right\| \le \frac{1}{10080}\sum_{i=0}^{n-1}\frac{\Phi(2^i x)}{2^i}\end{aligned}$$

for all $x \in V\backslash\{0\}$, which follows (2.19) as $n \to \infty$.

(2) Let φ satisfy the condition (2.13), then F_{-1}, F_2, F_3, F_4, and F_5 are defined by (2.27). Putting a general quintic mapping $F : V \to Y$ by

$$F(x) := F_{-1}(x) + F_2(x) + F_3(x) + F_4(x) + F_5(x)$$

for all $x \in V$. Then we have

$$\begin{aligned}&\left\|\tilde f(x) - 2^n\tilde f_1\left(\frac{x}{2^n}\right) - \sum_{k=2}^{5}\frac{\tilde f_k(2^n x)}{2^{kn}}\right\|\\ &\qquad\le \sum_{i=0}^{n-1}\left\|2^i\tilde f_1\left(\frac{x}{2^i}\right) - 2^{i+1}\tilde f_1\left(\frac{x}{2^{i+1}}\right)\right\| + \sum_{i=0}^{n-1}\left\|\sum_{k=2}^{5}\left(\frac{\tilde f_k(2^i x)}{2^{ki}} - \frac{\tilde f_k(2^{i+1}x)}{2^{k(i+1)}}\right)\right\|\\ &\qquad\le \sum_{i=0}^{n-1}\frac{2^i}{5040}\left\|\Gamma\tilde f\left(\frac{x}{2^{i+1}}\right)\right\|\end{aligned}$$

$$+\sum_{i=0}^{n-1}\left(\frac{1}{10752\cdot 4^i}-\frac{1}{36864\cdot 8^i}+\frac{1}{344064\cdot 16^i}-\frac{1}{10321920\cdot 32^i}\right)\|\Gamma\tilde{f}(2^ix)\|$$
$$\leq\frac{1}{5040}\sum_{i=0}^{n-1}2^i\Phi\left(\frac{x}{2^{i+1}}\right)+\frac{1}{10752}\sum_{i=0}^{n-1}\frac{\Phi(2^ix)}{4^i}$$

for all $x\in V\backslash\{0\}$ by (2.2), (2.3), (2.4), (2.5), and (2.6), which follows (2.20) as $n\to\infty$.

(3) Let φ satisfy the condition (2.14), then F_{-1}, F_{-2}, F_3, F_4, and F_5 are defined by (2.27). Putting a general quintic mapping

$$F(x):=F_{-1}(x)+F_{-2}(x)+F_3(x)+F_4(x)+F_5(x)$$

for all $x\in V$. We have the inequality

$$\left\|\tilde{f}(x)-\sum_{k=1}^{2}2^{kn}\tilde{f}_k\left(\frac{x}{2^n}\right)-\sum_{k=3}^{5}\frac{\tilde{f}_k(2^nx)}{2^{kn}}\right\|$$
$$\leq\sum_{i=0}^{n-1}\left\|\sum_{k=1}^{2}\left(2^{ki}\tilde{f}_k\left(\frac{x}{2^i}\right)-2^{k(i+1)}\tilde{f}_k\left(\frac{x}{2^{i+1}}\right)\right\|+\sum_{i=0}^{n-1}\left\|\sum_{k=3}^{5}\left(\frac{\tilde{f}_k(2^ix)}{2^{ki}}-\frac{\tilde{f}_k(2^{i+1}x)}{2^{k(i+1)}}\right)\right\|$$
$$\leq\sum_{i=0}^{n-1}\left(\frac{2^i}{5040}-\frac{4^i}{2688}\right)\left\|\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\|$$
$$+\sum_{i=0}^{n-1}\left(\frac{1}{36864\cdot 8^i}-\frac{1}{344064\cdot 16^i}+\frac{1}{10321920\cdot 32^i}\right)\|\Gamma\tilde{f}(2^ix)\|$$
$$\leq\frac{1}{2688}\sum_{i=0}^{n-1}4^i\Phi\left(\frac{x}{2^{i+1}}\right)+\frac{1}{36864}\sum_{i=0}^{n-1}\frac{\Phi(2^ix)}{8^i}$$

for all $x\in V\backslash\{0\}$ by (2.2), (2.3), (2.4), (2.5), and (2.6), which follows (2.21) as $n\to\infty$.

(4) Let φ satisfy the condition(2.15), then F_{-1}, F_{-2}, F_{-3}, F_4, and F_5 are defined by (2.27). Putting a general quintic mapping

$$F(x):=F_{-1}(x)+F_{-2}(x)+F_{-3}(x)++F_4(x)+F_5(x)$$

for all $x\in V$. We have the inequality

$$\left\|\tilde{f}(x)-\sum_{k=1}^{3}2^{kn}\tilde{f}_k\left(\frac{x}{2^n}\right)-\sum_{k=4}^{5}\frac{\tilde{f}_k(2^nx)}{2^{kn}}\right\|$$
$$\leq\sum_{i=0}^{n-1}\left\|\sum_{k=1}^{3}\left(2^{ki}\tilde{f}_k\left(\frac{x}{2^i}\right)-2^{k(i+1)}\tilde{f}_k\left(\frac{x}{2^{i+1}}\right)\right\|+\sum_{i=0}^{n-1}\left\|\sum_{k=4}^{5}\left(\frac{\tilde{f}_k(2^ix)}{2^{ki}}-\frac{\tilde{f}_k(2^{i+1}x)}{2^{k(i+1)}}\right)\right\|$$
$$\leq\sum_{i=0}^{n-1}\left(\frac{2^i}{5040}-\frac{4^i}{2688}+\frac{8^i}{4608}\right)\left\|\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\|$$
$$+\sum_{i=0}^{n-1}\left(\frac{1}{344064\cdot 16^i}-\frac{1}{10321920\cdot 32^i}\right)\|\Gamma\tilde{f}(2^ix)\|$$

$$\leq \frac{1}{4608}\sum_{i=0}^{n-1} 8^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{344064}\sum_{i=0}^{n-1}\frac{\Phi(2^i x)}{16^i}$$

for all $x \in V\backslash\{0\}$ by (2.2), (2.3), (2.4), (2.5), and (2.6), which follows (2.22) as $n \to \infty$.

(5) Let φ satisfy the condition (2.16), then F_{-1}, F_{-2}, F_{-3}, F_{-4}, and F_5 are defined by (2.27). Putting a general quintic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_{-3}(x) + F_{-4}(x) + F_5(x)$$

for all $x \in V$. We have the inequality

$$\begin{aligned}
&\left\|\tilde{f}(x) - \sum_{k=1}^{4} 2^{kn}\tilde{f}_k\left(\frac{x}{2^n}\right) - \frac{\tilde{f}_5(2^n x)}{2^{5n}}\right\| \\
&\leq \sum_{i=0}^{n-1}\left\|\sum_{k=1}^{4}\left(2^{ki}\tilde{f}_k\left(\frac{x}{2^i}\right) - 2^{k(i+1)}\tilde{f}_k\left(\frac{x}{2^{i+1}}\right)\right)\right\| + \sum_{i=0}^{n-1}\left\|\frac{\tilde{f}_5(2^i x)}{2^{5i}} - \frac{\tilde{f}_5(2^{i+1}x)}{2^{5(i+1)}}\right\| \\
&\leq \sum_{i=0}^{n-1}\left\|\left(\frac{2^i}{5040} - \frac{4^i}{2688} + \frac{8^i}{4608} - \frac{16^i}{21504}\right)\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| + \sum_{i=0}^{n-1}\frac{1}{10321920\cdot 32^i}\|\Gamma\tilde{f}(2^i x)\| \\
&\leq \frac{1}{21504}\sum_{i=0}^{n-1} 16^i\Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{10321920}\sum_{i=0}^{n-1}\frac{\Phi(2^i x)}{32^i}
\end{aligned}$$

for all $x \in V\backslash\{0\}$ by (2.2), (2.3), (2.4), (2.5), and (2.6), which follows (2.23) as $n \to \infty$.

(6) Let φ satisfy the condition (2.17), then F_{-1}, F_{-2}, F_{-3}, F_{-4}, and F_{-5} are defined by (2.27). Putting a general quintic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_{-3}(x) + F_{-4}(x) + F_{-5}(x)$$

for all $x \in V$. We have the inequality

$$\begin{aligned}
\left\|\tilde{f}(x) - \sum_{k=1}^{5} 2^{kn}\tilde{f}_k\left(\frac{x}{2^n}\right)\right\| &\leq \sum_{i=0}^{n-1}\left\|\sum_{k=1}^{5}\left(2^{ki}\tilde{f}_k\left(\frac{x}{2^i}\right) - 2^{k(i+1)}\tilde{f}_k\left(\frac{x}{2^{i+1}}\right)\right)\right\| \\
&\leq \sum_{i=0}^{n-1}\left\|\left(\frac{2^i}{5040} - \frac{4^i}{2688} + \frac{8^i}{4608} - \frac{16^i}{21504} + \frac{32^i}{322560}\right)\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \\
&\leq \sum_{i=4}^{n-1}\left(\frac{2^i}{5040} - \frac{4^i}{2688} + \frac{8^i}{4608} - \frac{16^i}{21504} + \frac{32^i}{322560}\right)\left\|\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \\
&\leq \frac{1}{322560}\sum_{i=4}^{n-1} 32^i\Phi\left(\frac{x}{2^{i+1}}\right)
\end{aligned}$$

for all $x \in V\backslash\{0\}$ by (2.2), (2.3), (2.4), (2.5), and (2.6), since $\frac{2^i}{5040} - \frac{4^i}{2688} + \frac{8^i}{4608} - \frac{16^i}{21504} + \frac{32^i}{322560} = 0$ when $i \in \{0, 1, 2, 3\}$, which follows (2.24) as $n \to \infty$.

Moreover, by the definition, we easily get

$$F_{\delta k}(2x) = 2^k F_{\delta k}(x)$$

and $\underset{y}{\overset{6}{\Delta}} F_{\delta k}(x) = 0$ for all $x, y \in V$. According to Lemma 2.4, F is the unique general quintic mapping. □

The stability results for the functional equation (1.1) proved by Y. H. Lee and S. M. Jung [7] and S. S. Jin and Y. H. Lee [3] only deal with the conditions (2.12) and (2.17) of Theorem 2.4. Compare the following concise theorem obtained from Theorem 2.4 with Theorem 1.1 obtained by Y. H. Lee [6].

Theorem 2.6 *Let θ be a positive real constant and p a real number such that $p \neq 1, 2, 3, 4, 5$. If $f : X \to Y$ satisfies the inequality*

$$\left\| \underset{y}{\overset{6}{\Delta}} f(x) \right\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X\backslash\{0\}$, then there exists a unique general quintic mapping F such that

$$\begin{aligned}
\|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{5040(2-2^p)} && \text{for} \quad p < 1,\\
\|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{5040(2^p-2)} + \frac{M\theta\|x\|^p}{2688(4-2^p)} && \text{for} \quad 1 < p < 2,\\
\|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{2688(2^p-4)} + \frac{M\theta\|x\|^p}{4608(8-2^p)} && \text{for} \quad 2 < p < 3,\\
\|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{4608(2^p-8)} + \frac{M\theta\|x\|^p}{21504(16-2^p)} && \text{for} \quad 3 < p < 4,\\
\|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{21504(2^p-16)} + \frac{M\theta\|x\|^p}{322560(32-2^p)} && \text{for} \quad 4 < p < 5,\\
\|\tilde{f}(x) - F(x)\| &\leq \frac{1024M\theta\|x\|^p}{315 \cdot 16^p(2^p-32)} && \text{for} \quad 5 < p
\end{aligned}$$

for all $x \in X\backslash\{0\}$, where

$$\begin{aligned}
M := &21 \cdot 24^p + 1680 \cdot 12^p + 57 \cdot 8^p + 8400 \cdot 6^p\\
&+ 1834 \cdot 4^p + 5376 \cdot 3^p + 19040 \cdot 2^p + 58624.
\end{aligned}$$

3 Conclusions

In this paper, we investigate the generalized stability of the general quintic functional equation (1.1). Precisely, if $f : V \to Y$ is a mapping such that $\left\| \Delta_y^6 f(x) \right\| \leq \varphi(x, y)$ for all $x, y \in V\backslash\{0\}$, where $\varphi : (V\backslash\{0\})^2 \to [0, \infty)$ holds the conditions (2.12), (2.13), (2.14), (2.15), (2.16), or (2.17), then there exists a unique general quintic mapping F such that the difference $\|\tilde{f}(x) - F(x)\|$ satisfies the conditions (2.19), (2.20), (2.21), (2.22), (2.23), or (2.24), respectively.

References

[1] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[2] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[3] S. S. Jin and Y.-H. Lee, *Stability of the general quintic functional equation*, Int. J. Math. Anal. (Ruse) 15 (2021), no. 6, 271–282.

[4] S. S. Jin and Y.-H. Lee, *Hyperstability of a general quintic functional equation and a general septic functional equation*, J. Chungcheong Math. Soc. 36 (2023), 107–123.

[5] S. M. Jung, Y. H. Lee and J. Roh, *A uniqueness theorem for stability problems of functional equations*, Symmetry **2024**, 16, 1298. https://doi.org/10.3390/sym16101298.

[6] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of a general quintic functional equation and a general sextic functional equation*, Mathematics 7(6) (2019), 510.

[7] Y.-H. Lee and S.-M. Jung, *A fixed point approach to the stability of a general quintic functional equation*, Nonlinear Funct. Anal. Appl. 28 (2023), 671–684.

[8] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[9] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.