Hierarchic control for a two-stroke linear system with missing data

Abstract

In this paper, we study a new hierarchical control problem for a linear two-stroke missing data problem, adjoint to an age and space structured single species population dynamics problem. We show that there are two controls such that the first control, called the follower, solves an optimal control problem which consists in bringing the state of the two-stroke linear system to a desired state, and the second control, called the leader, solves a null controllability problem. The results are obtained by means of an observability inequality associated with a homogeneous Dirichlet boundary condition.

Keywords: Optimal control, Carleman inequality, Null controlability, Missing data, Population dynamics, Low regret control, Euler- lagrange formula.

AMS Subject Classification 35Q93, 49J20, 93C41, 93B05, 92D25

1 Introduction

We consider a population with age dependence and spatial structure, and we assume that the population lives in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}\setminus \{0\}$ with boundary Γ of class C^2. Let $q = q(t, a, x)$ be the distribution of individuals of age $a \in [0, A]$ at time $t \in [0, T]$ and location $x \in \Omega$. Let also $A \geq 0$ be the life expectancy of an individual and the final time $T \geq 0$. Let ω_1 and ω_2 be nonempty subsets of Ω such that $\omega_1 \subset \omega_2$. We set $Q = (0, T) \times (0, A) \times \Omega$,

$\Omega^T = (0,T) \times \Omega$, $\Omega^A = (0,A) \times \Omega$, $\omega_i^{TA} = (0,T) \times (0,A) \times \omega_i$ with $i \in \{1,2\}$, $\Sigma = (0,T) \times (0,A) \times \Gamma$ and $\Sigma_1 = (0,T) \times (0,A) \times \Gamma_1$. We denote by $\mu = \mu(t,a,x) \geq 0$, the natural death rate of individuals of age a at time t and location x. Then, we consider the following linear system:

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - \Delta q + \mu q &= g + h\chi_{\omega 1} + k\chi_{\omega 2} \quad \text{in} \quad Q, \\ q(t,a,x) &= 0 \quad \text{on} \quad \Sigma, \\ q(T,a,x) &= 0 \quad \text{in} \quad \Omega^A, \\ q(t,A,x) &= 0 \quad \text{in} \quad \Omega^T, \end{cases} \tag{1}$$

where the controls v and k belong respectively to $L^2(\omega_1^{TA})$ and $L^2(\omega_2^{TA})$, $\chi_{\mathbb{X}}$ denotes the characteristic function on the open set $\mathbb{X}$. The function g is unknow and represents the supply of individuals and brows the set $\mathcal{O}$ a closed vector subspace of $L^2(Q)$. We make the following assumptions

$$\begin{cases} \mu(t,a,x) = \mu_0(a) + \mu_1(t,a,x) \text{ in } Q, \\ \mu_1 \in L^\infty(Q);\ \mu_1(t,a,x) \geq 0 \text{ for } (t,a,x) \text{ in } Q, \\ \mu_0 > 0,\ \mu_0 \in L^1_{loc}(0,A),\ \lim_{a\to A} \int_0^a \mu_0(s)ds = +\infty. \end{cases} \tag{2}$$

Under the above assumptions on the data as in [12], it is well known that the system (1) has a unique solution : $q(k,h,g) = q(t,a,x;k,h,g) \in L^2((0,T) \times (0,A); H_0^1(\Omega))$. Moreover, there exists a constante $C = C(T) > 0$ such that

$$\|q\|^2_{L^2((0,T)\times(0,A);H_0^1(\Omega))} \leq C(\|g\|^2_{L^2(Q)} + \|h\|^2_{L^2(\omega_1^{TA})} + \|k\|^2_{L^2(\omega_2^{TA})}). \tag{3}$$

Remark 1 *The variable $\mu = \mu(t,a,x)$ represents the natural mortality rate of individuals as a function of their age a, time t, and position x in a given domain. μ models the natural mortality rate of the population, meaning it acts as a diminishing factor for the population. Mortality depends on three variablestime, age, and positionallowing for the representation of a population with spatial structure and dependencies on time and age. It is decomposed into $\mu_0(a)$, which depends solely on age a, and a function $\mu_1(t,a,x)$, which depends on all three variables. This decomposition distinguishes an intrinsic mortality component related to age from the temporal and spatial components.*
$\mu_1(t,a,x) \geq 0$, ensuring that natural mortality does not negatively reduce the population.
$\mu_0(a) > 0$, so that intrinsic age-based mortality is positive. A boundary condition is imposed on μ_0, meaning that $\int_0^a \mu_0(s)\, ds$ tends to infinity as a approaches the maximum life expectancy A, indicating that survival probability decreases with age.

The system (1) is the adjoint of a population dynamics problem which, based on environmental sciences, can model an optimization process aimed, for example, at eradicating a harmful population such as armyworms in a cotton field. The presence of these pests can have devastating effects on crops, leading to significant losses in terms of yield and quality. This model plays a crucial role in a biodiversity management system, as it seeks to maximize the protection of agricultural crops while minimizing the damage caused by these pests. The region Ω represents the cultivated cotton field where the goal is to control and eventually eradicate the armyworms. This domain includes two specific subregions represented by the sets ω_1 and ω_2.

- The exact source of the armyworm infestation is unknown and is represented by the unknown g. This could be due to eggs randomly laid by migrating moths or to unidentified local egg-laying sources.

- Initially, the control v is applied in the subdomain ω_1, a specific part of the field's interior, to regulate the density of armyworms and achieve a desired state. This control can take the form of pheromone traps, installed in a targeted area to attract and capture adult male moths, thereby limiting their reproduction without requiring full-field coverage.

- Subsequently, in a second phase, the control k is applied in the subdomain ω_2, a region encompassing ω_1, to gradually reduce the population of armyworms until their extinction at the initial time 0. This control may involve the use of specific parasitoids, such as Trichogramma parasitoid wasps, to target and eradicate armyworm eggs. By significantly reducing the armyworm population in the treated areas, their overall reproductive capacity is disrupted, leading to complete eradication across the entire field.

In this work, we focus on the following problems:

Problem 1 *The control k being fixed in $L^2(\omega_2^{TA})$. For $\gamma > 0$, find the control $\breve{h}^\gamma = \breve{h}^\gamma(k) \in L^2(\omega_1^{TA})$ solution of*

$$\inf_{h \in L^2(\omega_1^{TA})} \sup_{g \in \mathcal{O}} J_1(k, h, g), \tag{4}$$

where

$$J_1(k, h, g) = J(k, h, g) - J(0, 0, g) - \gamma \|g\|^2_{L^2(Q)}, \tag{5}$$

and

$$J(k, h, g) = \|q(k, h, g) - z_d\|^2_{L^2(Q)} + \alpha \|h\|^2_{L^2(\omega_1^{TA})}, \tag{6}$$

with $z_d \in L^2(Q)$ and $\alpha > 0$.

Problem 2 *Let $\breve{h}^\gamma(k)$ be the control obtain in the first objective and $\breve{q}^\gamma = q(t, a, x; k, \breve{h}^\gamma(k))$ be the associated state. Find the control $k \in L^2(\omega_2^T)$ such that*

$$\breve{q}^\gamma(0) = q(0, a, x; k, \breve{h}^\gamma) = 0 \quad \text{in} \quad \Omega^A. \tag{7}$$

Remark 2 *In this paper:*

- *α represents a weighting parameter in the cost function. It controls the importance of minimizing the term $\|h\|^2$, which corresponds to the effort or intensity of the control h.*

- *$z_d \in L^2(Q)$ is a target state or a desired density for the harmful population in the domain Q.*
- *The relation (6) measures the difference between the current state q and the target state z_d, while adding a term to minimize the control effort h. The goal is to minimize J by finding an optimal control h.*
- *$\gamma > 0$ is a parameter that controls the impact of the uncertainty g in the cost function J_1.*

The Stackelberg leadership model is a multiple-objective optimization approach proposed by H. Von Stackelberg in [1]. This model involves two companies (controls) which compete on the market of the same product. The first(leader) to act must integrate the reaction of the other firms (followers) in the choices it makes in the amount of product that it decides to put on the market. There are several works in the literature dealing with Stackelberg strategy for distributed systems. J. L. Lions in [2] used the Stackelberg strategy on a system governed by a parabolic equation subjected to two controls. For more literature on stackelberg control of parabolic systems, we refer the reader to [3, 4, 5, 6, 7, 8, 9, 10]. Concerning Stackelberg control in population dynamics, we can cite the work of M. Mercan et al. in [11] and G. Mophou et al. studied in [12] the hierarchical control for a population dynamics model with the distribution of newborns as unknown.
In this paper, we propose a Stackelberg control problem with the supply of the invasive species as unknown. We believe that such a Stackelberg control problem has not yet been considered . This problem is ill-posed, we can not directly solve the associated optimal control problem. We use of the low-regret control developed by J. L. Lions for the follower, and appropriate Carleman for the leader. The main difficulty lies in obtaining the Carleman-type observability inequality associated with the adjoint system, where we have sometimes had to resort to the Poincaré inequality. More precisely, we prove the following results. The result obtain when solving problem(1) is as follows

Theorem 1 *Let Ω be a bounded subset of $\mathbb{R}^n$, $n \geq 1$ with boundary Γ of class C^2. Let ω_1 and ω_2 be nonempty subsets of Ω with $\omega_1 \subset \omega_2$. Let also $k \in L^2(\omega_2^{TA})$ and $\gamma > 0$. Then there exits $\left(p^\gamma \ , \tau^\gamma, \delta^\gamma \ \right)$ such that the optimization problem (4) has a unique solution $h^\gamma = h^\gamma(k) \in L^2(\omega_1^{TA})$ which is characterized by the following optimality system :*

$$\left\{ \begin{array}{rcll} -\dfrac{\partial q^\gamma}{\partial t} - \dfrac{\partial q^\gamma}{\partial a} - \Delta q^\gamma + \mu q^\gamma & = & h^\gamma \chi_{\omega 1} + k \chi_{\omega 2} & \text{in} \quad Q, \\ q^\gamma & = & 0 & \text{on} \quad \Sigma, \\ q^\gamma(T,.,.) & = & 0 & \text{in} \quad \Omega^A, \\ q^\gamma(.,A,.) & = & 0 & \text{in} \quad \Omega^T, \end{array} \right. \tag{8}$$

$$\left\{\begin{array}{rcll}\dfrac{\partial p^\gamma}{\partial t}+\dfrac{\partial p^\gamma}{\partial a}-\Delta p^\gamma+\mu p^\gamma &=& q^\gamma - z_d+\dfrac{1}{\sqrt{\gamma}}\delta^\gamma & \text{in} \quad Q,\\ p^\gamma &=& 0 & \text{on} \quad \Sigma,\\ p^\gamma(0,.,.) &=& 0 & \text{in} \quad \Omega^A,\\ p^\gamma(.,0,.) &=& 0 & \text{in} \quad \Omega^T,\end{array}\right. \tag{9}$$

$$\left\{\begin{array}{rcll}-\dfrac{\partial \tau^\gamma}{\partial t}-\dfrac{\partial \tau^\gamma}{\partial a}-\Delta \tau^\gamma+\mu \tau^\gamma &=& q^\gamma & \text{in} \quad Q,\\ \tau^\gamma &=& 0 & \text{on} \quad \Sigma,\\ \tau^\gamma(0,.,.) &=& 0 & \text{in} \quad \Omega^A,\\ \tau^\gamma(.,0,.) &=& 0 & \text{in} \quad \Omega^T,\end{array}\right. \tag{10}$$

$$\left\{\begin{array}{rcll}-\dfrac{\partial \delta^\gamma}{\partial t}-\dfrac{\partial \delta^\gamma}{\partial a}-\Delta \delta^\gamma+\mu \delta^\gamma &=& \dfrac{1}{\sqrt{\gamma}}\tau^\gamma & \text{in} \quad Q,\\ \delta^\gamma &=& 0 & \text{on} \quad \Sigma,\\ \delta^\gamma(T,.,.) &=& 0 & \text{in} \quad \Omega^A,\\ \delta^\gamma(.,A,.) &=& 0 & \text{in} \quad \Omega^T,\end{array}\right. \tag{11}$$

$$h^\gamma=-\frac{p^\gamma}{\alpha} \text{ in } \omega_1^{TA}. \tag{12}$$

Moreover there exists a constant $C=C(\gamma,\alpha,T)>0$ such that

$$\|h^\gamma\|_{L^2(\omega_1^{TA})}\leq C\big(\|k\|_{L^2(\omega_2^{TA})}+\|z_d\|_{L^2(Q)}\big). \tag{13}$$

Remark 3

- *In relation (12), the control h^γ applied in a subregion ω_1^{TA} is linked to a "correction pressure" p^γ, which reflects the deviation between the current state of the system and a target (such as reducing the invasive population or achieving a desired distribution). p^γ can be interpreted as a measure of sensitivity or the marginal cost associated with the control decisions h^γ. The factor $\alpha>0$ acts as a weighting parameter, limiting the intensity of the control. Physically, this could represent a constraint on the resources available to implement the control (e.g., limits on pesticides or intervention measures).*
  In summary, the control h^γ is proportional to a local measure of effort (p^γ), modulated by the available resources (α).

- *Relation (13) imposes a global limitation on the total intensity of the control h^γ, as a function of the control k and the desired state z_d. Physically, this could correspond to coordination between two management teams operating in their respective regions (ω_1^{TA} and ω_2^{TA}).*

The result obtain when solving problem(2) is as follows

Theorem 2 *Assume that the assumptions of Theorem 1 hold. Then there exists a positive real weight function θ to be define later by (68) such that, for any function $z_d\in L^2(Q)$ with $\theta z_d\in L^2(Q)$, then there exists a unique control $\breve{k}^\gamma\in L^2(\omega_2^{TA})$ such that $\big(\breve{k}^\gamma,\breve{q}^\gamma,\breve{p}^\gamma,\breve{\tau}^\gamma,\breve{\delta}^\gamma\big)$ is the solution of the null controllability problem (7)-(8). Moreover*

$$\breve{k}^\gamma=\breve{\rho}^\gamma \text{ in } \omega_2^{TA}, \tag{14}$$

$$\begin{cases} \dfrac{\partial \breve{\rho}^\gamma}{\partial t} + \dfrac{\partial \breve{\rho}^\gamma}{\partial a} - \Delta \breve{\rho}^\gamma + \mu \breve{\rho}^\gamma & = & \breve{\Psi}^\gamma + \breve{\varpi}^\gamma & \text{in} & Q, \\ \breve{\rho}^\gamma & = & 0 & \text{on} & \Sigma, \\ \breve{\rho}^\gamma(.,0,.) & = & 0 & \text{in} & \Omega^T, \end{cases} \tag{15}$$

$$\begin{cases} -\dfrac{\partial \breve{\Psi}^\gamma}{\partial t} - \dfrac{\partial \breve{\Psi}^\gamma}{\partial a} - \Delta \breve{\Psi}^\gamma + \mu \breve{\Psi}^\gamma & = & -\dfrac{\rho^\gamma}{\alpha}\chi_{\omega_1} & \text{in} & Q, \\ \breve{\Psi}^\gamma & = & 0 & \text{on} & \Sigma, \\ \breve{\Psi}^\gamma(T,.,.) & = & 0 & \text{in} & \Omega^A, \\ \breve{\Psi}^\gamma(.,A,.) & = & 0 & \text{in} & \Omega^T, \end{cases} \tag{16}$$

$$\begin{cases} -\dfrac{\partial \breve{\varpi}^\gamma}{\partial t} - \dfrac{\partial \breve{\varpi}^\gamma}{\partial a} - \Delta \breve{\varpi}^\gamma + \mu \breve{\varpi}^\gamma & = & \dfrac{1}{\sqrt{\gamma}}\breve{\zeta}^\gamma & \text{in} & Q, \\ \breve{\varpi}^\gamma & = & 0 & \text{on} & \Sigma, \\ \breve{\varpi}^\gamma(T,.,.) & = & 0 & \text{in} & \Omega^A, \\ \breve{\varpi}^\gamma(.,A,.) & = & 0 & \text{in} & \Omega^T, \end{cases} \tag{17}$$

$$\begin{cases} \dfrac{\partial \breve{\zeta}^\gamma}{\partial t} + \dfrac{\partial \breve{\zeta}^\gamma}{\partial a} - \Delta \breve{\zeta}^\gamma + \mu \breve{\zeta}^\gamma & = & \dfrac{1}{\sqrt{\gamma}}\breve{\Psi}^\gamma & \text{in} & Q, \\ \breve{\zeta}^\gamma & = & 0 & \text{on} & \Sigma, \\ \breve{\zeta}^\gamma(0,.,.) & = & 0 & \text{in} & \Omega^A, \\ \breve{\zeta}^\gamma(.,0,.) & = & 0 & \text{in} & \Omega^T, \end{cases} \tag{18}$$

and

$$\|\breve{k}^\gamma\|_{L^2(\omega_2^{TA})} \leq C\|\theta z_d\|_{L^2(Q)}. \tag{19}$$

Remark 4

- *In relation* (14)*,* k^γ *acts as a direct response to the needs expressed by* ρ^γ*, which reflects the effort required to meet the control objectives.*
- *In relation* (19)*, the total intensity of* k^γ *is bounded by the needs expressed through* z_d*, weighted by* θ*. This bound ensures that the control remains proportional to the set objectives while maintaining overall system stability.*

Remark 5 *The rest of the work will be organized as follows. In Section 2, we study Problem* (1) *corresponding to solving the optimal control problem. In Section 3 we establish an appropriate inequality of the Carleman type and give the proof of Theorem 2. Finally, in Section 4 we conclude.*

Let us now move on to a detailed analysis of the first objective of the problem, which aims to minimize the regret associated with the follower's control in a hierarchical framework.

2 Study of Problem 1: low-regret problem

2.1 Reformulation of the optimization Problem 1

Lemma 2.1 *For $g \in \mathcal{O}$, $h \in L^2(\omega_1^{TA})$ and $k \in L^2(\omega_2^{TA})$. Then, we have:*

$$J(k,h,g) - J(0,0,g) = J(k,h,0) - \|z_d\|^2_{L^2(Q)} + 2\int_Q \tau(t,a,x;k,h)g \; dtdadx, \quad (20)$$

where $\tau = \tau(t,a,x;k,h) \in L^2((0,T)\times(0,A);H_0^1(\Omega))$ is solution of :

$$\left\{\begin{array}{rcll} \dfrac{\partial \tau}{\partial t} - \dfrac{\partial \tau}{\partial a} - \Delta\tau + \mu\tau &=& q(k,h,0) & \text{in} \quad Q, \\ \tau &=& 0 & \text{on} \quad \Sigma, \\ \tau(T,.,.) &=& 0 & \text{in} \quad \Omega^A, \\ \tau(.,A,.) &=& 0 & \text{in} \quad \Omega^T. \end{array}\right. \quad (21)$$

Proof. From the uniqueness of the solutions (1), we have the following decomposition:

$$q(k,h,g) = q(k,h,0) + q(0,0,g). \quad (22)$$

It is important to note that

$$J(k,h,0) = \|q(k,h,0) - z_d\|^2_{L^2(Q)} + \alpha\|h\|^2_{L^2(\omega_1^{TA})}, \quad (23)$$

$$J(0,0,g) = \|q(0,0,g) - z_d\|^2_{L^2(Q)}. \quad (24)$$

According to (4), (22), (23) and (24), we have:

$$J(k,h,g) - J(0,0,g) = J(k,h,0) - \|z_d\|^2_{L^2(Q)} + 2\langle q(k,h,0); q(0,0,g)\rangle_{L^2(Q)}. \quad (25)$$

Now, if we multiply the first equation of (21) by $q(0,0,g)$ and integrate by parts over Q, we obtain

$$\langle q(k,h,0); q(0,0,g)\rangle_{L^2(Q)} = \int_Q \tau(t,a,x;k,h)g \; dtdadx.$$

Combining the latter inequality with (25), we obtain (20) ■

Lemma 2.2 *For k fixed in $L^2(\omega_2^{TA})$ and $\gamma > 0$. Then the optimisation problem (4) is equivalent to the following optimal control problem: find the control $h^\gamma := h^\gamma(k) \in L^2(\omega_1^{TA})$ such that*

$$\Upsilon(h^\gamma) = \inf_{h \in L^2(\omega_1^{TA})} \Upsilon(h), \quad (26)$$

where

$$\Upsilon(h) = J(k,h,0) - \|z_d\|^2_{L^2(Q)} + \frac{1}{\gamma}\|\tau(t,a,x;k,h)\|^2_{L^2(Q)}. \quad (27)$$

Proof. According to (20), the optimization problem (4) is equivalent to

$$\inf_{h\in L^2(\omega_1^{TA})}[J(k,h,0)-\|z_d\|^2_{L^2(Q)}+2\sup_{g\in\mathcal{O}}\left(\int_Q \tau(t,a,x;k,h)g\ dtdadx-\frac{\gamma}{2}\|g\|^2_{L^2(Q)}\right)]. \tag{28}$$

Using the Frenchel-legendre transform we have the following result

$$\sup_{g\in\mathcal{O}}\left(\langle\tau(t,a,x,k,h),g\rangle_{L^2(Q)}-\frac{\gamma}{2}\|g\|^2_{L^2(Q)}\right)=\frac{1}{2\gamma}\|\tau(t,a,x;k,h)\|^2_{L^2(Q)}. \tag{29}$$

Therefore, Using the previous equality the optimization problem (4) becomes:

$$\inf_{h\in L^2(\omega_1^{TA})}[J(k,h,0)-\|z_d\|^2_{L^2(Q)}+\frac{1}{\gamma}\|\tau(t,a,x;k,h)\|^2_{L^2(Q)}]. \tag{30}$$

■

We now demonstrate that the reformulated problem admits a unique solution, characterized by optimality conditions that we will establish.

2.2 Proof of Theorem 1

Let $\gamma>0$. We have $\Upsilon(h)\geq -\|z_d\|^2_{L^2(Q)}$ $\forall h\in L^2(\omega_1^{TA})$, hence the set $\{\Upsilon(h), h\in L^2(\omega_1^{TA})\}$ is nonempty and lowered by $\mathbb{R}$, and therefore the $\inf_{h\in L^2(\omega_1^{TA})}\Upsilon(h)$ exists. We can prove using minimizing sequences and standart arguments that there exists a unique Low-regret control h^γ solution to problem (26).
Now, Let us write the Euler Lagrange optimality conditions which characterize the optimal control h^γ:

$$\lim_{\lambda\to 0}\frac{\Upsilon(h^\gamma+\lambda h)-\Upsilon(h^\gamma)}{\lambda}=0, \qquad \forall \quad h\in L^2(\omega_1^{TA}). \tag{31}$$

After some calculations, (31) gives

$$\langle\bar{q};q^\gamma-z_d\rangle_{L^2(Q)}+\alpha\langle h^\gamma;h\rangle_{L^2(\omega_1^{TA})}+\langle\frac{1}{\sqrt{\gamma}}\bar{\tau};\frac{1}{\sqrt{\gamma}}\tau^\gamma\rangle_{L^2(Q)}=0, \tag{32}$$

where $\bar{q}=\bar{q}(t,a,x;0,h,0)\in L^2((0,T)\times(0,A);H_0^1(\Omega)))$ and $\bar{\tau}=\bar{\tau}(t,a,x;0,h)\in L^2((0,T)\times(0,A);H_0^1(\Omega))$ are respectively solution of

$$\left\{\begin{array}{rcll} -\dfrac{\partial\bar{q}}{\partial t}-\dfrac{\partial\bar{q}}{\partial a}-\Delta\bar{q}+\mu\bar{q} &=& h\chi_{\omega 1} & \text{in} \quad Q,\\ \bar{q} &=& 0 & \text{on} \quad \Sigma,\\ \bar{q}(T,.,.) &=& 0 & \text{in} \quad \Omega^A,\\ \bar{q}(.,A,.) &=& 0 & \text{in} \quad \Omega^T, \end{array}\right. \tag{33}$$

and

$$\left\{\begin{array}{rcll} \dfrac{\partial\bar{\tau}}{\partial t}+\dfrac{\partial\bar{\tau}}{\partial a}-\Delta\bar{\tau}+\mu\bar{\tau} &=& \bar{q} & \text{in} \quad Q,\\ \bar{\tau} &=& 0 & \text{on} \quad \Sigma,\\ \bar{\tau}(0,.,.) &=& 0 & \text{in} \quad \Omega^A,\\ \bar{\tau}(.,0,.) &=& 0 & \text{in} \quad \Omega^T. \end{array}\right. \tag{34}$$

To interpret (32), we consider the adjoint states p^γ and δ^γ respectively solution of (9) and (11). If we multiply the first equation of (34) and (33) respectively by $\dfrac{1}{\sqrt{\gamma}}\delta^\gamma$ and p^γ, integrate by parts over Q and using (32), we obtain:

$$h^\gamma = -\frac{p^\gamma}{\alpha} \text{ in } \omega_1^{TA}.$$

At present, we decompose q^γ and τ^γ the respectively solution of (1) and (21) as follows:

$$q^\gamma = \bar{q}(h^\gamma) + \mathcal{L}(k) \quad \textit{and} \quad \tau^\gamma = \bar{\tau}(h^\gamma) + \Theta, \tag{35}$$

where $\mathcal{L}(k)$ and Θ are respectively solution of

$$\begin{cases} -\dfrac{\partial \mathcal{L}}{\partial t} - \dfrac{\partial \mathcal{L}}{\partial a} - \Delta\mathcal{L} + \mu\mathcal{L} &= k\chi_{\omega 2} \quad \text{in} \quad Q, \\ \mathcal{L} &= 0 \quad \text{on} \quad \Sigma, \\ \mathcal{L}(T,.,.) &= 0 \quad \text{in} \quad \Omega^A, \\ \mathcal{L}(.,A,.) &= 0 \quad \text{in} \quad \Omega^T, \end{cases} \tag{36}$$

and

$$\begin{cases} \dfrac{\partial \Theta}{\partial t} + \dfrac{\partial \Theta}{\partial a} - \Delta\Theta + \mu\Theta &= \bar{q} \quad \text{in} \quad Q, \\ \Theta &= 0 \quad \text{on} \quad \Sigma, \\ \Theta(0,.,.) &= 0 \quad \text{in} \quad \Omega^A, \\ \Theta(.,0,.) &= 0 \quad \text{in} \quad \Omega^T. \end{cases} \tag{37}$$

Moreover there exits a constant $C = C(T) > 0$ such that

$$\|\bar{q}\|_{L^2((0,T);H_0^1(\Omega))} \leq C\|h^\gamma\|_{L^2(\omega_1^{TA})}, \tag{38}$$

$$\|\mathcal{L}\|_{L^2((0,T);H_0^1(\Omega))} \leq C\|k\|_{L^2(\omega_2^{TA})}, \tag{39}$$

$$\|\bar{\tau}\|_{L^2((0,T);H_0^1(\Omega))} \leq C\|h^\gamma\|_{L^2(\omega_1^{TA})}, \tag{40}$$

$$\|\Theta\|_{L^2((0,T);H_0^1(\Omega))} \leq C\|k\|_{L^2(\omega_2^{TA})}. \tag{41}$$

According to the Euler Lagrange conditions given by (32) and the decomposition of q^γ and τ^γ given by (35), for any $h \in L^2(\omega_1^{TA})$, we have

$$0 = \pi(h, h^\gamma) + \langle \bar{q}(h); \mathcal{L}(k) - z_d\rangle_{L^2(Q)} + \frac{1}{\gamma}\langle \bar{\tau}(h); \Theta\rangle_{L^2(Q)}, \tag{42}$$

where

$$\pi(h^\gamma, h) = \langle \bar{q}(h); \bar{q}(h^\gamma)\rangle_{L^2(Q)} + \alpha\langle h^\gamma; h\rangle_{L^2(\omega_1^T)} + \frac{1}{\gamma}\langle \bar{\tau}(h); \bar{\tau}(h^\gamma)\rangle_{L^2(Q)}.$$

Taking $h = h^\gamma$ in (42), using Cauchy Schwarz inequality and (38)-(41), we have

$$\pi(h^\gamma, h^\gamma) \leq C(\gamma, T)\|h^\gamma\|_{L^2(\omega_1^{TA})}\big(\|k\|_{L^2(\omega_1^T)} + \|z_d\|_{L^2(Q)}\big).$$

Then (13) is true.

The analysis of leader control is based on obtaining a Carleman-type observability inequality, which we present and prove in this section.

3 Carleman inequality

We use Carleman inequalities to provide an observability estimate, which is essential for solving the null controllability problem. To this end, for $\rho_0 \in L^2(\Omega^A)$, we consider the adjoint systems of (8)-(11):

$$\begin{cases} \dfrac{\partial \rho^\gamma}{\partial t} + \dfrac{\partial \rho^\gamma}{\partial a} - \Delta \rho^\gamma + \mu \rho^\gamma &= \Psi^\gamma + \varpi^\gamma \quad \text{in} \quad Q, \\ \rho^\gamma &= 0 \quad \text{on} \quad \Sigma, \\ \rho^\gamma(0,.,.) &= \rho^0 \quad \text{in} \quad \Omega^A, \\ \rho^\gamma(.,0,.) &= 0 \quad \text{in} \quad \Omega^T, \end{cases} \tag{43}$$

$$\begin{cases} -\dfrac{\partial \Psi^\gamma}{\partial t} - \dfrac{\partial \Psi^\gamma}{\partial a} - \Delta \Psi^\gamma + \mu \Psi^\gamma &= -\dfrac{1}{\alpha}\rho^\gamma \chi_{\omega 1} \quad \text{in} \quad Q, \\ \Psi^\gamma &= 0 \quad \text{on} \quad \Sigma, \\ \Psi^\gamma(T,.,.) &= 0 \quad \text{in} \quad \Omega^A, \\ \Psi^\gamma(.,A,.) &= 0 \quad \text{in} \quad \Omega^T, \end{cases} \tag{44}$$

$$\begin{cases} -\dfrac{\partial \varpi^\gamma}{\partial t} - \dfrac{\partial \varpi^\gamma}{\partial a} - \Delta \varpi^\gamma + \mu \varpi^\gamma &= \dfrac{1}{\sqrt{\gamma}}\zeta^\gamma \quad \text{in} \quad Q, \\ \varpi^\gamma &= 0 \quad \text{on} \quad \Sigma, \\ \varpi^\gamma(T,.,.) &= 0 \quad \text{in} \quad \Omega^A, \\ \varpi^\gamma(.,A,.) &= 0 \quad \text{in} \quad \Omega^T, \end{cases} \tag{45}$$

$$\begin{cases} \dfrac{\partial \varsigma^\gamma}{\partial t} + \dfrac{\partial \varsigma^\gamma}{\partial a} - \Delta \varsigma^\gamma + \mu \varsigma^\gamma &= \dfrac{1}{\sqrt{\gamma}}\Psi^\gamma \quad \text{in} \quad Q, \\ \varsigma^\gamma &= 0 \quad \text{on} \quad \Sigma, \\ \varsigma^\gamma(0,.,.) &= 0 \quad \text{in} \quad \Omega^A, \\ \varsigma^\gamma(.,0,.) &= 0 \quad \text{in} \quad \Omega^T. \end{cases} \tag{46}$$

If we set $\varrho^\gamma = \Psi^\gamma + \varpi^\gamma$, we have

$$\begin{cases} -\dfrac{\partial \varrho^\gamma}{\partial t} - \dfrac{\partial \varsigma^\gamma}{\partial a} - \Delta \varrho^\gamma + \mu \varrho^\gamma &= \dfrac{1}{\sqrt{\gamma}}\varsigma^\gamma - \dfrac{1}{\alpha}\rho^\gamma \chi_{\omega 1} \quad \text{in} \quad Q, \\ \varrho^\gamma &= 0 \quad \text{on} \quad \Sigma, \\ \varrho^\gamma(A,.,.) &= 0 \quad \text{in} \quad \Omega^A, \\ \varrho^\gamma(.,T,.) &= 0 \quad \text{in} \quad \Omega^T. \end{cases} \tag{47}$$

Let ω_0 be an open set such that $\omega_0 \subset \omega' \subset \omega_2 \subset\subset \Omega$. Then there exists $\psi \in C^2(\overline{\Omega})$ such that:

$$\begin{cases} \psi(x) > 0 \quad \forall x \in \Omega; \quad \nabla \psi(x) \neq 0 \quad \forall\, x \in \overline{\Omega - \omega_0} \\ \psi(x) = 0 \quad \forall x \in \Gamma. \end{cases} \tag{48}$$

For any $\lambda > 0$, we use the function ψ defined previously to construct the weight functions φ and η define by:

$$\varphi(t,a,x) = \frac{e^{\lambda \psi(x)}}{t(T-t)a(A-a)}, \tag{49}$$

$$\eta(t,a,x) = \frac{e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}}{t(T-t)a(A-a)}. \tag{50}$$

Let $f \in L^2(Q)$ and $z \in L^2((0,T)\times(0,A);H_0^1(\Omega))$ be the solution of

$$\begin{cases} Lz &= f \text{ in } Q, \\ z &= 0 \text{ on } \Sigma. \end{cases} \tag{51}$$

Then, the following proposition provides the expression of a global Carleman inequality associated with system (51).

Proposition 3.1 *[12] Let $\omega_0 \subset \omega' \subset \omega_2 \subset\subset \Omega$. Let also ψ, φ and η be defined as in* (48), (49) *and* (50) *respectively. There exist constants $\lambda_0 > 1$ and $C = C(\psi) > 0$ such that for all $\lambda > \lambda_0$, for all $s > s_0$ and for all $z \in L^2((0,T)\times(0,A);H_0^1(\Omega))$, we have :*

$$\mathcal{K}(z) \leq C_1\Big(\int_Q e^{-2s\eta} \mid f \mid^2 \ dtdadx + s^3\lambda^4 \int_0^T\int_0^A\int_{\omega'} e^{-2s\eta}\varphi^3 \mid z \mid^2 \ dtdadx\Big). \tag{52}$$

where

$$s_0(\lambda) = C_1(\psi)\frac{TA}{4}e^{2\lambda\|\psi\|_\infty}\big(\frac{T^2A^2}{4} + T^2A^3 + T^3A^2 + T + A\big) \text{ and}$$

$$\mathcal{K}(z) = s\lambda\int_Q e^{-2s\eta}\varphi|\nabla z|^2 \ dtdadx + s^3\lambda^4\int_Q e^{-2s\eta}\varphi^3 \mid z \mid^2 \ dtdadx. \tag{53}$$

The following proposition provides the expression of a Carleman inequality adapted to the adjoint systems (43)-(46)

Proposition 3.2 *Let $\omega_0 \subset \omega' \subset \omega_2 \subset\subset \Omega$. Let also ψ, φ and η be defined respectively by* (48), (49) *and* (50) *then there exist a constant $C = C(\psi,\gamma,\alpha,T,s,\lambda) > 0$ and positives weight functions κ and θ to be define respectively by* (68) *and* (70) *such that the following estimate holds for all $(\rho^\gamma,\Psi^\gamma)$ solutions of* (43) *and* (44)

$$\|\kappa\Psi^\gamma\|^2_{L^2(Q)} + \left\|\frac{1}{\theta^2}\rho^\gamma\right\|^2_{L^2(Q)} \leq C\int_{\omega_2^{TA}} \mid \rho^\gamma \mid^2 \ dtdadx, \tag{54}$$

Proof. In the first time we apply inequality (52) to $(\rho^\gamma,\varrho^\gamma)$, the respective solutions of (43) and (47). We obtain the following expressions:

$$\mathcal{K}(\rho^\gamma) \leq C(\psi)\Big(\int_Q e^{-2s\eta} \mid \varrho^\gamma \mid^2 \ dtdadx + s^3\lambda^4\int_0^T\int_0^A\int_{\omega'} e^{-2s\eta}\varphi^3 \mid \rho^\gamma \mid^2 \ dtdadx\Big), \tag{55}$$

and

$$\mathcal{K}(\varrho^\gamma) \leq C(\psi)\Big(\frac{1}{(\alpha\gamma)^2}C(T)\|\rho^\gamma\|^2_{L^2(\omega_1^{TA})} + \frac{1}{\alpha^2}\int_0^T\int_0^A\int_{\omega_1} e^{-2s\eta} \mid \rho^\gamma \mid^2 \; dtdadx$$
$$+ s^3\lambda^4\int_0^T\int_0^A\int_{\omega'} e^{-2s\eta}\varphi^3 \mid \varrho^\gamma \mid^2 \; dtdadx\Big). \tag{56}$$

Indeed, taking into account Poincar inequality, there exists a positive constant $C = C(T)$ such that

$$\frac{1}{\gamma}\|e^{-s\eta}\varsigma^\gamma\|^2_{L^2((0,T)\times(0,A);H_0^1(\Omega))} \leq \frac{1}{\gamma^2}C(T)\|\Psi^\gamma\|^2_{L^2((0,T)\times(0,A);H_0^1(\Omega))},$$
$$\leq \frac{1}{(\alpha\gamma)^2}C(a_0,T)\|\rho^\gamma\|^2_{L^2(\omega_1^{TA})}. \tag{57}$$

Since $e^{-2s\eta}\varphi^3 \in L^\infty(Q)$, inequality (56) becomes

$$\mathcal{K}(\varrho^\gamma) \leq C(\psi)\Big(\frac{1+\gamma^2}{(\alpha\gamma)^2}C(T)\int_0^T\int_0^A\int_{\omega_1} e^{-2s\eta}\varphi^3 \mid \rho^\gamma \mid^2 \; dtdadx$$
$$+ s^3\lambda^4\int_0^T\int_0^A\int_{\omega_0} e^{-2s\eta}\varphi^3 \mid \varrho^\gamma \mid^2 \; dtdadx\Big). \tag{58}$$

Since $\omega_1 \subset \Omega$, $s, \lambda > 0$, and $\varphi^{-1} \in L^\infty(Q)$, combining the inequalities given by relations (55) and (58), we obtain

$$\mathcal{K}(\rho^\gamma) + \mathcal{K}(\varrho^\gamma) \leq C(\psi)s^3\lambda^4\int_0^T\int_0^A\int_{\omega_0} e^{-2s\eta}\varphi^3\big(\mid \varrho^\gamma \mid^2 + \mid \rho^\gamma \mid^2 \big)\, dtdadx$$
$$+ \Big(\frac{1+\gamma^2}{(\alpha\gamma)^2}+1\Big)C(\psi)s^2\lambda^4\int_Q e^{-2s\eta}\varphi^3\big(\mid \varrho^\gamma \mid^2 + \mid \rho^\gamma \mid^2 \big)\, dtdadx.$$

Since $\omega_0 \subset \omega_2$, by choosing s such that $s \geq s_1 = \max\Big\{s_0, 2\Big(\frac{1+\gamma^2}{(\alpha\gamma)^2}+1\Big)C(\psi)\Big\}$, we have

$$\mathcal{K}(\rho^\gamma) + \mathcal{K}(\varrho^\gamma) \leq C_1 s^3\lambda^4\int_0^T\int_0^A\int_{\omega_2} e^{-2s\eta}\varphi^3\big(\mid \rho^\gamma \mid^2 + \mid \varrho^\gamma \mid^2 \big)\, dtdadx, \tag{59}$$

where $C = C(\psi, \gamma, \alpha, T) > 0$.

In the second time, we consider as in [12] the function $\theta \in C_0^\infty(\Omega)$ and such that

$$0 \leq \theta \leq 1, \quad \theta = 1 \quad in \quad \omega', \quad \theta = 0 \quad in \quad \Omega \setminus \omega_2,$$
$$\frac{\Delta\theta}{\theta^{\frac{1}{2}}} \in L^\infty(\omega_2), \quad \frac{\nabla\theta}{\theta^{\frac{1}{2}}} \in [L^\infty(\omega_2)]^N. \tag{60}$$

Set $u = s^3\lambda^4\varphi^3 e^{-2s\eta}$. If we multiply the first equation of (43) by $u\theta\varrho^\gamma$ and integrate by parts over Q, using the Young inequality, we obtain

$$\begin{aligned}\int_{\omega'^{TA}} u \mid \varrho^\gamma \mid^2 dxdt \leq & \sum_{i=1}^{5} \frac{\mu_i}{2} \int_{\omega'^{TA}} u \mid \varrho^\gamma \mid^2 dtdadx \\ & + \frac{1}{2}\int_Q s\lambda\varphi e^{-2s\eta} \mid \nabla\varrho^\gamma \mid^2 dtdadx \\ & + C(\psi)\int_{\omega_2^{TA}} s^7\lambda^{10}\varphi^7 e^{-2s\eta} \mid \rho^\gamma \mid^2 dtdadx \\ & + \frac{1}{\alpha}\int_{\omega_2^{TA}} s^3\lambda^4\varphi^3 e^{-2s\eta} \mid \rho^\gamma \mid^2 dtdadx \\ & + \frac{\mu_2}{2}\int_{\omega'^{TA}} s^3\lambda^4\varphi^3 e^{-2s\eta} \mid \varsigma^\gamma \mid^2 dtdadx. \qquad (61)\end{aligned}$$

Since $e^{-2s\eta}\varphi^3 \in L^\infty(Q)$. Using (46),(3) and the Poincaré inequality, we have

$$\int_{\omega'^{TA}} s^3\lambda^4\varphi^3 e^{-2s\eta} \mid \varsigma^\gamma \mid^2 dtdadx \leq \frac{1}{\gamma\alpha^2} C(T)\int_{\omega_2^{TA}} s^3\lambda^4\varphi^3 e^{-2s\eta} \mid \rho^\gamma \mid^2 dtdadx. \qquad (62)$$

Choosing $\sum_{i=1}^{5} \frac{\mu_i}{2} = \frac{1}{2}$. If we Combining (61), (62) with (**??**), we obtain

$$\begin{aligned}\mathcal{K}(\rho^\gamma) + \mathcal{K}(\varrho^\gamma) \leq & C(\psi,\gamma,\alpha,T)\int_Q s\lambda\varphi e^{-2s\eta} \mid \nabla\varrho^\gamma \mid^2 dtdadx \\ & + C(\psi,\gamma,\alpha,T)\int_{\omega_2^{TA}} s^7\lambda^9\varphi^7 e^{-2s\eta} \mid \rho^\gamma \mid^2 dtdadx \\ & + \left(\frac{1}{\gamma\alpha^2} + \frac{2}{\alpha} + 1\right) C(\psi,\gamma,\alpha,T)\int_{\omega_2^{TA}} s^3\lambda^4\varphi^3 e^{-2s\eta} \mid \rho^\gamma \mid^2 dtdadx.\end{aligned}$$

Using (53) and choosing $\lambda \geq \lambda_1 = max(\lambda_0, 2C(\psi,\gamma,\alpha,T)$, we have

$$\begin{aligned}\mathcal{K}(\rho^\gamma) + \mathcal{K}(\varrho^\gamma) \leq & C(\psi,\gamma,\alpha,T)\int_{\omega_2^{TA}} s^7\lambda^9\varphi^7 e^{-2s\eta} \mid \rho^\gamma \mid^2 dtdadx \\ & + \left(\frac{1}{\gamma\alpha^2} + \frac{2}{\alpha} + 1\right) C(\psi,\gamma,T)\int_{\omega_2^{TA}} s^3\lambda^4\varphi^3 e^{-2s\eta} \mid \rho^\gamma \mid^2 dtdadx.\end{aligned}$$

Set $s \geq s_1 = max(s_0, 2C(\psi,\gamma,\alpha,T)$, and $\lambda \geq \lambda_2 = max(\lambda_0, 2C(\psi,\gamma,\alpha,T)$ and using (53), we have

$$\int_{\omega_2^{TA}} e^{-2s\eta}\varphi^3 \mid \rho^\gamma \mid^2 dtdadx \leq C(\psi,\gamma,\alpha,T)s^7\lambda^9 \int_{\omega_2^{TA}} \varphi^7 e^{-2s\eta} \mid \rho^\gamma \mid^2 dtdadx. \qquad (63)$$

In the three time, we set

$$D = \{(t,a) \in [0;T] \times [0;A] \; such\ that \; t \geq \frac{T}{2} \; and \; a \geq \frac{A}{2}\}.$$

We define functions $\breve{\varphi}$ and $\breve{\eta}$ by:

$$\breve{\varphi}(t,a,x) = \begin{cases} \varphi\big(\frac{T}{2}, \frac{A}{2}, x\big), & \forall t \in \big[[0;T] \times [0;A]\big] \setminus D, \\ \varphi(t,a,x), & \forall (t,a) \in D, \end{cases} \tag{64}$$

and

$$\breve{\eta}(t,a,x) = \begin{cases} \eta\big(\frac{T}{2}, \frac{A}{2}, x\big), & \forall (t,a) \in \big[[0;T] \times [0;A]\big] \setminus D, \\ \eta(t,a,x), & \forall (t,a) \in D. \end{cases} \tag{65}$$

If we replace respectively η and φ by $\breve{\eta}$ and $\breve{\varphi}$ in (63), we have

$$\int_Q e^{-2s_1\breve{\eta}} \breve{\varphi}^3 \mid \rho^\gamma \mid^2 dtdadx \leq C(\psi,\gamma,\alpha,T) s^7 \lambda^9 \int_{\omega_2^{TA}} \breve{\varphi}^7 e^{-2s_1\breve{\eta}} \mid \rho^\gamma \mid^2 dtdadx. \tag{66}$$

We introduce the function

$$\hat{\eta}(t,a) = \underset{x\in\Omega}{max} \breve{\eta}(t,a,x), \tag{67}$$

and we set

$$\kappa(t,a) = e^{-s_1\hat{\eta}(t,a)}. \tag{68}$$

Multiply (44) by $\kappa^2\Psi^\gamma$ and integrate by part over Ω, we have

$$-\frac{1}{2}\frac{\partial}{\partial t}\|\kappa\Psi^\gamma\|^2_{L^2(\Omega)} - \frac{1}{2}\frac{\partial}{\partial a}\|\kappa\Psi^\gamma\|^2_{L^2(\Omega)} - \int_\Omega s_1 \frac{\partial \hat{\eta}}{\partial t}(\kappa\Psi^\gamma)^2 dx - \int_\Omega s_1 \frac{\partial \hat{\eta}}{\partial a}(\kappa\Psi^\gamma)^2 dx$$
$$+ \|\kappa\nabla\Psi^\gamma\|^2_{L^2(\Omega)} + \int_\Omega \mu\kappa^2 \mid \Psi^\gamma \mid^2 dx = -\frac{1}{\alpha}\int_\Omega \kappa^2 \rho^\gamma \Psi^\gamma dx.$$

Integrating this latter inequality over $(0,T)$, we obtain

$$\begin{aligned} &\|\kappa(0,.)\Psi^\gamma(0,a,.)\|_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\kappa\Psi^\gamma\|^2_{L^2(\Omega^A)} + \|\kappa(.,0)\Psi^\gamma(t,0,.)\|_{L^2(\Omega)} \\ &+ \frac{1}{2}\frac{d}{dt}\|\kappa\Psi^\gamma\|^2_{L^2(\Omega^T)} - \int_Q s_1 \frac{\partial\hat{\eta}}{\partial t}(\kappa\Psi^\gamma)^2 dtdadx - \int_Q s_1 \frac{\partial\hat{\eta}}{\partial a}(\kappa\Psi^\gamma)^2 dtdadx \\ &+ \|\kappa\nabla\Psi^\gamma\|^2_{L^2(Q)} + \int_Q \mu\kappa^2 \mid \Psi^\gamma \mid^2 dtdadx = -\frac{1}{\alpha}\int_Q \kappa^2\rho^\gamma\Psi^\gamma dtdadx. \end{aligned}$$

since $\mu > 0$, $s_1 > 0$, $\frac{\partial}{\partial t}\hat{\eta}(t,a)$ and $\frac{\partial}{\partial a}\hat{\eta}(t,a)$ are positive functions on $[0,T] \times [0,A]$, using the Young inequality and the Poincaré inequality, we obtain

$$\|\kappa(0,.)\Psi^\gamma(0,a,.)\|_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\kappa\Psi^\gamma\|^2_{L^2(\Omega^A)} + \|\kappa(.,0)\Psi^\gamma(t,0,.)\|_{L^2(\Omega)} + \frac{1}{2}\frac{d}{da}\|\kappa\Psi^\gamma\|^2_{L^2(\Omega)}$$
$$+ \left(\frac{1}{2} - \big(C(T,s_1) + C(A,s_1) + \|\mu\|_\infty\big)\right)\|\kappa\Psi^\gamma\|^2_{L^2((0,T)\times(0,A);H_0^1(\Omega))} \leq \frac{1}{2\alpha^2}\|\kappa\rho^\gamma\|^2_{L^2(Q)}.$$

We deduce that for $\left(\frac{1}{2} - \left(C(T,s_1) + C(A,s_1) + \|\mu\|_\infty\right)\right) > 0$, we have

$$\int_Q \kappa^2 \mid \Psi^\gamma \mid^2 dtdadx \leq \frac{1}{2\alpha^2} \int_Q e^{-2s\breve{\eta}} \breve{\varphi}^3 \mid \rho^\gamma \mid^2 dtdadx.$$

Combining this later inequality with (66), we obtain

$$\int_Q \kappa^2 \mid \Psi^\gamma \mid^2 dtdadx + \int_Q e^{-2s\breve{\eta}} \breve{\varphi}^3 \mid \rho^\gamma \mid^2 dtdadx \leq$$
$$C(\psi,\gamma,\alpha,T) s^7 \lambda^9 \varphi^7 \int_{\omega_2^{TA}} e^{-2s\breve{\eta}} \mid \rho^\gamma \mid^2 dtdadx \tag{69}$$

We set

$$\frac{1}{\theta^2} = e^{-2s\breve{\eta}} \breve{\varphi}^3. \tag{70}$$

Then it folows from (69) that there exists $C = C(\psi,\gamma,\alpha,T,s,\lambda) > 0$ such that (54) holds true ■

Building on the previous results, we now address the resolution of the second problem by demonstrating that the leader control enables the state to reach nullity from the initial condition.

4 Resolution of Problem 2:

In this section, we are concerned with the proof of Theorem 2. Recall that the main objective is to prove the null controllability of q^γ at time 0. For any $\gamma > 0$, we look for a control $k \in L^2(\omega_2^{TA})$ such that the solutions of (8)-(12) satisfies (7). To prove this null controllability problem, we proceed in three steps using a penalization method.

Step 1 *For any $\epsilon > 0$, we define the cost function:*

$$J_\epsilon(k) = \frac{1}{2\epsilon} \int_{\Omega^A} \mid q(0,.,.;k,h^\gamma(k),0) \mid^2 dadx + \frac{1}{2} \int_{\omega_2^{TA}} \mid k \mid^2 dtdadx. \tag{71}$$

Then we consider the optimal control problem: find $k_\epsilon^\gamma \in L^2(\omega_2^{TA})$ such that

$$J_\epsilon(k_\epsilon^\gamma) = \inf_{k \in L^2(\omega_2^{TA})} J_\epsilon(k). \tag{72}$$

Using minimizing sequences, we can prove that there exists a unique solution k_ϵ^γ to (72). Using an Euler-Lagrange first order optimality condition that characterizes the solution k_ϵ, we can prove that

$$k_\epsilon^\gamma = \rho_\epsilon^\gamma \quad \text{in} \quad \omega_2^{TA}, \tag{73}$$

$$\left\{\begin{array}{rcll}\dfrac{\partial\rho_\epsilon^\gamma}{\partial t}+\dfrac{\partial\rho_\epsilon^\gamma}{\partial a}-\Delta\rho_\epsilon^\gamma+\mu\rho_\epsilon^\gamma &=& \Psi_\epsilon^\gamma+\varpi_\epsilon^\gamma & \text{in} \quad Q,\\ \rho_\epsilon^\gamma &=& 0 & \text{on} \quad \Sigma,\\ \rho_\epsilon^\gamma(0,.,.) &=& -\dfrac{1}{\epsilon}q_\epsilon(0,.,.,k_\epsilon^\gamma,h(k_\epsilon^\gamma),0) & \text{in} \quad \Omega^A,\\ \rho_\epsilon^\gamma(.,0,.) &=& 0 & \text{in} \quad \Omega^T,\end{array}\right. \tag{74}$$

$$\left\{\begin{array}{rcll}-\dfrac{\partial\Psi_\epsilon^\gamma}{\partial t}-\dfrac{\partial\Psi_\epsilon^\gamma}{\partial a}-\Delta\Psi_\epsilon^\gamma+\mu\Psi_\epsilon^\gamma &=& -\dfrac{1}{\alpha}\rho_\epsilon^\gamma\chi_{\omega_1} & \text{in} \quad Q,\\ \Psi_\epsilon^\gamma &=& 0 & \text{on} \quad \Sigma,\\ \Psi_\epsilon^\gamma(T,.,.) &=& 0 & \text{in} \quad \Omega^A,\\ \Psi_\epsilon^\gamma(.,A,.) &=& 0 & \text{in} \quad \Omega^T,\end{array}\right. \tag{75}$$

$$\left\{\begin{array}{rcll}-\dfrac{\partial\varpi_\epsilon^\gamma}{\partial t}-\dfrac{\partial\varpi_\epsilon^\gamma}{\partial a}-\Delta\varpi_\epsilon^\gamma+\mu\varpi_\epsilon^\gamma &=& \dfrac{1}{\sqrt{\gamma}}\zeta_\epsilon^\gamma & \text{in} \quad Q,\\ \varpi_\epsilon^\gamma &=& 0 & \text{on} \quad \Sigma,\\ \varpi_\epsilon^\gamma(T,.,.) &=& 0 & \text{in} \quad \Omega^A,\\ \varpi_\epsilon^\gamma(.,A,.) &=& 0 & \text{in} \quad \Omega^T,\end{array}\right. \tag{76}$$

$$\left\{\begin{array}{rcll}\dfrac{\partial\varsigma_\epsilon^\gamma}{\partial t}+\dfrac{\partial\varsigma_\epsilon^\gamma}{\partial a}-\Delta\varsigma_\epsilon^\gamma+\mu\varsigma_\epsilon^\gamma &=& \dfrac{1}{\sqrt{\gamma}}\Psi_\epsilon^\gamma & \text{in} \quad Q,\\ \varsigma_\epsilon^\gamma &=& 0 & \text{on} \quad \Sigma,\\ \varsigma_\epsilon^\gamma(0,.,.) &=& 0 & \text{in} \quad \Omega^A,\\ \varsigma_\epsilon^\gamma(.,0,.) &=& 0 & \text{in} \quad \Omega^T,\end{array}\right. \tag{77}$$

and q_ϵ^γ, p_ϵ^γ, τ_ϵ^γ, δ_ϵ^γ *is the solution of systems* (8)-(11).

Step 2 *If we multipliy the first line in* (74)- (77) *respectively by* q_ϵ^γ, p_ϵ^γ, τ_ϵ^γ, δ_ϵ^γ *solution of* (8)-(11)*, and integrate by parts over* Q*, we obtain:*

$$\frac{1}{\epsilon}\|q_\epsilon(0,.,k_\epsilon^\gamma,h(k_\epsilon^\gamma),0)\|^2_{L^2(\Omega^A)}+\int_{\omega_2^{TA}}k_\epsilon^\gamma\rho_\epsilon^\gamma dtdadx=\int_Q z_d\Psi_\epsilon^\gamma dtdadx. \tag{78}$$

Since θz_d *and* $\frac{1}{\kappa}z_d$ *belongs to* $L^2(Q)$*. If we use Cauchy Schwarz inequality, we have*

$$\int_Q z_d\Psi_\epsilon^\gamma dtdadx\le\left\|\frac{1}{\kappa}z_d\right\|_{L^2(Q)}\|\kappa\Psi_\epsilon^\gamma\|_{L^2(Q)}. \tag{79}$$

Now, if we apply the carleman inequality (54) *to* Ψ_ϵ^γ *and* ρ_ϵ^γ*, using* (73), (79) *then there exits* $C=C(\psi,\gamma,\alpha,T,s,\lambda)$ *such that* (78) *can be rewritten*

$$\frac{1}{\epsilon}\|q_\epsilon(0,.,.,k_\epsilon^\gamma,h(k_\epsilon^\gamma),0)\|^2_{L^2(\Omega^A)}+\|k_\epsilon^\gamma\|^2_{L^2(\omega_2^{TA})}\le C\left\|\frac{1}{\kappa}z_d\right\|_{L^2(Q)}\|k_\epsilon^\gamma\|_{L^2(\omega_2^{TA})}. \tag{80}$$

Hence, it follows that

$$\|k_\epsilon^\gamma\|_{L^2(\omega_2^{TA})}\le C\left\|\frac{1}{\kappa}z_d\right\|_{L^2(Q)} \quad \textit{and} \tag{81}$$

$$\|q_\epsilon(0,.,.,k_\epsilon^\gamma,h(k_\epsilon^\gamma),0)\|_{L^2(\Omega^A)} \leq C\sqrt{\epsilon}\left\|\frac{1}{\kappa}z_d\right\|_{L^2(Q)}. \tag{82}$$

Using (13) *and* (81)*, there exists* $C = C(\psi,\gamma,\alpha,T,s,\lambda) > 0$ *such that*

$$\|h_\epsilon^\gamma\|_{L^2(\omega_1^{TA}))} \leq C\left(\left\|\frac{1}{\kappa}z_d\right\|_{L^2(Q)} + \|z_d\|_{L^2(Q)}\right). \tag{83}$$

Using the fact that k_ϵ^γ *satisfies* (81)*, we deduce* (q_ϵ^γ, p_ϵ^γ, τ_ϵ^γ, δ_ϵ^γ) *is solution of systems* (8)-(11)*, associated to the control* h_ϵ^γ *satisfies* (83)*. Then we can extract sub-sequences still denoted* (k_ϵ^γ)*,* (h_ϵ^γ)*,* (q_ϵ^γ)*,* (p_ϵ^γ)*,* (τ_ϵ^γ) *and* (δ_ϵ^γ) *when* $\epsilon \to 0$*, we have*

$$k_\epsilon^\gamma \rightharpoonup \breve{k}^\gamma \quad \text{weakly in} \quad L^2(\omega_2^{TA}), \tag{84}$$

$$h_\epsilon^\gamma \rightharpoonup \breve{h}^\gamma \quad \text{weakly in} \quad L^2(\omega_1^{TA}), \tag{85}$$

$$q_\epsilon^\gamma \rightharpoonup \breve{q}^\gamma \quad \text{weakly in} \quad L^2((0,T)\times(0,A);H_0^1(\Omega)), \tag{86}$$

$$p_\epsilon^\gamma \rightharpoonup \breve{p}^\gamma \quad \text{weakly in} \quad L^2((0,T)\times(0,A);H_0^1(\Omega)), \tag{87}$$

$$\tau_\epsilon^\gamma \rightharpoonup \breve{\tau}^\gamma \quad \text{weakly in} \quad L^2((0,T)\times(0,A);H_0^1(\Omega)), \tag{88}$$

$$\frac{1}{\sqrt{\gamma}}\tau_\epsilon^\gamma \rightharpoonup \frac{1}{\sqrt{\gamma}}\breve{\tau}_\epsilon^\gamma \quad \text{weakly in} \quad L^2(Q), \tag{89}$$

$$\delta_\epsilon^\gamma \rightharpoonup \breve{\delta}^\gamma \quad \text{weakly in} \quad L^2((0,T)\times(0,A);H_0^1(\Omega)), \tag{90}$$

$$\frac{1}{\sqrt{\gamma}}\delta_\epsilon^\gamma \rightharpoonup \frac{1}{\sqrt{\gamma}}\breve{\delta}_\epsilon^\gamma \quad \text{weakly in} \quad L^2(Q), \tag{91}$$

$$q^\gamma(0,.,.,k_\epsilon^\gamma,h_\epsilon(k_\epsilon^\gamma),0) \longrightarrow 0 \quad \text{strongly in} \quad L^2(\Omega^A). \tag{92}$$

Proceed as in[[12], Section 3, Step 2] and using (84) - (92)*, we prove that* $(\breve{q}^\gamma,\breve{p}^\gamma,\breve{\tau}^\gamma,\breve{\delta}^\gamma)$ *is a solution of* (8)-(11) *and we have*

$$\breve{q}^\gamma(0,.,.) = 0 \quad \text{in} \quad \Omega^A.$$

Step 3 *We study the convergence when* $\epsilon \longrightarrow 0$ *of the sequence* ρ_ϵ^γ*,* Ψ_ϵ^γ*,* ϖ_ϵ^γ *and* ζ_ϵ^γ*. Using the definition of* $\breve{\varphi}$ *and* $\breve{\eta}$*, it can be readly seen that there exists a constant* $C > 0$ *such that*

$$\kappa \geq C \quad \textit{and} \quad \frac{1}{\theta} \geq C.$$

And therefore using (73)*,* (81) *and the carleman inequality given by* (54)*, we obtain*

$$\|\Psi_\epsilon^\gamma\|^2_{L^2(Q)} + \|\rho_\epsilon^\gamma\|^2_{L^2(Q)} \leq C\left\|\frac{1}{\kappa}z_d\right\|^2_{L^2(Q)}, \tag{93}$$

where $C = C(\psi, \gamma, \alpha, T, s, \lambda) > 0$. *Using hence, we deduce that* (74)-(77) *and* (93)*, we can extract subsequences still denoted* ρ_ϵ^γ, Ψ_ϵ^γ, ϖ_ϵ^γ, ζ_ϵ^γ *such that when* $\epsilon \to 0$*, we obtain*

$$\rho_\epsilon^\gamma \rightharpoonup \breve{\rho}^\gamma \quad \text{weakly in} \quad L^2(Q), \tag{94}$$

$$\Psi_\epsilon^\gamma \rightharpoonup \breve{\Psi}^\gamma \quad \text{weakly in} \quad L^2((0,T) \times (0,A); H_0^1(\Omega)), \tag{95}$$

$$\zeta_\epsilon^\gamma \rightharpoonup \breve{\zeta}^\gamma \quad \text{weakly in} \quad L^2((0,T) \times (0,A); H_0^1(\Omega)), \tag{96}$$

$$\varpi_\epsilon^\gamma \rightharpoonup \breve{\varpi}^\gamma \quad \text{weakly in} \quad L^2((0,T) \times (0,A); H_0^1(\Omega)). \tag{97}$$

Using (73)*,* (84) *and* (94)*, we deduce that*

$$\breve{\rho}^\gamma = \breve{k}^\gamma \quad \text{in} \quad \omega_2^{TA}.$$

Now, if we use (94)-(97)*, we can prove by passing to the limit in systems* (74)-(77) *that the functions* ($\breve{\rho}^\gamma$, $\breve{\Psi}^\gamma$, $\breve{\varpi}^\gamma$, $\breve{\zeta}^\gamma$) *is solution of* (15)-(18)*. Moreover, using* (84) *and* (81)*, we deduce the weak lower semi-continuity of the norm the inequality* (19)*.*

5 Conclusion

Through hierarchical control, we have demonstrated the existence and uniqueness of two controls. This problem was ill-posed. This is why we have used the least regret control developed by Lions and adapted to problems with missing data. After establishing a Carleman observability inequality, we were able to solve the null controllability problem associated with (1). In perspective we apply the Stackelberg-Nash strategy to this problem.

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