<u>Original Research Article</u>

Fundamental Properties of Generalized *n*-th Roots of Real Numbers

Abstract

In this paper we generalize our recent results related to the question "For which $x \in \mathbf{R}$ and $n \in \mathbf{N}$, $n \geq 2$, $\sqrt[n]{x} \in \mathbf{Q}$ holds?". We now tackle the more general question "For which $x \in \mathbf{R}$ and $r \in \mathbf{Q}$, $x^r \in \mathbf{Q}$ holds?". We choose a step-wise approach to answer this generalized question starting with x representing an irrational number, followed by x representing a negative real number. Finally, we comprehensively answer the question for x representing a natural number, which also directly leads to solutions covering the practically relevant case of x representing a positive rational number. To demonstrate the potential usage of our results related to fundamental properties of *n*-th roots of real numbers, by way of example, we indicate how our results allow us to obtain astonishingly simple solutions for several rather challenging tasks related to irrational $\sqrt[n]{x}$.

Keywords: *Number theory, prime factorization, generalization of Euclid's proof, $\sqrt[n]{x}$ rational or irrational number? (for $n \in \mathbf{N}$, $n \geq 2$, $x \in \mathbf{R}$).*

1. Introduction.

In number theory, e.g. important properties of numbers are of interest. As an example, people are interested in knowing whether an integer number $x \in \mathbf{Z}$ can be divided by a given integer $c \in \mathbf{Z}$ yielding a remainder of 0, i.e. x is an integer multiple of c. It is well-known that, if the last two digits of x and the sum of its digits are given, rather simple tests exist to determine whether x is an integer multiple of c, e.g. those tests are available for divisors c, $2 \leq c \leq 6$ or $c \in \{9, 10\}$.

A similar question of interest is: "Is the root of a natural number c, i.e. $\sqrt{c}$, still a rational number?". For example, already more than 2000 years ago, Euclid was able to prove that $\sqrt{2}$ is an irrational number [1, 2]. Euclid's results have been generalized over the centuries to some extent (cf., e.g. [3]).

As $\sqrt{2}$ can also be written in the form $2^{1/2}$ it is interesting to tackle the question of rationality in a much more general manner, i.e. the question we study in this paper is:

"For given $x \in \mathbf{R}$ and $r \in \mathbf{Q}$, is x^r still a rational number?". (Q)

Before answering this very general question, let us show that this question (Q) is equivalent to the significantly simpler question (Q*):

"For given $x \in \mathbf{R}$ and $n \in \mathbf{N}$, is $\sqrt[n]{x}$ still a rational number?". $(Q^{\bullet})$

This simplification can be proven by the following reasoning:

Consider two arbitrarily chosen values $y \in \mathbf{R}$ and $r \in \mathbf{Q}$. As $r \in \mathbf{Q}$, $\exists\ p \in \mathbf{Z}$ and $q \in \mathbf{N}$ such that $r = p/q$. Thus, $y^r = y^{p/q} = \sqrt[q]{y^p}$. If we now set $n = q$ and $x = y^p$, we see that $y^r = \sqrt[n]{x}$ with $n \in \mathbf{N}$ and $x \in \mathbf{R}$. This proves the simplification claimed. $\square$

So, in this paper it will be sufficient to investigate the properties of $\sqrt[n]{x}$, $n \in \mathbf{N}$, $x \in \mathbf{R}$. Nevertheless, the general question (Q) will be covered completely.

The paper is structured as follows: In sections 2 to 5 we will investigate the fundamental properties of $\sqrt[n]{x}$, x being element of different number domains, namely $x \in \mathbf{R} \setminus \mathbf{Q}$ (in Section 2), $x \in \mathbf{R}^-$ (in Section 3), $x \in \mathbf{N}$ (in Section 4) and $x \in \mathbf{Q}^+$ (in Section 5). Section 6 will demonstrate, by way of example, the potential usage of the results obtained by us regarding fundamental properties of *n*-th roots. Section 7 will conclude this paper by providing a short summary and outlook.

2. Properties of $\sqrt[n]{x}$ for $x \in \mathbf{R} \setminus \mathbf{Q}$, $x \in \mathbf{N}$, $n \geq 2$

In order to make the results of this paper more easily understandable for the readers, we have decided to present the results regarding $\sqrt[n]{x}$ in the following sections in such a manner that different sections always cover values of x being element of different domains of numbers (e.g. **N**, **Q**, **R**). In combination the domains of numbers as covered by Sections 2 to 5 will represent a complete coverage of the total set **R** of real numbers. For a more in-depth treatment we refer the readers to [4, 5, 6].

Let us begin this section by assuming x being irrational. To simplify our notation in the following we denote by $\mathbf{N}_{\geq 2}$ the set $\mathbf{N}_{\geq 2} := \{\, n \in \mathbf{N} \mid n \geq 2 \,\}$.

Theorem: If $x \in \mathbf{R} \setminus \mathbf{Q}$ then $\sqrt[n]{x}$ is irrational $\forall\ n \in \mathbf{N}_{\geq 2}$.

Proof: Let $x \in \mathbf{R} \setminus \mathbf{Q}$ and assume $\sqrt[n]{x}$ is rational, i.e., $\exists p \in \mathbf{Z}$, $q \in \mathbf{N}$ such that $\sqrt[n]{x} = p/q$. Then $x = p^n / q^n$, which implies $x \in \mathbf{Q}$ and this contradicts the assumption $x \in \mathbf{R} \setminus \mathbf{Q}$. $\square$

3. Properties of $\sqrt[n]{x}$ for $x \in \mathbf{R}^-$, $n \in \mathbf{N}_{\geq 2}$

Let us now cover $\sqrt[n]{x}$ for $x \in \mathbf{R}^-$**.** We want to investigate the question of whether there exists an $n \in \mathbf{N}_{\geq 2}$ for which $\sqrt[n]{x}$ is rational**.**

- **Case C1:** Let *n* be an even number.

Let z denote $z := \sqrt[n]{x}$. Then, $z^n = x$. Assume $z \in \mathbf{Q}$**.**

a) $z>0 \Rightarrow z^n > 0$ and therefore $z^n \neq x$, because $x<0$.
b) $z=0 \Rightarrow z^n = 0$ and therefore $z^n \neq x$.
c) $z<0 \Rightarrow z^n > 0$ (because n is even). Therefore $z^n \neq x$

To summarize: We have proven that, if n is even, then $\sqrt[n]{x} \notin \mathbf{Q}$, $\forall n \in \mathbf{N}_{\geq 2}$.

- **Case C2:** Let n be an odd number.

As x<0 we obtain: $\sqrt[n]{x}$ = . $\sqrt[n]{-|x|}$.

a) Some mathematicians consider $\sqrt[n]{-a}$ to be undefined for a>0. This implies that $\sqrt[n]{x} \notin$ **Q,** also for all n, if n is odd.
b) For other mathematicians $\sqrt[n]{-a}$ is equal to $-\sqrt[n]{a}$ for a>0, e.g. $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$. Thus, this alternative view will imply that $\sqrt[n]{x} \in \mathbf{Q} \Leftrightarrow \sqrt[n]{-x} \in \mathbf{Q}$ (if n is odd and $x<0$). Note that Section 5 covers the analysis of $\sqrt[n]{y}$ **,** $y \in \mathbf{Q}^+$, which becomes relevant in Case C2, part b).

4. Properties of $\sqrt[n]{x}$ for $x \in \mathbf{N}$, $n \in \mathbf{N}_{\geq 2}$

We now consider the case that the set of numbers to which x in $\sqrt[n]{x}$ belongs is $\mathbf{Q}^+$. All other domains of numbers for x have been covered already in Sections 2 and 3. As the case $x \in \mathbf{Q}^+ \setminus \mathbf{N}$ can be simply treated by using the results of case $x \in \mathbf{N}$ we will start in Section 4 by assuming $x \in \mathbf{N}$, i.e. here we look at $\sqrt[n]{x}$ for $x \in \mathbf{N}$, $n \in \mathbf{N}_{\geq 2}$.

With these assumptions for x and n, fundamental properties of $\sqrt[n]{x}$ can be determined in a quite straightforward manner by making use of the prime factorization of x which always exists and is unique [7, 8]..

So, let the prime factorization of x be as follows:

$$x = p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet\ p_m^{k\_m} \tag{1}$$

where p_i denote prime numbers, $p_i \neq p_j$, $\forall i \neq j$ and k_i (to be read as k_i): $k_i \in \mathbf{N}$, $m \geq 1$.

Consider $\nu \in \{1, 2, \ldots, m\}$ and, for a given $x \in \mathbf{N}_{\geq 2}$, let denote $D_\nu(x) := \{\mu \in \mathbf{N}_{\geq 2} \mid \exists\ \alpha \in \mathbf{N}$ with $\alpha \bullet \mu = k_\nu\}$.

In [4] it has been proven that

$$\sqrt[n]{x} \in \mathbf{Q},\ n \in \mathbf{N}_{\geq 2},\ x \in \mathbf{N} \ \Leftrightarrow\ n \in D_1(x) \cap D_2(x) \cap \ldots \cap D_m(x) \tag{2}$$

Moreover, if $\sqrt[n]{x} \in \mathbf{Q}$ then $\sqrt[n]{x} = p_1^{k\_1/n} \bullet p_2^{k\_2/n} \bullet\bullet\bullet\ p_m^{k\_m/n}$ (3)

An important consequence of (2) is that:

$\sqrt[n]{x} \notin \mathbf{Q}, \forall\ n \in \mathbf{N}_{\geq 2}$, if $\exists\ \nu \in \{1, 2, \ldots, m\}$ such that $D_\nu(x) = \emptyset$. which, e.g. is fulfilled if $k_\nu = 1$ for at least one value of ν.

5. Properties of $\sqrt[n]{x}$ for $x \in \mathbf{Q}^+$, $n \in \mathbf{N}_{\geq 2}$

Though Section 4 is already covering a subset of $x \in \mathbf{Q}^+$, the task to treat ratios which do not represent natural numbers still remains.

As now $x \in \mathbf{Q}^+$ is assumed, x can be represented as

$x = p/q$ with $p, q \in \mathbf{N}$, p and q being coprime.

Let denote D_p and D_q by

$$D_p := \{ n \in \mathbf{N} \mid \sqrt[n]{p} \in \mathbf{Q} \} \quad \text{and} \quad D_q := \{ n \in \mathbf{N} \mid \sqrt[n]{q} \in \mathbf{Q} \}$$

and for $x = p/q$ let D_x denote $D_x := D_p \cap D_q$

$$\text{Then, for } \sqrt[n]{x} = \sqrt[n]{\frac{p}{q}}\text{, we obtain: } \sqrt[n]{\frac{p}{q}} \in \mathbf{Q} \iff n \in D_x. \qquad (4)$$

And if $\exists\, n \in \mathbf{N}_{\geq 2}$ such that $\sqrt[n]{x} \in \mathbf{Q}$ for $x \in \mathbf{Q}$, then the value of $\sqrt[n]{x}$ can be easily determined as:

$$\sqrt[n]{x} = (p_1^{k\_1/n} \bullet p_2^{k\_2/n} \bullet\bullet\bullet\ p_s^{k\_s/n}) \,/\, (q_1^{k\_1/n} \bullet q_2^{k\_2/n} \bullet\bullet\bullet\ q_t^{k\_t/n}) \qquad (5)$$

if the prime factorizations of p and q are:

$$p = p_1^{k\_1} \bullet p_2^{k\_2} \bullet\bullet\bullet\ p_s^{k\_s} \quad \textit{and} \quad q = q_1^{k\_1} \bullet q_2^{k\_2} \bullet\bullet\bullet\ q_t^{k\_t}$$

Example: For $n=2$ and $x=4/9$ we obtain $\sqrt{4/9} = 2^{2/2} / 3^{2/2} = 2/3$ by just using the prime factorization of 4 and 9.

For a more detailed proof of these facts, cf. [4].

6. Simple Methods to Create Irrational Numbers all *n*-th Roots of which are Irrational

In this contribution we have shown, e.g. that a sufficient condition for the n-th root of a natural number to be irrational is: the prime factorization of x (cf. eq. (1)) comprises at least one prime number p_i, $1 \leq i \leq m$ which appears with an exponent $k_i=1$. A direct consequence of this fact is that for all prime numbers p: " $\sqrt[n]{p} \notin \mathbf{Q}, \forall\, n \in \mathbf{N}_{\geq 2}$ " holds.

We consider two further examples in this section to demonstrate how the insights we gained with respect to the irrationality of n-th roots of natural numbers can be used to solve problems (being non-trivial a priori) in an astonishingly simple way:

Task T1: Find an extremely large natural number x for which " $\sqrt[n]{x} \notin \mathbf{Q}, \forall\, n \in \mathbf{N}_{\geq 2}$ " holds.

Solution: The task can be solved directly by choosing $x = p_1 \bullet p_2^c$ with p_1 and p_2 as arbitrary prime numbers, $p_1 \neq p_2$ and $c \in \mathbf{N}$, c sufficiently large.

Task T2: By means of appending a single digit x_0 to a given (arbitrary) natural number x, a new natural number $y = 10 \bullet x + x_0$ should be constructed for which " $\sqrt[n]{y} \notin \mathbf{Q}, \forall\, n \in \mathbf{N}_{\geq 2}$ " holds.

Solution: It is sufficient to make sure that the newly constructed number y satisfies, e.g. one of the following Conditions C_1 or C_2:

C_1: y is an integer multiple of 2 (i.e. y is an even number) but y is not an integer multiple of 4.

C_2: y is an integer multiple of 5 (i.e. the last digit of y is 0 or 5) but y is not an integer multiple of 25.

It is evident that condition C_1 implies that the prime factorization of y is:

$$y = 2 \bullet p_2^{k\_2} \bullet\bullet\bullet\ p_m^{k\_m}$$

where p_i denote prime numbers, $p_i \neq 2, \forall i \in \{2, \ldots, m\}$,

and condition C_2 implies that the prime factorization of y is:

$$y = 5 \bullet p_2^{k\_2} \bullet\bullet\bullet\ p_m^{k\_m}$$

where p_i denote prime numbers, $p_i \neq 5, \forall i \in \{2, \ldots, m\}$.

So, in both cases, evidently $\sqrt[n]{y} \notin \mathbf{Q}, \forall\ n \in \mathbf{N}_{\geq 2}$ holds.

To determine the digit x_0 to be appended to x it is sufficient to consider just the last digit of x.

(a) Satisfying condition C_1:
Let us denote the last digit of x by x_1. Then, x_0 is only dependent on x_1 and we can set x_0 as follows:
  - *if* $x_1 \in \{0, 2, 4, 6, 8\} \Rightarrow x_0 \in \{2, 6\}$, i.e. we can choose either $x_0 = 2$ or $x_0 = 6$.
  - *if* $x_1 \in \{1, 3, 5, 7, 9\} \Rightarrow x_0 \in \{0, 4, 8\}$.

(b) Satisfying condition C_2:
Dependent on the value of x_1 we can set x_0 as follows:
  - *if* $x_1 \in \{1, 3, 4, 6, 8, 9\} \Rightarrow x_0 \in \{0, 5\}$,
  - *if* $x_1 \in \{2, 7\} \Rightarrow x_0 = 0$,
  - *if* $x_1 \in \{0, 5\} \Rightarrow x_0 = 5$.

To summarize, we see that the choice of the value of x_0 to be appended to x is a really simple task if fundamental properties of n-th roots are understood to a sufficient extent. Then, choice of x_0 is just a matter of seconds.

7. Summary and Conclusions

In this contribution we have demonstrated that fundamental properties of n-th roots of rational numbers can be discovered in a rather straightforward manner if we take into account the prime factorization of integer numbers. Moreover, irrationality of n-th roots of irrational numbers can be proven in an astonishingly simple manner. From a general point of view, we have completely solved the quite challenging problem “$x^r \in Q$ *or* $x^r \notin Q$?“ for arbitrary $x \in \mathbf{R}$ and $r \in \mathbf{Q}$.

In Section 6, preliminary examples have illustrated how our results can be applied to answer rather challenging questions.

Our hope is that our innovative results will lead to a much better understanding of fundamental properties of the *n*-th roots of real numbers.

Declarations

- **Ethical approval**
  not applicable
- **Availability of data and materials**
  not applicable

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