Original Research Article

Generalized stability of a general quartic functional equation

Abstract. The general quartic functional equation is a generalization of many functional equations such as a Jensen functional equation and a general quadratic functional equation. In this paper, we investigate the generalized stability of the general quartic functional equation

$$\sum_{i=0}^{5} \binom{5}{i} (-1)^{5-i} f(x+iy) = 0.$$

Key Words: stability of a functional equation; general quartic functional equation; general quartic mapping

AMS subject classification numbers: Primary 39B82; 39B52

1 Introduction

In this paper, let V, X, and Y be a real vector space, a real normed space, and a real Banach space, respectively. In response to the question of the stability of group isomorphism raised by Ulam [7] in 1940, the result of the stability of additive functional equation obtained by Hyers [2] became the starting point for the stability of functional equations (see [1, 6] for more generalized results), and many mathematicians subsequently studied the stability of various functional equations.

Consider the general quartic functional equation

$$\sum_{i=0}^{5} \binom{5}{i} (-1)^{5-i} f(x+iy) = 0 \tag{1.1}$$

for all $x, y \in V$. We call the solution mapping of the functional equation (1.1) a general quartic mapping. Y. H. Lee has proved the following theorems for the Hyers-Ulam-Rassas stability and the hyperstability of the general quartic functional equation, respectively.

Theorem 1.1 (Theorem 2.2 in [4]) *Let $p \neq 1, 2, 3, 4$ be a real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\left\| \sum_{i=0}^{5} \binom{5}{i} (-1)^{5-i} f(x+iy) \right\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.2}$$

for all $x, y \in X \setminus \{0\}$. *Then there exists a unique general quartic mapping* F *such that*

$$\|f(x) - f(0) - F(x)\| \leq \left(\frac{7 + 5 \cdot 2^p}{|4 - 2^p||16 - 2^p|} + \frac{7 + 5 \cdot 2^p}{|8 - 2^p||4 - 2^p|}\theta \|x\|^p\right)$$

for all $x \in X \setminus \{0\}$ *and* $F(0) = 0$.

Theorem 1.2 (Theorem 2.3 in [4]) *Let* $p < 0$ *be a real number. Suppose that* $f : X \to Y$ *is a mapping satisfying the inequality* (1.2) *for all* $x, y \in X \setminus \{0\}$. *Then* f *satisfies the functional equation* (1.1).

Y. H. Lee and S. M. Jung [5] obtained partial results of the generalized stability of the functional equation (1.1) using the fixed point method.

In this paper, by proving the generalized stability of the general quadratic function equation, we will give improved results that generalize the existing results shown by Y. H. Lee and S. M. Jung.

2 Stability of a general quartic functional equation

Throughout this paper, for a given mapping $f : V \to Y$, we use the following abbreviations:

$$\begin{aligned}
\tilde{f}(x) :=& f(x) - f(0),\\
f_1(x) :=& -\frac{1}{168}(f(8x) - 28f(4x) + 224f(2x) - 512f(x)),\\
f_2(x) :=& \frac{1}{96}(f(8x) - 26f(4x) + 176f(2x) - 256f(x)),\\
f_3(x) :=& -\frac{1}{192}(f(8x) - 22f(4x) + 104(2x) - 128f(x)),\\
f_4(x) :=& \frac{1}{1344}(f(8x) - 14f(4x) + 56f(2x) - 64f(x)),\\
\overset{5}{\underset{y}{\Delta}} f(x) :=& \sum_{i=0}^{5} \binom{5}{i} (-1)^{5-i} f(x + iy),\\
\Gamma f(x) :=& f(16x) - 30f(8x) + 280f(4x) - 960f(2x) + 1024f(x)
\end{aligned}$$

for all $x, y \in V$. By laborious computation we can get some useful equalities in the following lemma.

Lemma 2.1 *For a given mapping* $f : V \to Y$, *the equalities*

$$\begin{aligned}
&\overset{5}{\underset{y}{\Delta}} \tilde{f}(x) = \overset{5}{\underset{y}{\Delta}} f(x),\\
&\Gamma \tilde{f}(x) = \overset{5}{\underset{2x}{\Delta}} f(6x) + 5 \overset{5}{\underset{2x}{\Delta}} f(4x) + 15 \overset{5}{\underset{2x}{\Delta}} f(2x) - 35 \overset{5}{\underset{-2x}{\Delta}} f(10x)\\
&\qquad + 40 \overset{5}{\underset{x}{\Delta}} f(3x) + 200 \overset{5}{\underset{x}{\Delta}} f(2x) + 376 \overset{5}{\underset{x}{\Delta}} f(x) - 280 \overset{5}{\underset{-x}{\Delta}} f(5x), \qquad (2.1)\\
&\tilde{f}_1(x) - \frac{\tilde{f}_1(2x)}{2} = \frac{\Gamma \tilde{f}(x)}{336}, \qquad \tilde{f}_1(x) - 2\tilde{f}_1\left(\frac{x}{2}\right) = -\frac{1}{168}\Gamma \tilde{f}\left(\frac{x}{2}\right), \qquad (2.2)\\
&\tilde{f}_2(x) - \frac{\tilde{f}_2(2x)}{4} = -\frac{\Gamma \tilde{f}(x)}{384}, \qquad \tilde{f}_2(x) - 4\tilde{f}\left(\frac{x}{2}\right) = \frac{1}{96}\Gamma \tilde{f}\left(\frac{x}{2}\right), \qquad (2.3)\\
&\tilde{f}_3(x) - \frac{\tilde{f}_3(2x)}{8} = \frac{\Gamma \tilde{f}(x)}{1536}, \qquad \tilde{f}_3(x) - 8\tilde{f}_3\left(\frac{x}{2}\right) = -\frac{1}{192}\Gamma \tilde{f}\left(\frac{x}{2}\right), \qquad (2.4)
\end{aligned}$$

$$\tilde{f}_4(x) - \frac{\tilde{f}_4(2x)}{16} = -\frac{\Gamma\tilde{f}(x)}{21504}, \qquad \tilde{f}_4(x) - 16\tilde{f}_4\left(\frac{x}{2}\right) = \frac{1}{1344}\Gamma\tilde{f}\left(\frac{x}{2}\right), \tag{2.5}$$

$$f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x) \tag{2.6}$$

hold for all $x, y \in V$.

Lemma 2.2 *Let* $f : V \to Y$ *satisfy the functional equation*

$$\mathop{\Delta}_{y}^{5} f(x) = 0$$

for all $x, y \in V$, *then we have*

$$\tilde{f}_k(2x) = 2^k \tilde{f}_k(x) \tag{2.7}$$

for all $x \in V$ *and each* $k \in \{1, 2, 3, 4\}$.

Proof. It is clear that $\Gamma\tilde{f}(x) = 0$ by (2.1). Therefore, the equality (2.7) follows from the equalities (2.2), (2.3), (2.4), and (2.5). $\square$

According to Corollary 6 in [3], we obtain following Lemma.

Lemma 2.3 [3] *For a given mapping* $f : V \to Y$, *if there exist a mapping* $F : V \to Y$ *and a function* $\phi : V \setminus \{0\} \to [0, \infty)$ *that satisfy*

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{1}{2^i}\phi(2^i x) < \infty \quad or$$

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{1}{2^{(\ell+1)i}}\phi(2^i x) + \sum_{i=0}^{\infty} 2^{\ell i}\phi\left(\frac{1}{2^i}x\right) < \infty \quad or$$

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} 2^{4i}\phi\left(\frac{1}{2^i}x\right) < \infty$$

for all $x \in V \setminus \{0\}$ *and for some* $\ell \in \{1, 2, 3\}$, *where* $F(x) = \sum_{k=1}^{4} F_k(x)$ *and every* F_k *has the property* (2.7), *i.e.,* $F_k(2x) = 2^k F_k(x)$ *for all* $x \in V$ *and* $k \in \{1, 2, 3, 4\}$, *then the mapping* F *is uniquely determined.*

Lemma 2.4 *If a mapping* $f : V \to Y$ *satisfies the functional equation* $\mathop{\Delta}_{y}^{5} f(x) = 0$ *for all* $x, y \in V \backslash \{0\}$, *then it is a general quartic mapping.*

Proof. It is clear that $\mathop{\Delta}_{0}^{5} f(x) = 0$ for all $x \in V$ and

$$\mathop{\Delta}_{y}^{5} f(0) = -\mathop{\Delta}_{-y}^{5} f(5y) = 0$$

for all $y \in V \backslash \{0\}$. So $\mathop{\Delta}_{y}^{5} f(x) = 0$ for all $x, y \in V$ as desired. $\square$

Now we show the generalized stability result of (1.1).

Theorem 2.5 *Let $\varphi : (V\backslash\{0\})^2 \to [0,\infty)$ be a function satisfying one of the following conditions*

$$\sum_{i=0}^{\infty} 2^{-i}\varphi(2^i x, 2^i y) < \infty, \tag{2.8}$$

$$\sum_{i=0}^{\infty} 4^{-i}\varphi(2^i x, 2^i y) < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} 2^{i}\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.9}$$

$$\sum_{i=0}^{\infty} 8^{-i}\varphi(2^i x, 2^i y) < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} 4^{i}\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.10}$$

$$\sum_{i=0}^{\infty} 16^{-i}\varphi(2^i x, 2^i y) < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} 8^{i}\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, \tag{2.11}$$

$$\sum_{i=0}^{\infty} 16^{i}\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{2.12}$$

for all $x, y \in V\backslash\{0\}$. Suppose that $f : V \to Y$ is a mapping such that

$$\left\| \underset{y}{\overset{5}{\Delta}} f(x) \right\| \le \varphi(x, y) \tag{2.13}$$

for all $x, y \in V\backslash\{0\}$. Then there exists a unique general quartic mapping F such that

$$\|\tilde{f}(x) - F(x)\| \le \frac{1}{336}\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{2^i}, \tag{2.14}$$

$$\|\tilde{f}(x) - F(x)\| \le \frac{1}{168}\sum_{i=0}^{\infty} 2^i\Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{384}\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{4^i}, \tag{2.15}$$

$$\|\tilde{f}(x) - F(x)\| \le \frac{1}{96}\sum_{i=0}^{\infty} 4^i\Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{1536}\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{8^i}, \tag{2.16}$$

$$\|\tilde{f}(x) - F(x)\| \le \frac{1}{192}\sum_{i=0}^{\infty} 8^i\Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{21504}\sum_{i=0}^{\infty} \frac{\Phi(2^i x)}{16^i}, \tag{2.17}$$

$$\|\tilde{f}(x) - F(x)\| \le \frac{1}{1344}\sum_{i=3}^{\infty} 16^i\Phi\left(\frac{x}{2^{i+1}}\right) \tag{2.18}$$

for all $x \in V\backslash\{0\}$, if φ satisfies (2.8), (2.9), (2.10), (2.11), or (2.12), respectively, where the function $\Phi : V\backslash\{0\} \to [0,\infty)$ is defined by

$$\begin{aligned}\Phi(x) :=& \varphi(6x, 2x) + 5\varphi(4x, 2x) + 15\varphi(2x, 2x) + 35\varphi(10x, -2x) \\ &+ 40\varphi(3x, x) + 200\varphi(2x, x) + 376\varphi(x, x) + 280\varphi(5x, -x).\end{aligned}$$

Proof. Notice that, from(2.1) and (2.13), we have

$$\begin{aligned}\|\Gamma\tilde{f}(x)\| =& \Big\| \underset{2x}{\overset{5}{\Delta}} f(6x) + 5\underset{2x}{\overset{5}{\Delta}} f(4x) + 15\underset{2x}{\overset{5}{\Delta}} f(2x) - 35\underset{-2x}{\overset{5}{\Delta}} f(10x) \\ &+ 40\underset{x}{\overset{5}{\Delta}} f(3x) + 200\underset{x}{\overset{5}{\Delta}} f(2x) + 376\underset{x}{\overset{5}{\Delta}} f(x) - 280\underset{-x}{\overset{5}{\Delta}} f(5x) \Big\| \le \Phi(x)\end{aligned} \tag{2.19}$$

for all $x \in V$. We prove the theorem in two steps.

Step 1. Let $k \in \{1, 2, 3, 4\}$ and $\delta \in \{-1, 1\}$, and let φ satisfy

$$\sum_{n=0}^{\infty} \frac{\varphi(2^{\delta n}x, 2^{\delta n}y)}{2^{\delta k n}} < \infty \tag{2.20}$$

for all $x, y \in V\backslash\{0\}$. Together with

$$\frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta kn}} - \frac{\tilde{f}_k(2^{\delta(n+m)}x)}{2^{\delta k(n+m)}} = \sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_k(2^{\delta i}x)}{2^{\delta ki}} - \frac{\tilde{f}_k(2^{\delta(i+1)}x)}{2^{\delta k(i+1)}}\right),$$

and (2.2), (2.3), (2.4), (2.5), (2.19), we have the inequalities

$$\left\|\frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m}x)}{2^{n+m}}\right\| \leq \frac{1}{336}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{2^i}\right\| \leq \frac{1}{336}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{2^i},$$

$$\left\|2^n\tilde{f}_1\left(\frac{x}{2^n}\right) - 2^{n+m}\tilde{f}_1\left(\frac{x}{2^{n+m}}\right)\right\| \leq \frac{1}{168}\sum_{i=n}^{n+m-1}\left\|2^i\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \leq \frac{1}{168}\sum_{i=n}^{n+m-1}2^i\Phi\left(\frac{x}{2^{i+1}}\right),$$

$$\left\|\frac{\tilde{f}_2(2^n x)}{4^n} - \frac{\tilde{f}_2(2^{n+m}x)}{4^{n+m}}\right\| \leq \frac{1}{384}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{4^i}\right\| \leq \frac{1}{384}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{4^i},$$

$$\left\|4^n\tilde{f}_2\left(\frac{x}{2^n}\right) - 4^{n+m}\tilde{f}_2\left(\frac{x}{2^{n+m}}\right)\right\| \leq \frac{1}{96}\sum_{i=n}^{n+m-1}\left\|4^i\,\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \leq \frac{1}{96}\sum_{i=n}^{n+m-1}4^i\Phi\left(\frac{x}{2^{i+1}}\right),$$

$$\left\|\frac{\tilde{f}_3(2^n x)}{8^n} - \frac{\tilde{f}_3(2^{n+m}x)}{8^{n+m}}\right\| \leq \frac{1}{1536}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{8^i}\right\| \leq \frac{1}{1536}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{8^i},$$

$$\left\|8^n\tilde{f}_3\left(\frac{x}{2^n}\right) - 8^{n+m}\tilde{f}_3\left(\frac{x}{2^{n+m}}\right)\right\| \leq \frac{1}{192}\sum_{i=n}^{n+m-1}\left\|8^i\,\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \leq \frac{1}{192}\sum_{i=n}^{n+m-1}8^i\Phi\left(\frac{x}{2^{i+1}}\right),$$

$$\left\|\frac{\tilde{f}_4(2^n x)}{16^n} - \frac{\tilde{f}_4(2^{n+m}x)}{16^{n+m}}\right\| \leq \frac{1}{21504}\sum_{i=n}^{n+m-1}\left\|\frac{\Gamma\tilde{f}(2^i x)}{16^i}\right\| \leq \frac{1}{21504}\sum_{i=n}^{n+m-1}\frac{\Phi(2^i x)}{16^i},$$

and

$$\begin{aligned}\left\|16^n\tilde{f}_4\left(\frac{x}{2^n}\right) - 16^{n+m}\tilde{f}_4\left(\frac{x}{2^{n+m}}\right)\right\| &\leq \frac{1}{1344}\sum_{i=n}^{n+m-1}\left\|16^i\,\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \\ &\leq \frac{1}{1344}\sum_{i=n}^{n+m-1}16^i\Phi\left(\frac{x}{2^{i+1}}\right)\end{aligned}$$

for all $x \in V\backslash\{0\}$ and $n, m \in \mathbb{N}\cup\{0\}$. It leads us to prove that, if φ satisfies (2.20) then the sequence $\left\{\frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta kn}}\right\}$ is a Cauchy sequence for all $x \in V\backslash\{0\}$. Moreover, since Y is complete and $\tilde{f}_k(0) = 0$, the sequence converges for all $x \in V$. It follows that we can define a mapping $F_{\delta k} : V \to Y$ by

$$F_{\delta k}(x) := \lim_{n\to\infty}\frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta kn}} \tag{2.21}$$

for all $x \in V$ if φ satisfies (2.20). Now we observe that the equality

$$\begin{aligned}\overset{5}{\underset{y}{\Delta}} F_{\delta k}(x) &= F_{\delta k}(x+5y) - 5F_{\delta k}(x+4y) + 10F_{\delta k}(x+3y) - 10F_{\delta k}(x+2y) + 5F_{\delta k}(x+y) - F_{\delta k}(x) \\ &= \lim_{n\to\infty}\left(\frac{\tilde{f}_k(2^{\delta n}(x+5y))}{2^{\delta n}} - 5\frac{\tilde{f}_k(2^{\delta n}(x+4y))}{2^{\delta n}} + 10\frac{\tilde{f}_k(2^{\delta n}(x+3y))}{2^{\delta n}} - 10\frac{\tilde{f}_k(2^{\delta n}(x+2y))}{2^{\delta n}}\right. \\ &\qquad \left. +5\frac{\tilde{f}_k(2^{\delta n}(x+y))}{2^{\delta n}} - \frac{\tilde{f}_k(2^{\delta n}x)}{2^{\delta n}}\right)\end{aligned}$$

holds for all $x, y \in V\backslash\{0\}$. Together with the definition of $\tilde{f}_1$, if φ satisfies (2.20) for $k = 1$, then we have

$$\begin{aligned}\left\|\overset{5}{\underset{y}{\Delta}} F_{\delta 1}(x)\right\| &= \lim_{n\to\infty}\left\| -\frac{1}{168}\left(\frac{\tilde{f}(2^{\delta n+3}(x+5y))}{2^{\delta n}} - 5\frac{\tilde{f}(2^{\delta n+3}(x+4y))}{2^{\delta n}} + 10\frac{\tilde{f}(2^{\delta n+3}(x+3y))}{2^{\delta n}}\right.\right. \\ &\quad \left. - 10\frac{\tilde{f}(2^{\delta n+3}(x+2y))}{2^{\delta n}} + 5\frac{\tilde{f}(2^{\delta n+3}(x+y))}{2^{\delta n}} - \frac{\tilde{f}(2^{\delta n+3}x)}{2^{\delta n}}\right) \\ &\quad + \frac{28}{168}\left(\frac{\tilde{f}(2^{\delta n+2}(x+5y))}{2^{\delta n}} - 5\frac{\tilde{f}(2^{\delta n+2}(x+4y))}{2^{\delta n}} + 10\frac{\tilde{f}(2^{\delta n+2}(x+3y))}{2^{\delta n}}\right. \\ &\quad \left. - 10\frac{\tilde{f}(2^{\delta n+2}(x+2y))}{2^{\delta n}} + 5\frac{\tilde{f}(2^{\delta n+2}(x+y))}{2^{\delta n}} - \frac{\tilde{f}(2^{\delta n+2}x)}{2^{\delta n}}\right) \\ &\quad - \frac{224}{168}\left(\frac{\tilde{f}(2^{\delta n+1}(x+5y))}{2^{\delta n}} - 5\frac{\tilde{f}(2^{\delta n+1}(x+4y))}{2^{\delta n}} + 10\frac{\tilde{f}(2^{\delta n+1}(x+3y))}{2^{\delta n}} - 10\frac{\tilde{f}(2^{\delta n+1}(x+2y))}{2^{\delta n}}\right. \\ &\quad \left. + 5\frac{\tilde{f}(2^{\delta n+1}(x+y))}{2^{\delta n}} - \frac{\tilde{f}(2^{\delta n+1}x)}{2^{\delta n}}\right) + \frac{512}{168}\left(\frac{\tilde{f}(2^{\delta n}(x+5y))}{2^{\delta n}} - 5\frac{\tilde{f}(2^{\delta n}(x+4y))}{2^{\delta n}}\right. \\ &\quad \left.\left. + 10\frac{\tilde{f}(2^{\delta n}(x+3y))}{2^{\delta n}} - 10\frac{\tilde{f}(2^{\delta n}(x+2y))}{2^{\delta n}} + 5\frac{\tilde{f}(2^{\delta n}(x+y))}{2^{\delta n}} - \frac{\tilde{f}(2^{\delta n}x)}{2^{\delta n}}\right)\right\| \\ &= \lim_{n\to\infty}\left\|\frac{\Delta^5_{2^{\delta n+3}y}f(2^{\delta n+3}x)}{-168\cdot 2^{\delta n}} + \frac{28\Delta^5_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{168\cdot 2^{\delta n}} - \frac{224\Delta^5_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{168\cdot 2^{\delta n}} + \frac{512\,\Delta^5_{2^{\delta n}y}f(2^{\delta n}x)}{168\cdot 2^{\delta n}}\right\| \\ &\le \lim_{n\to\infty}\left(\frac{\varphi(2^{\delta n+3}x, 2^{\delta+3}y)}{168\cdot 2^{\delta n}} + \frac{\varphi(2^{\delta n+2}x, 2^{\delta+2}y)}{6\cdot 2^{\delta n}} + \frac{4\varphi(2^{\delta n+1}x, 2^{\delta n+1}y)}{3\cdot 2^{\delta n}} + \frac{64\varphi(2^{\delta n}x, 2^{\delta n}y)}{21\cdot 2^{\delta n}}\right) \\ &= 0\end{aligned}$$

for all $x, y \in V\backslash\{0\}$. In a similar way, by the definition of $\tilde{f}_k$, if φ satisfies (2.20) for $k = 2, 3, 4$, respectively, then we get

$$\begin{aligned}&\left\|\overset{5}{\underset{y}{\Delta}} F_{\delta 2}(x)\right\| \\ &= \lim_{n\to\infty}\left\|\frac{\Delta^5_{2^{\delta n+3}y}f(2^{\delta n+3}x)}{96\cdot 4^{\delta n}} - \frac{26\Delta^5_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{96\cdot 4^{\delta n}} + \frac{176\Delta^5_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{96\cdot 4^{\delta n}} - \frac{256\,\Delta^5_{2^{\delta n}y}f(2^{\delta n}x)}{96\cdot 4^{\delta n}}\right\| \\ &\le \lim_{n\to\infty}\left(\frac{\varphi(2^{\delta n+3}x, 2^{\delta n+3}y)}{96\cdot 4^{\delta n}} + \frac{13\varphi(2^{\delta n+2}x, 2^{\delta n+2}y)}{48\cdot 4^{\delta n}} + \frac{11\varphi(2^{\delta n+1}x, 2^{\delta n+1}y)}{6\cdot 4^{\delta n}} + \frac{8\varphi(2^{\delta n}x, 2^{\delta n}y)}{3\cdot 4^{\delta n}}\right) \\ &= 0,\end{aligned}$$

$$\begin{aligned}&\left\|\overset{5}{\underset{y}{\Delta}} F_{\delta 3}(x)\right\| \\ &= \lim_{n\to\infty}\left\|\frac{\Delta^5_{2^{\delta n+3}y}f(2^{\delta n+3}x)}{-192\cdot 8^{\delta n}} + \frac{22\Delta^5_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{192\cdot 8^{\delta n}} - \frac{104\Delta^5_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{192\cdot 8^{\delta n}} + \frac{128\,\Delta^5_{2^{\delta n}y}f(2^{\delta n}x)}{192\cdot 8^{\delta n}}\right\|\end{aligned}$$

$$\le \lim_{n\to\infty}\left(\frac{\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{192\cdot 8^{\delta n}}+\frac{11\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{96\cdot 8^{\delta n}}+\frac{13\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{24\cdot 8^{\delta n}}+\frac{2\varphi(2^{\delta n}x,2^{\delta n}y)}{3\cdot 8^{\delta n}}\right)$$
$$= 0,$$

and

$$\left\|\underset{y}{\overset{5}{\Delta}} F_{\delta 4}(x)\right\|$$
$$= \lim_{n\to\infty}\left\|\frac{\Delta^5_{2^{\delta n+3}y}f(2^{\delta n+3}x)}{1344\cdot 8^{\delta n}}-\frac{14\Delta^5_{2^{\delta n+2}y}f(2^{\delta n+2}x)}{1344\cdot 8^{\delta n}}+\frac{56\Delta^5_{2^{\delta n+1}y}f(2^{\delta n+1}x)}{1344\cdot 8^{\delta n}}-\frac{64\,\Delta^5_{2^{\delta n}y}f(2^{\delta n}x)}{1344\cdot 8^{\delta n}}\right\|$$
$$\le \lim_{n\to\infty}\left(\frac{\varphi(2^{\delta n+3}x,2^{\delta n+3}y)}{1344\cdot 8^{\delta n}}+\frac{\varphi(2^{\delta n+2}x,2^{\delta n+2}y)}{96\cdot 8^{\delta n}}+\frac{\varphi(2^{\delta n+1}x,2^{\delta n+1}y)}{24\cdot 8^{\delta n}}+\frac{\varphi(2^{\delta n}x,2^{\delta n}y)}{21\cdot 8^{\delta n}}\right)$$
$$= 0$$

for all $x, y \in V\backslash\{0\}$.

And then, since $\Delta^5_y F_{\delta k}(x) = 0$ for all $x, y \in V\backslash\{0\}$, the mapping $F_{\delta k}$ is a general quartic mapping for all $k = 1, 2, 3, 4$ and $\delta = \pm 1$ by Lemma 2.4.

Step 2. Now we define the desired general quartic mapping F for all cases.

(1) Let φ satisfy the condition (2.8), then F_1, F_2, F_3, and F_4 are defined by (2.21). We put a general quartic mapping $F : V \to Y$ by

$$F(x) := F_1(x) + F_2(x) + F_3(x) + F_4(x)$$

for all $x \in V$. Observe that. by (2.2), (2.3), (2.4), and (2.5), we have

$$\begin{aligned}\left\|\tilde f(x)-\sum_{k=1}^{4}\frac{\tilde f_k(2^n x)}{2^{kn}}\right\| &\le \sum_{i=0}^{n-1}\left\|\sum_{k=1}^{4}\Big(\frac{\tilde f_k(2^i x)}{2^{ki}}-\frac{\tilde f_k(2^{i+1}x)}{2^{k(i+1)}}\Big)\right\|\\ &= \sum_{i=0}^{n-1}\left(\frac{1}{336\cdot 2^i}-\frac{1}{384\cdot 4^i}+\frac{1}{1536\cdot 8^i}-\frac{1}{21504\cdot 16^i}\right)\|\Gamma\tilde f(2^i x)\|\\ &\le \sum_{i=0}^{n-1}\left\|\frac{\Gamma\tilde f(2^i x)}{336\cdot 2^i}\right\|\\ &\le \frac{1}{336}\sum_{i=0}^{n-1}\frac{\Phi(2^i x)}{2^i}\end{aligned}$$

for all $x \in V\backslash\{0\}$, which follows (2.14) as $n \to \infty$.

(2) Let φ satisfy the condition (2.9), then F_{-1}, F_2, F_3, and F_4 are defined by (2.21). Putting a general quartic mapping $F : V \to Y$ by

$$F(x) := F_{-1}(x) + F_2(x) + F_3(x) + F_4(x)$$

for all $x \in V$. Then we have

$$\begin{aligned}&\left\|\tilde f(x)-2^n\tilde f_1\left(\frac{x}{2^n}\right)-\sum_{k=2}^{4}\frac{\tilde f_k(2^n x)}{2^{kn}}\right\|\\ &\quad\le \sum_{i=0}^{n-1}\left\|2^i\tilde f_1\left(\frac{x}{2^i}\right)-2^{i+1}\tilde f_1\left(\frac{x}{2^{i+1}}\right)\right\|+\sum_{i=0}^{n-1}\left\|\sum_{k=2}^{4}\Big(\frac{\tilde f_k(2^i x)}{2^{ki}}-\frac{\tilde f_k(2^{i+1}x)}{2^{k(i+1)}}\Big)\right\|\end{aligned}$$

$$\leq \sum_{i=0}^{n-1} \frac{2^i}{168} \left\| \Gamma \tilde{f}\left(\frac{x}{2^{i+1}}\right) \right\| + \sum_{i=0}^{n-1} \left(-\frac{1}{384 \cdot 4^i} + \frac{1}{1536 \cdot 8^i} - \frac{1}{21504 \cdot 16^i} \right) \|\Gamma \tilde{f}(2^i x)\|$$

$$\leq \frac{1}{168} \sum_{i=0}^{n-1} 2^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{384} \sum_{i=0}^{n-1} \frac{\Phi(2^i x)}{4^i}$$

for all $x \in V\backslash\{0\}$ by (2.2), (2.3), (2.4), and (2.5) again, which follows (2.15) as $n \to \infty$.

(3) Let φ satisfy the condition (2.10), then F_{-1}, F_{-2}, F_3, and F_4 are defined by (2.21). Putting a general quartic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_3(x) + F_4(x)$$

for all $x \in V$. We have the inequality by (2.2), (2.3), (2.4), and (2.5) that

$$\begin{aligned}
&\left\| \tilde{f}(x) - \sum_{k=1}^{2} 2^{kn} \tilde{f}_k\left(\frac{x}{2^n}\right) - \sum_{k=3}^{4} \frac{\tilde{f}_k(2^n x)}{2^{kn}} \right\| \\
&\quad \leq \sum_{i=0}^{n-1} \left\| \sum_{k=1}^{2} \left(2^{ki} \tilde{f}_k\left(\frac{x}{2^i}\right) - 2^{k(i+1)} \tilde{f}_k\left(\frac{x}{2^{i+1}}\right) \right) \right\| + \sum_{i=0}^{n-1} \left\| \sum_{k=3}^{4} \left(\frac{\tilde{f}_k(2^i x)}{2^{ki}} - \frac{\tilde{f}_k(2^{i+1} x)}{2^{k(i+1)}} \right) \right\| \\
&\quad \leq \sum_{i=0}^{n-1} \left(-\frac{2^i}{168} + \frac{4^i}{96} \right) \left\| \Gamma \tilde{f}\left(\frac{x}{2^{i+1}}\right) \right\| + \sum_{i=0}^{n-1} \left(\frac{1}{1536 \cdot 8^i} - \frac{1}{21504 \cdot 16^i} \right) \|\Gamma \tilde{f}(2^i x)\| \\
&\quad \leq \frac{1}{96} \sum_{i=0}^{n-1} 4^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{1536} \sum_{i=0}^{n-1} \frac{\Phi(2^i x)}{8^i}
\end{aligned}$$

for all $x \in V\backslash\{0\}$ by (2.2), (2.3), (2.4), and (2.5), which follows (2.16) as $n \to \infty$.

(4) Let φ satisfy the condition (2.11), then F_{-1}, F_{-2}, F_{-3}, and F_4 are defined by (2.21). Putting a general quartic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_{-3}(x) + F_4(x)$$

for all $x \in V$. We have the inequality by (2.2), (2.3), (2.4), and (2.5) that

$$\begin{aligned}
&\left\| \tilde{f}(x) - \sum_{k=1}^{3} 2^{kn} \tilde{f}_k\left(\frac{x}{2^n}\right) - \frac{\tilde{f}_4(2^n x)}{2^{4n}} \right\| \\
&\quad \leq \sum_{i=0}^{n-1} \left\| \sum_{k=1}^{3} \left(2^{ki} \tilde{f}_k\left(\frac{x}{2^i}\right) - 2^{k(i+1)} \tilde{f}_k\left(\frac{x}{2^{i+1}}\right) \right) \right\| + \sum_{i=0}^{n-1} \left\| \frac{\tilde{f}_4(2^i x)}{2^{4i}} - \frac{\tilde{f}_4(2^{i+1} x)}{2^{4(i+1)}} \right\| \\
&\quad \leq \sum_{i=0}^{n-1} \left(\frac{2^i}{168} - \frac{4^i}{96} + \frac{8^i}{192} \right) \left\| \Gamma \tilde{f}\left(\frac{x}{2^{i+1}}\right) \right\| + \sum_{i=0}^{n-1} \frac{1}{21504 \cdot 16^i} \|\Gamma \tilde{f}(2^i x)\| \\
&\quad \leq \frac{1}{192} \sum_{i=0}^{n-1} 8^i \Phi\left(\frac{x}{2^{i+1}}\right) + \frac{1}{21504} \sum_{i=0}^{n-1} \frac{\Phi(2^i x)}{16^i}
\end{aligned}$$

for all $x \in V\backslash\{0\}$, which follows (2.17) as $n \to \infty$.

(5) Let φ satisfy the condition (2.12), then F_{-1}, F_{-2}, F_{-3}, and F_{-4} are defined by (2.21). Putting a general quartic mapping

$$F(x) := F_{-1}(x) + F_{-2}(x) + F_{-3}(x) + F_{-4}(x)$$

for all $x \in V$. We have the inequality by (2.2), (2.3), (2.4), and (2.5) that

$$\begin{aligned}\left\|\tilde{f}(x) - \sum_{k=1}^{4} 2^{kn}\tilde{f}_k\left(\frac{x}{2^n}\right)\right\| &\leq \sum_{i=0}^{n-1}\left\|\sum_{k=1}^{4}\left(2^{ki}\tilde{f}_k\left(\frac{x}{2^i}\right) - 2^{k(i+1)}\tilde{f}_k\left(\frac{x}{2^{i+1}}\right)\right)\right\| \\ &\leq \sum_{i=0}^{n-1}\left\|\left(-\frac{2^i}{168}\frac{4^i}{96} - \frac{8^i}{192} + \frac{16^i}{1344}\right)\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \\ &\leq \sum_{i=3}^{n-1}\left(-\frac{2^i}{168} + \frac{4^i}{96} - \frac{8^i}{192} + \frac{16^i}{1344}\right)\left\|\Gamma\tilde{f}\left(\frac{x}{2^{i+1}}\right)\right\| \\ &\leq \frac{1}{1344}\sum_{i=3}^{n-1} 16^i\Phi\left(\frac{x}{2^{i+1}}\right)\end{aligned}$$

for all $x \in V\backslash\{0\}$, since $-\frac{2^i}{168} + \frac{4^i}{96} - \frac{8^i}{192} + \frac{16^i}{1344} = 0$ when $i \in \{0, 1, 2\}$, which follows (2.18) as $n \to \infty$. Moreover, by the definition, we easily get

$$F_{\delta k}(2x) = 2^k F_{\delta k}(x)$$

and $\overset{5}{\underset{y}{\Delta}} F_{\delta k}(x) = 0$ for all $x, y \in V$. According to Lemma 2.4, F is the unique general quartic mapping, and we complete the proof of the theorem. □

Recall the stability result for the functional equation (1.1) proved by Y. H. Lee and S. M. Jung [5] only deal with the conditions (2.8) and (2.12) of Theorem 2.4.

Compare the following concise theorem obtained from Theorem 2.4 with Theorem 1.1 obtained by Y. H. Lee [4].

Theorem 2.6 *Let θ be a positive real constant and p a real number such that $p \neq 1, 2, 3, 4$. If $f : X \to Y$ satisfies the inequality*

$$\left\|\overset{5}{\underset{y}{\Delta}} f(x)\right\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X\backslash\{0\}$, then there exists a unique general quartic mapping F such that

$$\begin{aligned}\|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{168(2-2^p)} && \text{for } \quad p < 1, \\ \|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{168(2^p-2)} + \frac{M\theta\|x\|^p}{96(4-2^p)} && \text{for } \quad 1 < p < 2, \\ \|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{96(2^p-4)} + \frac{M\theta\|x\|^p}{192(8-2^p)} && \text{for } \quad 2 < p < 3, \\ \|\tilde{f}(x) - F(x)\| &\leq \frac{M\theta\|x\|^p}{192(2^p-8)} + \frac{M\theta\|x\|^p}{1344(16-2^p)} && \text{for } \quad 3 < p < 4, \\ \|\tilde{f}(x) - F(x)\| &\leq \frac{64M\theta\|x\|^p}{21\cdot 8^p(2^p-16)} && \text{for } \quad 4 < p\end{aligned}$$

for all $x \in X\backslash\{0\}$, where

$$M := 35\cdot 10^p + 6^p + 280\cdot 5^p + 5\cdot 4^p + 40\cdot 3^p + 271\cdot 2^p + 1272.$$

3 Conclusions

In this paper, we investigate the generalized stability of the general quartic functional equation (1.1). Precisely, if $f : V \to Y$ is a mapping such that $\left\| \Delta_y^5 f(x) \right\| \leq \varphi(x, y)$ for all $x, y \in V \backslash \{0\}$, where $\varphi : (V\backslash\{0\})^2 \to [0, \infty)$ holds the conditions (2.8), (2.9), (2.10), (2.11), or (2.12), then there exists a unique general quartic mapping F such that the difference $\|\tilde{f}(x) - F(x)\|$ satisfies the conditions (2.14), (2.15), (2.16), (2.17), or (2.18), respectively.

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