Fundamental Properties of Generalized *n*-th Roots

of Real Numbers

Abstract

In this paper we generalize our recent results related to the question "For which $x \in \mathbf{R}$ and $n \in \mathbf{N}$, $n \geq 2$, $\sqrt[n]{x} \in \mathbf{Q}$ holds?". We now address the more general question "For which $x \in \mathbf{R}$ and $r \in \mathbf{Q}$, $x^r \in \mathbf{Q}$ holds?". We choose a step-wise approach to answer this generalized question starting with *x* representing an irrational number, followed by *x* representing a negative real number. Finally, we comprehensively answer the question for *x* representing a natural number, which also directly leads to solutions covering the practically relevant case of *x* representing a positive rational number. One important goal of our research has been to improve the understanding of the properties of irrational numbers as well as simplifying existing solutions for *n*-th root problems.

To demonstrate the potential usage of our results related to fundamental properties of *n*-th roots of real numbers, by way of example, we indicate how our results allow us to obtain elegant solutions for several rather challenging tasks related to irrational $\sqrt[n]{x}$.

Keywords: *Number theory, prime factorization, generalization of Euclid's proof, rationality of n-th roots, simplification of algorithms to calculate the value of a root.*

1. Introduction.

In number theory, e.g. important properties of numbers are of interest. As an example, people are interested in knowing whether an integer number $x \in \boldsymbol{Z}$ can be divided by a given integer $c \in \boldsymbol{Z}$ yielding a remainder of 0, i.e. *x* is an integer multiple of *c*. It is well-known that, if the last two digits of *x* and the sum of its digits are given, rather simple tests exist to determine whether *x* is an integer multiple of *c*, e.g. those tests are available for divisors *c*, $2 \leq c \leq 6$ or $c \in \{9, 10\}$.

A similar question of interest is: "Is the root of a natural number c, i.e. $\sqrt{c}$, still a rational number?". For example, already more than 2000 years ago, Euclid was able to prove that $\sqrt{2}$ is an irrational number [1, 2]. Euclid's results have been generalized over the centuries to some extent (cf., e.g. [3]).

As $\sqrt{2}$ can also be written in the form $2^{1/2}$ it is interesting to tackle the question of rationality in a much more general manner, i.e. the question we study in this paper is:

"For given $x \in \mathbf{R}$ and $r \in \mathbf{Q}$, is x^r still a rational number?". (Q)

Before answering this very general question, let us show that this question (Q) is equivalent to the significantly simpler question (Q*):

"For given $x \in$ **R** and $n \in$ **N**, is $\sqrt[n]{x}$ still a rational number?". (Q•)

This simplification can be proven by the following reasoning:

Consider two arbitrarily chosen values $y \in$ **R** and $r \in$ **Q**. As $r \in$ **Q**, $\exists\ p \in$ **Z** and $q \in$ **N** such that $r = p/q$. Thus, $y^r = y^{p/q} = \sqrt[q]{y^p}$. If we now set $n = q$ and $x = y^p$, we see that $y^r = \sqrt[n]{x}$ with $n \in$ **N** and $x \in$ **R**. This proves the simplification claimed. □

So, in this paper it will be sufficient to investigate the properties of $\sqrt[n]{x}$, $n \in$ **N**, $x \in$ **R**. Nevertheless, the general question (Q) will be covered completely.

The paper is structured as follows: In sections 2 to 5 we will investigate the fundamental properties of $\sqrt[n]{x}$, x being element of different number domains, namely $x \in$ **R \ Q** (in Section 2), $x \in$ **R**$^-$ (in Section 3), $x \in$ **N** (in Section 4) and $x \in$ **Q**$^+$ (in Section 5). Section 6 will demonstrate, by way of example, the potential usage of the results obtained by us regarding fundamental properties of *n*-th roots. Section 7 will conclude this paper by providing a short summary and outlook.

2. Properties of $\sqrt[n]{x}$ for $x \in$ R \ Q , $x \in$ N, $n \geq 2$

In order to make the results of this paper more easily understandable for the readers, we have decided to present the results regarding $\sqrt[n]{x}$ in the following sections in such a manner that different sections always cover values of *x* being element of different domains of numbers (e.g. **N**, **Q**, **R**). In combination the domains of numbers as covered by Sections 2 to 5 will represent a complete coverage of the total set **R** of real numbers. For a more in-depth treatment we refer the readers to [4, 5, 6].

Let us begin this section by assuming *x* being irrational. To simplify our notation in the following we denote by $\mathbf{N}_{\geq 2}$ the set $\mathbf{N}_{\geq 2} := \{ n \in \mathbf{N} \mid n \geq 2 \}$.

Theorem: If $x \in$ **R \ Q** then $\sqrt[n]{x}$ is irrational $\forall\ n \in \mathbf{N}_{\geq 2}$.

Proof: Let $x \in$ **R \ Q** and assume $\sqrt[n]{x}$ is rational, i.e., $\exists p \in$ **Z,** $q \in$ **N** such that $\sqrt[n]{x} = p/q$. Then $x = p^n / q^n$, which implies $x \in$ **Q** and this contradicts the assumption $x \in$ **R \ Q** . □

3. Properties of $\sqrt[n]{x}$ for $x \in$ R$^-$, $n \in \mathbf{N}_{\geq 2}$

Let us now cover $\sqrt[n]{x}$ for $x \in$ **R**$^-$**.** We want to investigate the question of whether there exists an $n \in \mathbf{N}_{\geq 2}$ for which $\sqrt[n]{x}$ is rational**.**

- **Case C1:** Let *n* be an even number.

Moreover, to simplify our argumentation, let us denote $\sqrt[n]{x}$ by z = z(*n*,x). Then, $z^n = x$ with $x<0$. Assume $z \in$ **Q.**

a) $z>0 \Rightarrow z^n > 0$ and therefore $z^n \neq x$, because $x<0$.
b) $z=0 \Rightarrow z^n = 0$ and therefore $z^n \neq x$.
c) $z<0 \Rightarrow z^n > 0$ (because *n* is even). Therefore $z^n \neq x$

To summarize: We have proven that, if *n* is even, then $\sqrt[n]{x} \notin$ **Q,** $\forall n \in \mathbf{N}_{\geq 2}$.

- **Case C2:** Let *n* be an odd number.

As x<0 we obtain: $\sqrt[n]{x} = \sqrt[n]{-|x|}$.

a) Some mathematicians consider $\sqrt[n]{-a}$ to be undefined for a>0. This implies that $\sqrt[n]{x} \notin$ **Q,** also for all *n*, if *n* is odd.

b) For other mathematicians $\sqrt[n]{-a}$ is equal to $-\sqrt[n]{a}$ for a>0, e.g. $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$. Thus, this alternative view will imply that $\sqrt[n]{x} \in$ **Q** $\Leftrightarrow \sqrt[n]{-x} \in$ **Q** (if *n* is odd and *x*<0). Note that Section 5 covers the analysis of $\sqrt[n]{y}$ **,** *y*∈**Q**$^+$, which becomes relevant in Case C2, part b) and which is the interpretation of $\sqrt[n]{-a}$, a>0 preferred by us and assumed in the rest of this paper.

4. Properties of $\sqrt[n]{x}$ for *x* ∈ **N,** *n* ∈ $\mathbf{N}_{\geq 2}$

We now consider the case that the set of numbers to which *x* in $\sqrt[n]{x}$ belongs is **Q**$^+$. All other domains of numbers for *x* have been covered already in Sections 2 and 3. As the case *x* ∈ **Q**$^+$ **\ N** can be simply treated by using the results of case *x* ∈ **N** we will start in Section 4 by assuming *x* ∈ **N**, i.e. here we look at $\sqrt[n]{x}$ for *x* ∈ **N,** *n* ∈ $\mathbf{N}_{\geq 2}$.

With these assumptions for *x* and *n*, fundamental properties of $\sqrt[n]{x}$ can be determined in a quite straightforward manner by making use of the prime factorization of *x* which always exists and is unique [7, 8]. Regarding the state-of-the-art of currently existing prime factorization algorithms the reader is referred to [9]. It is well-known that prime factorization of a very large natural number can become a very difficult task which may be even practically infeasible using currently available (super) computers. Computer scientists have made use of this fact in the development of cryptographic algorithms such as RSA algorithm (see [10]) by taking into account that prime factorization of extremely large numbers is practically infeasible. However, we have shown in [11] that even without availability of the prime factorization of a given *x* ∈ $\mathbf{N}_{\geq 2}$ it may still be very easy to decide whether $\sqrt[n]{x} \notin$ **Q** for a given *n* ∈ $\mathbf{N}_{\geq 2}$. A simple example for this fact is that, even without knowing the exact prime factorization of *x*, we may know at least that the prime factorization of *x* contains the factor p^1 (*p* being a prime number) but not a factor p^i, with i ≥ 2, the consequence of which is: $\sqrt[n]{x} \notin$ **Q**.

So, let the prime factorization of *x* be as follows:

$$x = p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet\ p_m^{k\_m} \qquad (1)$$

where p_i denote prime numbers, $p_i \neq p_j$, ∀i ≠ j and k_i (to be read as k_i): $k_i \in$ **N,** m≥1.

Consider ν∈{1, 2, …, m} and, for a given *x* ∈ $\mathbf{N}_{\geq 2}$, let denote $D_\nu(x)$:= {*μ* ∈ $\mathbf{N}_{\geq 2}$ | ∃ *α* ∈ **N** with $\alpha \bullet \mu = k_\nu$}.

In [4] it has been proven that

$$\sqrt[n]{x} \in \mathbf{Q},\ n \in \mathbf{N}_{\geq 2},\ x \in \mathbf{N} \ \Leftrightarrow\ n \in D_1(x) \cap D_2(x) \cap \ldots \cap D_m(x) \qquad (2)$$

Moreover, if $\sqrt[n]{x} \in$ **Q** then $\sqrt[n]{x} = p_1^{k\_1/n} \bullet p_2^{k\_2/n} \bullet\bullet\bullet\ p_m^{k\_m/n}$ (3)

An important consequence of (2) is that:

$\sqrt[n]{x} \notin \mathbf{Q}, \forall\, n \in \mathbf{N}_{\geq 2}$, if $\exists\, \nu \in \{1, 2, \ldots, m\}$ such that $D_\nu(x) = \varnothing$. which, e.g. is fulfilled if $k_\nu = 1$ for at least one value of ν.

5. Properties of $\sqrt[n]{x}$ for $x \in \mathbf{Q}^+$, $n \in \mathbf{N}_{\geq 2}$

Though Section 4 is already covering a subset of $x \in \mathbf{Q}^+$, the task to treat ratios which do not represent natural numbers still remains.

As now $x \in \mathbf{Q}^+$ is assumed, x can be represented as

$x = p/q$ with p, $q \in \mathbf{N}$, p and q being coprime.

Let denote D_p and D_q by

$$D_p := \{ n \in \mathbf{N} \mid \sqrt[n]{p} \in \mathbf{Q} \} \quad \text{and} \quad D_q := \{ n \in \mathbf{N} \mid \sqrt[n]{q} \in \mathbf{Q} \}$$

and for $x = p/q$ let D_x denote $D_x := D_p \cap D_q$

$$\text{Then, for } \sqrt[n]{x} = \sqrt[n]{\frac{p}{q}}, \text{ we obtain: } \sqrt[n]{\frac{p}{q}} \in \mathbf{Q} \iff n \in D_x. \qquad (4)$$

And if $\exists\, n \in \mathbf{N}_{\geq 2}$ such that $\sqrt[n]{x} \in \mathbf{Q}$ for $x \in \mathbf{Q}$, then the value of $\sqrt[n]{x}$ can be easily determined as:

$$\sqrt[n]{x} = (p_1^{k_1/n} \bullet p_2^{k_2/n} \bullet\bullet\bullet\; p_s^{k_s/n}) \,/\, (q_1^{k_1/n} \bullet q_2^{k_2/n} \bullet\bullet\bullet\; q_t^{k_t/n}) \qquad (5)$$

if the prime factorizations of p and q are:

$$p = p_1^{k_1} \bullet p_2^{k_2} \bullet\bullet\bullet\; p_s^{k_s} \quad \textit{and} \quad q = q_1^{k_1} \bullet q_2^{k_2} \bullet\bullet\bullet\; q_t^{k_t}$$

Example: For n=2 and x=4/9 we obtain $\sqrt{4/9} = 2^{2/2} / 3^{2/2} = 2/3$ by just using the prime factorization of 4 and 9.

For a more detailed proof of these facts, cf. [4].

6. Simple Methods to Create Irrational Numbers all *n*-th Roots of which are Irrational

In this contribution we have shown, e.g. that a sufficient condition for the n-th root of a natural number to be irrational is: the prime factorization of x (cf. eq. (1)) comprises at least one prime number p_i, $1 \leq i \leq m$ which appears with an exponent k_i=1. A direct consequence of this fact is that for all prime numbers p: " $\sqrt[n]{p} \notin \mathbf{Q}, \forall\, n \in \mathbf{N}_{\geq 2}$ " holds.

We consider two further examples in this section to demonstrate how the insights we gained with respect to the irrationality of n-th roots of natural numbers can be used to solve problems (being non-trivial a priori) in a rather simple way:

Task T1: Find an extremely large natural number x for which " $\sqrt[n]{x} \notin \mathbf{Q}, \forall\, n \in \mathbf{N}_{\geq 2}$ " holds.

Solution: The task can be solved directly by choosing $x = p_1 \bullet p_2^c$ with p_1 and p_2 as arbitrary prime numbers, $p_1 \neq p_2$ and $c \in \mathbf{N}$, c sufficiently large.

Task T2: By means of appending a single digit x_0 to a given (arbitrary) natural number x, a new natural number $y=10 \bullet x + x_0$ should be constructed for which " $\sqrt[n]{y} \notin \mathbf{Q}, \forall n \in \mathbf{N}_{\geq 2}$ " holds.

Solution: It is sufficient to make sure that the newly constructed number y satisfies, e.g. one of the following Conditions C_1 or C_2:

C_1: y is an integer multiple of 2 (i.e. y is an even number) but y is not an integer multiple of 4.

C_2: y is an integer multiple of 5 (i.e. the last digit of y is 0 or 5) but y is not an integer multiple of 25.

It is evident that condition C_1 implies that the prime factorization of y is:

$$y = 2 \bullet p_2^{k\_2} \bullet\bullet\bullet \; p_m^{k\_m}$$

where p_i denote prime numbers, $p_i \neq 2, \forall i \in \{2, \ldots, m\}$,

and condition C_2 implies that the prime factorization of y is:

$$y = 5 \bullet p_2^{k\_2} \bullet\bullet\bullet \; p_m^{k\_m}$$

where p_i denote prime numbers, $p_i \neq 5, \forall i \in \{2, \ldots, m\}$.

So, in both cases, evidently $\sqrt[n]{y} \notin \mathbf{Q}, \forall n \in \mathbf{N}_{\geq 2}$ holds.

To determine the digit x_0 to be appended to x it is sufficient to consider just the last digit of x.

(a) Satisfying condition C_1:
Let us denote the last digit of x by x_1. Then, x_0 is only dependent on x_1 and we can set x_0 as follows:
- *if* $x_1 \in \{0, 2, 4, 6, 8\} \Rightarrow x_0 \in \{2, 6\}$, i.e. we can choose either $x_0 = 2$ or $x_0=6$.
- *if* $x_1 \in \{1, 3, 5, 7, 9\} \Rightarrow x_0 \in \{0, 4, 8\}$.

(b) Satisfying condition C_2:
Dependent on the value of x_1 we can set x_0 as follows:
- *if* $x_1 \in \{1, 3, 4, 6, 8, 9\} \Rightarrow x_0 \in \{0, 5\}$,
- *if* $x_1 \in \{2, 7\} \Rightarrow x_0 = 0$,
- *if* $x_1 \in \{0, 5\} \Rightarrow x_0 = 5$.

To summarize, we see that the choice of the value of x_0 to be appended to x is a really simple task if fundamental properties of n-th roots are understood to a sufficient extent. Then, choice of x_0 is just a matter of seconds.

7. Summary and Conclusions

In this contribution we have demonstrated that fundamental properties of n-th roots of rational numbers can be discovered in a rather straightforward manner if we take into account the prime factorization of integer numbers. Moreover, irrationality of n-th roots of irrational numbers can be proven in an astonishingly simple manner. From a general point of view, we have completely solved the quite challenging problem "$x^r \in Q$ *or* $x^r \notin Q$?" for arbitrary $x \in \mathbf{R}$ and $r \in \mathbf{Q}$.

In Section 6, preliminary examples have illustrated how our results can be applied to answer rather challenging questions.

Our hope is that our innovative results will lead to a much better understanding of fundamental properties of the *n*-th roots of real numbers and thus shed light on the investigation of fundamental aspects of numbers, being one of the important goals of the Number Theory community.

Declarations

- **Ethical approval**
  not applicable
- **Availability of data and materials**
  not applicable
- **Disclaimer (Artificial intelligence)**
  The author hereby declares that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

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