Mathematical Models in Economics: Applications of Sequences, Derivatives, and Differential Equations

Abstract

This paper examines the application of mathematical tools—numerical sequences, derivatives, and differential equations—that have important uses within economic analyses on population modeling, investment growth, and population cost. Numerical examples shall be used to illustrate how such a mathematical approach helps in financial decision-making and long-term forecasting. It's expected that this work will provide insight into the benefits of mathematical models in addressing complex economic challenges and make a case for interdisciplinary approaches in economics.

Keywords: sequences, derivatives, differential equations, marginal cost, average cost, GDP

Introduction

This manuscript is important to the scientific community as it reaches back into the very fundamental mathematics underlying economics, an area so key for policy analysis, financial forecasting, and strategic economic planning. By linking the mathematical models with real-world economic scenarios, like those that are concerned with population growth, as well as the calculation of interest, a foundation for making more robust, data-driven decisions is laid. Another strength of the paper is its focus on mathematical applications that help reach an interdisciplinary understanding, showing economists and mathematicians that their work has relevance to problems in economics.

1. **Application of sequences in economy**

Population changes depend upon such factors as nutrition, natural catastrophes, wars, economic wealth, cultural perceptions of family, and especially on time. Over time, the population may remain constant, increase, or decrease.

This process can be formulated using mathematical terms, with the use of some Math functions and how to visualize it with the connection between Math, Economy, and Informatics or programming[12]. The number of population in a certain year is a natural number, so if we want to study the development during in period of time, then the numbers of population in this period of time are a subset of natural numbers [7,8].

Suppose in this year we have N_0 inhabitants in the Republic of North Macedonia, then after a year we have N_1 inhabitants, after two years N_2, and after i years we have N_i inhabitants. So we can generate the finite sequence

$$N_1, N_2, \dots, N_i$$

where N_k tell the number of inhabitants after k-years, $k = 1, 2, 3, \dots i$

Below, we present the population in the Republic of North Macedonia, over the years:

	2009	2014	2019
Population at the end of the year	2052722	2069172	2076255

Table1. Population at the end of the certian year, in North Macedonia

From the table, we can construct a sequence, in this way:

$$N_{2009} = 2052722, \qquad N_{2014} = 2069172, \qquad N_{2019} = 2076255$$

when N_i denoted the number of the population at the end of the year i. So, based on this finite sequence we have an increased sequence, which means the population in North Macedonia from 2009 to 2019 is increasing continuously.

Now, we can give the formal definition of finite sequence:

Definition1 A sequence is a function whose domain is the positive integers.

The following are examples of sequences:

i. $\mathrm{f(n)} = 2\mathrm{n}$ or $2, 4, 6, 8, 10, \dots$

ii. $\mathrm{f(n)} = \frac{1}{\mathrm{n}}$ or $1, 1/2, 1/3, 1/4, \dots$

iii. $\mathrm{f(n)} = (-1)^{\mathrm{n}}$ or $-1, 1, -1, 1, \dots$

The reason for giving the population was because it has an important impact on the economy of a state and the direction of the economics and politics of a government. So if we study like sequences we can predict consequences and the state's institutions could take adequate action [9,10].

Another well-known sequence in mathematics, which is used in economics, especially in finance is the sequence, when Jacob Bernoulli, in 1683 was studying compound interest, especially continuous compound interest.

Suppose you lend money to your friend at a 100% interest rate, compounded continuously, year by year. In the next year, your money would double. If the interest rate would be 50% at every six months, then your money would grow 225% in one year.

So, as the interval gets smaller, the total return gets slightly higher. If the interest rate calculated n times per year with a rate $100\%/n$, then the total wealth at the end of the first year will growth 2.7 times, compared with the initial wealth, when n is sufficiently large.

Let's take an example. Suppose you have 20000€ paying 6% interest with continuous compounding, after ten years, how many euros do you have? ([5])

This financial problem is solved by the following formula

$$FV = PVe^{rt} \ldots (1),$$

where

$$FV = \text{ future value}$$

$$PV = \text{ present value of balance or sum}$$

$$e = Euler's\ number, (e = 2.71828 \ldots)$$

$$r = interest\ rate\ being\ compounded$$

$$t = time\ in\ years$$

So, by the problem, we have that $PV = 20000€$, $r = 0.06$, $t = 10$, then

$$FV = 20000 \cdot 2.71828^{0.06 \cdot 10} = 36442.3613€ \ldots (2)$$

This amount of interest can calculated in another way, with the formula

$$FV = PV\left(1 + \frac{r}{n}\right)^{nt} \ldots (3)$$

where n is represented the number of compounding periods in a year, in our case 12, then

$$FV = 20000\left(1 + \frac{0.06}{12}\right)^{12 \cdot 10} = 20000(1 + 0.005)^{120} =$$

$$= 20000 \cdot 1.005^{120} = 36387.93468$$

If we compare the two last results the difference is small, considering the number of years.

But, if we want to know how long we should wait to double our initial investment, so if we save 20000€, fixed interest rate 6% paid annually?

Now, we have invested an amount P and interest is paid annually at interest rate i,i $-$ the interest rate, after one year we have $P(1+i)$, after two years

$\left(P(1+i)\right)(1+i) = P(1+i)^2$,

and after t years $P(1+i)^t$, so we can write the equation, by the condition of the problem:

$$20000(1+0.06)^t = 40000$$

$$1.06^t = 2$$

$$t \log 1.06 = \log 2$$

$$t = \frac{log2}{log1.06} = 11.8956610459$$

So we should wait 12 years to double our initial investment. (see [3], [4])

2. Application of derivative in Economics

In the 1970's in economic world happened a revolution, which is called marginal revolution. From production to consumption, from supply to demand, and from cost to utility were some of the features of this revolution. In economics terminology, it contains an analysis of the marginal utility theory of value and the extensive application of the marginal analysis method. In fact, the last method is based on the mathematical analysis method, to express more explicitly on concept of derivative.

The mathematical definition of the derivative of a function is defined by:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x+x_0)-f(x_0)}{x-x_0} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where f is a function defined in a subset of real numbers and for its exists the above limit.

In economic terminology, the derivative of a function is called its marginal function and the value at a specific point is called the marginal value.

Marginal cost is related closely to cost minimization, in other words, is the change of total cost for each additional unit of goods that are produced if the other conditions remain the same.

If the product put [put is Q and its corresponding cost function is $C(Q)$ is differentiable, hence the marginal cost is the derivative of the cost function with respect to output.

Now, let the problem of a factory of kinds of pasta be given as follows:

Suppose that their output is Q and the cost function

$$C(Q) = 0{,}7Q^2 + 234 \quad \ldots (4),$$

then the marginal cost function will be $C'(Q) = 1.4Q$. So if we take $C'(5) = 1.4 \cdot 5$=7, $C'(50) = 70$ and $C'(500) = 700$, so the marginal cost of producing 5 pasta is 7, 50 is 70, and 500 is 700.

Now, let's see the average cost:

$$\frac{C(5)}{5} = 50.3 \quad \ldots (5)$$

$$\frac{C(50)}{50} = 39.68 \quad \ldots (6)$$

and

$$\frac{C(500)}{500} = 350.468 \quad \ldots (7)$$

When $Q = 5$ the output must be increased because the marginal cost is lower than the average cost, but in two other cases is not good to increase the output. (see [2], [6], [10])

3. Application of differential equations in economics

Ordinary differential equations have different applications, as in mathematics, and also in other fields outside of mathematics, in our case even economics, respectively in the analysis of market models.

Mathematically a linear differential equation is expressed with the formula:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

where

y is an unknown function, x is an independent variable a_i represents the coefficient functions and $g(x)$ is forced function.

3.1 To calculate the change in Gross Domestic Product with time

If Y(t)is the current state of GDP, then $\frac{dY}{dt}$ is the rate of change of the current state with respect to time and it is proportional to the current GDP, thus mathematically can we write

$$\frac{dY}{dt} = gY(t) \quad \dots(8)$$

g -is the growth rate.

3.2. To explain an economy's growth rate

These purposes economists have created different models, in which model they used differential equations, we will mention two of them.

3.2.1 Harrod-Domar model

This model was constructed by Roy. F. Harrod in 1939 and by Evsey Domar in 1946, in an independent manner. On based of this model, an economy's growth rate is explained in terms of saving and productivity of capital.

Its mathematical representation is

$$\frac{\dot{Y}}{Y} = sc - \delta \quad \dots(9)$$

Where $\frac{\dot{Y}}{Y}$ represents the output growth rate,

$\dot{Y}$represents thederivative of Y,

c represents the marginal product of capital,

δ represents the rate of depreciation of capital stock, and

s represents the saving growth rate. [1]

3.2.2. Solow-Swan Model

Even this model was developed independently by Robert Solow and Trevor Swan in 1956. They try to explain the long-run economic growth, in terms of capital accumulation, population growth, and technological progress.

The equation which describes this model is

$$\mathrm{Y} = \mathrm{K}^{\alpha}(\mathrm{AL})^{1-\alpha}, 0 < \alpha < 1 \quad \dots(10)$$

Y-output, K and L labour and capital used,α elasticity of output with respect to capital, A labour augmenting technology, AL effective labour.

The marginal product of capital is given as

$$MP_k = \frac{\partial Y}{\partial K} = \frac{\alpha A^{1-\alpha}}{(\frac{K}{L})^{1-\alpha}} \quad \ldots(11)$$

This is a single ordinary differential equation, which is non-linear, in pursuit of giving insight into long-run economic growth. (see [1], [8])

3.2.3 Samuelson

Paul Samuelson, in 1941, used in his paper differential equation in order to study the stability of equilibrium for several demand-supply scenarios.

Let $D(p, \alpha)$ and $S(p)$ be demand and supply functions price p, α shift parameter, which represents taste. At equilibrium, the price p^* and quantity q^*, are given by

$$q^* = D(p^*, \alpha) = S(p^*), \ldots(12)$$

Where derivative of D with respect to α is greater than 0, but with respect to p is less than 0.

3.2.4. Phelps

He has developed a model based on the neoclassical growth model to study the consumption per unit of labor at equilibrium. The consumption per unit of labor is given by

$$c(t) = f(k) - nk \ldots(13)$$

While for maximum consumption per unit of labor is given by:

$$\frac{dc}{dk} = \frac{df}{dk} - n = 0 \ldots(14)$$

The second derivative of f with respect to k is less than zero, it means that the point of maximum is given by $\frac{df}{dk} = n$.

Therefore, we can conclude that marginal output for workers must equal the growth rate of the labour force. ([1],[2])

Furthermore, differential equations have a wide range of applications in the economy: to study trade cycles, economic chaos, dynamic stability conditions of equilibrium et cetera.

4. Application of statistical and zero-sum games in economics

This concept is present in different fields, where we have to do "decision making", in our case it is widespread in the economy because it is part of every day of businessmen, and scientists of the economy. It uses optimal allocation of resources in different assets. Some authors consider the application of a game-theoretical approach can impact in security risks.

Distribution of the economic environment allows the decision-maker to have greater flexibility when analyzing different economic systems.

The application of game theory in decision-making consists of the construction of a payoff matrix, which is in the most time-consuming phase. In the game-theoretic modeling in the economy, not all entries of the matrix are known, even some of them are unknown absolutely. This model helps us to estimate the economic environment for finding them.

Studying these unknown probability values is based on some type of ordering relation, whose studied by Fishburn and Trukhaev. In this direction have to mention Fishburn point estimates, Fishburn arithmetic progression, and Fishburn geometric progression, ([11], [12])

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