A ONE-SIDED CUMULATIVE SUM CONTROL CHART FOR MONITORING SHIFT IN THE PARAMETERS OF THE WEIGHTEDWEIBULLDISTRIBUTION USING V-MASK APPROACH

ABSTRACT

In our research, we have developed a cumulative sum control chart to identify changes in the parameter of the weighted Weibull distribution. This chart incorporates the V-mask technique and a sequential ratio test. Our analysis of the V-mask revealed that even minor shifts in the weighted Weibull distribution's parameters led to significant variations in the mask's angle, lead distance, and average run length. Finally, we applied the cumulative sum chart to real data from the Kabsad Scientific Hospital to illustrate the sensitivity of the proposed cumulative sum control chart.

Keywords: Deviations, flexibility, inadequacy, sensitivity, and variability

1. Introduction

Product quality features have recently received much attention from design engineers, production personnel, and consumers. It has increasingly become apparent that improving product quality can decreasethe cost of production and increase consumer satisfaction, culminatingin a high percentage of profits.

Control charts are used widely as a diagnostic tool for monitoring production and services to identify instability and circumstances that are unusual in the process.A defect of the traditional Shewhart control chart is its inadequacy in detecting a relatively small change in a process mean. This is largely the consequence of whether a process is judged out of control at a particular time depends only on the sample at that time and not on the history of the process (Devore, 2010). Cumulative sum control charts are statistical process control tools used to monitor shifts in the process mean based on samples taken from the system at given intervals. The intervals could be hours, days, weeks, or months. It makes use of the cumulative sum of deviations from a given point.It is a powerful technique for monitoring and controllingprocesses, particularly when the shift is small compared to the process variability. The CUSUM charts plot the cumulative sum of deviations from the target point for individual measurements or subgroup means.It indicates the accumulative information of previous and current samples. Given this reason, the CUSUM control charts are more effective than the Shewhart control charts in detecting small shifts in the process mean. The CUSUM control charts depend on the specification of a target value and a known or reliable estimate of the variance. This makes the CUSUM chart a better tool to be used after process control has been established. The merit of using CUSUM charts is due to their flexibility, sensitivity, and easeof use. It is good for monitoring processes with small shifts, improving process stability, and very good in reducing false alarms in systems.The CUSUM charts have applications in manufacturing, chemical processing, finance, and healthcare among others. A one-sided CUSUM chart is used to detect shifts in one direction, either positive or negative. It is particularly useful when only one type of shift is of interest or when the process is more sensitive to shifts in one direction.The first CUSUM control chart was introduced by Page (1954) and has since been widely used in various industries for monitoring and controlling processes. Other published works on CUSUM included combined Shewhart-CUSUM quality control scheme (Lucas, 1982), cumulative sum control charting (Hawkins, 1993), statistical design of CUSUM charts (Woodall and Adams, 1993), exact results for Shewhart controlcharts with supplementaryrun rules (Champ and Woodall, 2001),one-sided CUSUM control chart for the Erlang-Truncated exponential distribution (Rao, 2013), one-sided cumulative sum control chart for monitoring shifts in the shape parameter of the Pareto distribution (Nasiru, 2016),one-sided cumulative sum control chart for monitoring shift in the scale parameter, delta of the new Weibull-Pareto distribution (Sayibu et al., 2017), unified sum control chart for monitoring shifts in the parameters of the Pareto distribution (Sayibu and Maahi, 2017),two-sided cumulative sum control chart for monitoring shifts in the shape parameter of the Pareto distribution (Sayibu and Luguterah, 2018), distribution-free CUSUM control charts using bootstrap-based control limits (Zhang and Woodall, 2019), how to use a CUSUM chart for process improvement (Doganaksov and Hahn, 2020), CUSUM analysis (Grigg and Farewell, 2020),and CGR-CUSUM:a continuous time generalized rapid response cumulative sum (Xie and Goh, 2022) among others.

The rest of the study is organized as follows: section 2 discusses the sequential probability ratio test, weighted Weibull distribution is covered under Section 3, the cumulative sum control chart is enumerated in Section 4, average run length is treated in section 5, practical application is illustrated in section 6 and section 7 covers conclusion.

2.Sequential Probability Ratio Test

The sequential probability ratio test (SPRT) is a statistical test used to determine whether a process or system is in control or out of control. The test is performed sequentially, with data points added one at a time. The test calculates the ratio of probabilitiesof the data under two hypotheses and the test statistic is the cumulative sum of the logarithms of the probability ratios. It includes the following steps: defining the null and alternative hypotheses and setting the significance level (δ), and the power $(1-\theta)$. The rest are calculating the test statistic for each new data point and comparing the test statistic to the upper and lower boundaries. If the test statistic crosses the upper boundary, reject the null hypothesis thus out of control. On the other hand, if the test statistic crosses the lower boundary, accept the null hypothesis thus the system is in control. If the test statistic remains within the boundaries, then continue the sampling process. The SPRT is efficient, flexible, and powerful. By way of illustration, Wald's SPRT is widely used for determining between two alternative hypotheses, $H_0 : \rho = \rho_0$ and $H_1 = \rho = \rho_1$. Let $x_1, x_2, ..., x_n$ denote successive observations of a random variable X which are independent and identically distributed and follow the weighted Weibull distribution (WWD). If the probability of observing $x_1, x_2, ..., x_n$ is given by $P_i(n)$ when H_i is true $(i = 0,1)$. If the pdf of the random variable is $f_i(x)$ when H_i is true, then

$$P_i(n) = f_i(x_1), ..., f_i(x_n)$$

Given two constants A and B such that $A > 0$ and $B > 0$, Wald's sequential test of H_i against H_0 is given as follows. At the n^{th} sampling stage, the logarithm of the probability ratio is

$$\eta(n) = \log \frac{P_1(n)}{P_0(n)} = \log \frac{L(\rho_1; X_1, ..., X_n)}{L(\rho_0; X_1, ..., X_n)} > K ,$$

is calculated. The Wald's SPRT has the following nature

- If $\eta(n) \geq A$, then terminate the observation and accept H_i as true
- $\eta(n) < A$ then decide that H_0 is true and terminate the process;
- $A < \eta(n) < B$, continue to collect more observations to obtain η_{n+1}.

The SPRT is optimal as it minimizes the average sample size before a decision can be made within all sequential tests which do not have larger error probabilities than the SPRT. Furthermore, the boundaries A and B can be determined with very good approximation as

$A = \log\frac{\theta}{1-\delta}$ and $B = \log\frac{1-\theta}{\delta}$.

3.Weighted Weibull distribution

The weighted Weibull (WW) distribution was developed by (Nasiru, 2015). It is said to be very flexible among other competing models. The cumulative density function (CDF) of the WWD is given by

$$F\left(x,\alpha,\gamma,\lambda\right) = 1 - e^{-\left(\gamma x^{\alpha} + \gamma(\lambda x)^{\alpha}\right)}, \tag{1}$$

Where $x > 0, \alpha > 0, \gamma > 0, \lambda > 0$, α is a shape parameter, β and λ are scale parameters.

The corresponding probability density function (pdf) is obtained by differentiating the CDF and is given as

$$f\left(x,\alpha,\gamma,\lambda\right) = \alpha\gamma\left(1+\lambda^{\alpha}\right)x^{\alpha-1}e^{-\left(\gamma x^{\alpha} + \gamma(\lambda x)^{\alpha}\right)}, \tag{2}$$

The hazard function is defined as the ratio of the PDF to the survival function. This is presented as

$$H\left(x\right) = \frac{\alpha\gamma\left(1+\lambda^{\alpha}\right)x^{\alpha-1}e^{-\left(\gamma x^{\alpha} + \gamma(\lambda x)^{\alpha}\right)},}{e^{-\left(\gamma x^{\alpha} + \gamma(\lambda x)^{\alpha}\right)}} \tag{3}$$

The quantile function which is very useful in generating random numbers from the distribution is obtained as

$$x(u) = \left[\frac{-\log(1-u)}{\gamma\left(1+\lambda^{\alpha}\right)}\right]^{1/\alpha} \tag{4}$$

Where u is the uniform distribution within the interval 0 and 1.

4. V-Mask

The CUSUM control chart is realized by plotting the sum $S_n = \sum_{i=1}^{n} \ln x$ versus the number of observations n . the V-mask is very important in determining the status of a system, whether it is incontrol or outofcontrol. The process is achieved by placing a V-mask on the final plotted CUSUM points in line with OS_1 or OS_{-1} parallel to the axis m. The points plotted earlier are then investigated to determine if they are found above or below the arms of the V-mask. If it happens that all the points fall inbetween the two arms of the V-mask, then the process is judged to be in-control. On the other hand, if the past plotted points happen to fall outside the arms of the V-mask, then the process is assumed to be out of out-of-control.

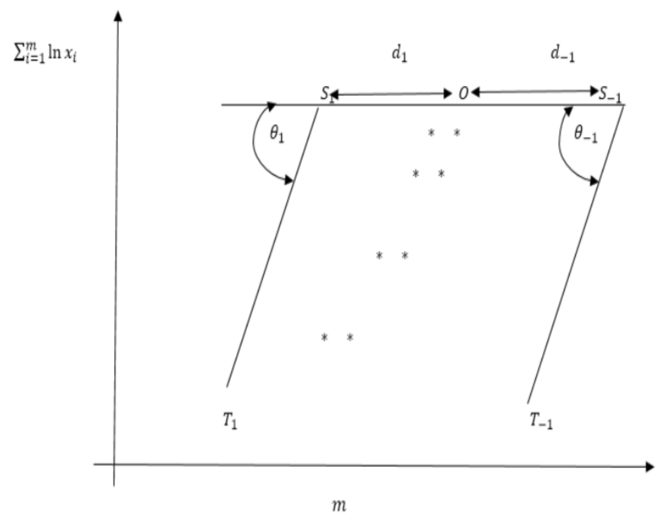

Figure 1: V-mask

5. Cumulative Sum Control Chart of the WW Distribution

If $x_1, x_2, ..., x_n$ are randomly independent and identically distributed variables and follow the WWD, then the likelihood ratio for testing the hypotheses that there is a shift in γ and keeping other parameters fixed, are given as

$$H_0 : \gamma = \gamma_0$$

$$H_1 : \gamma = \gamma_1$$

Likelihood function for equation (2) when there is no shift in the scale parameters is

$$L_0 = \prod_{i=1}^{n} \left[\alpha\gamma_0 \left(1+\lambda^\alpha\right) x^{\alpha-1} e^{-\left(\gamma_0 x^\alpha + \gamma_0 (\lambda x)^\alpha\right)} \right] \tag{5}$$

And the likelihood function for a shift in the scale parameters is

$$L_1 = \prod_{i=1}^{n} \left[\alpha\gamma_1 \left(1+\lambda^\alpha\right) x^{\alpha-1} e^{-\left(\gamma_1 x^\alpha + \gamma_1 (\lambda x)^\alpha\right)} \right] \tag{6}$$

The likelihood ratio $\left(L_R\right)$ of equations (5) and (6) is given as

$$L_R = \frac{\prod_{i=1}^{n} \left[\alpha\gamma_1 \left(1+\lambda^\alpha\right) x^{\alpha-1} e^{-\left(\gamma_1 x^\alpha + \gamma_1 (\lambda x)^\alpha\right)} \right]}{\prod_{i=1}^{n} \left[\alpha\gamma_0 \left(1+\lambda^\alpha\right) x^{\alpha-1} e^{-\left(\gamma_0 x^\alpha + \gamma_0 (\lambda x)^\alpha\right)} \right]} \tag{7}$$

This is simplified as

$$L_R = \prod_{i=1}^{n}\left[\frac{\gamma_1}{\gamma_0} e^{-\left(\gamma_1 x^{\alpha} + \gamma_1 (\lambda x)^{\alpha}\right) + \left(\gamma_0 x^{\alpha} + \gamma_0 (\lambda x)^{\alpha}\right)}\right] \tag{8}$$

Further simplification yields

$$L_R = \left(\frac{\gamma_1}{\gamma_0}\right)^{n} e^{-\left(\gamma_1 x^{\alpha} + \gamma_1 \lambda^{\alpha} \sum_{i=1}^{n} x_i^{\alpha}\right) + \left(\gamma_0 x^{\alpha} + \gamma_0 \lambda^{\alpha} \sum_{i=1}^{n} x_i^{\alpha}\right)} \tag{9}$$

The continuation region of the SPTR differentiating between the two hypotheses is given by

$$\frac{\delta}{1-\theta} < L_R < \frac{1-\delta}{\theta}$$

Where δ and θ are the types I and II errors. Substituting equation (9) in the continuation region gives

$$\frac{\delta}{1-\theta} < \left(\frac{\gamma_1}{\gamma_0}\right)^{n} e^{-\left(\gamma_1 x^{\alpha} + \gamma_1 \lambda^{\alpha} \sum_{i=1}^{n} x_i^{\alpha}\right) + \left(\gamma_0 x^{\alpha} + \gamma_0 \lambda^{\alpha} \sum_{i=1}^{n} x_i^{\alpha}\right)} < \frac{1-\delta}{\theta} \tag{10}$$

Taking the natural logarithm of both sides gives

$$\log\left(\frac{\delta}{1-\theta}\right) < \left(\frac{\gamma_1}{\gamma_0}\right)^{n} e^{-\left(\gamma_1 x^{\alpha} + \gamma_1 \lambda^{\alpha} \sum_{i=1}^{n} x_i^{\alpha}\right) + \left(\gamma_0 x^{\alpha} + \gamma_0 \lambda^{\alpha} \sum_{i=1}^{n} x_i^{\alpha}\right)} < \log\left(\frac{1-\delta}{\theta}\right) \tag{11}$$

If $\delta = 0$, then

$$n\log\left(\frac{\gamma_1}{\gamma_0}\right) - \left(\gamma_1 x^{\alpha} + \gamma_1 \lambda^{\alpha} \sum_{i=1}^{n} x_i^{\alpha}\right) + \left(\gamma_0 x^{\alpha} + \gamma_0 \lambda^{\alpha} \sum_{i=1}^{n} x_i^{\alpha}\right) < \log\left(\frac{1}{\theta}\right) \tag{12}$$

Factorizing yields

$$\sum_{i=1}^{n} x_i^{\alpha}\left(\gamma_0 \lambda^{\alpha} - \gamma_1 \lambda^{\alpha} + \gamma_0 - \gamma_1\right) < \log\left(\frac{1}{\theta}\right) - n\log\left(\frac{\gamma_1}{\gamma_0}\right) \tag{13}$$

simplifying gives

$$\sum_{i=1}^{n} x_i < \frac{{}^{1}\!/_{\alpha} \log\left(\frac{1}{\theta}\right) - n^{1/\alpha}\ {}^{1}\!/_{\alpha} \log\left(\frac{\gamma_1}{\gamma_0}\right)}{\left(\gamma_0 \lambda^{\alpha} - \gamma_1 \lambda^{\alpha} + \gamma_0 - \gamma_1\right)^{1/\alpha}} \tag{14}$$

Writing equation (14) in the form $\sum_{i=1}^{n} x_i \le bx + q$, gives where $q = \dfrac{-\frac{1}{\alpha}\left(\log\left(\frac{1}{\theta}\right)\right)}{-\left[\lambda^{\alpha}\left(\gamma_0-\gamma_1\right)-\left(\gamma_0-\gamma_1\right)\right]^{\frac{1}{\alpha}}}$ and $b = \dfrac{n^{\frac{1}{\alpha}}\frac{1}{\alpha}\log\left(\frac{\gamma_1}{\gamma_0}\right)}{-\left[\lambda^{\alpha}\left(\gamma_0-\gamma_1\right)+\left(\gamma_0-\gamma_1\right)\right]^{\frac{1}{\alpha}}}$

The mark angle is obtained as

$$\theta = \tan^{-1}\left[\frac{\frac{1}{\alpha}\log\left(\frac{\gamma_1}{\gamma_0}\right)}{-\left[\lambda^{\alpha}\left(\gamma_0-\gamma_1\right)+\left(\gamma_0-\gamma_1\right)\right]^{\frac{1}{\alpha}}}\right]$$

5.1. Effects of positive and negative shifts in the value of gamma on the mask angle

It can be deduced from Table 1 that as the shift in the value of gamma widens towards the right-hand side, the angle of the mask decreases in value. On the other hand, when there is a negative shift in the value of gamma, the angle increases in value as displayed in Table 2.

Table 1: Positive shift in the value of gamma

γ_0	γ_1	α	λ	θ
0.001	0.01	2	5.0	78.5
0.001	0.05	2	5.0	56.9
0.001	0.09	2	5.0	44.2
0.001	0.12	2	5.0	37.7
0.001	0.18	2	5.0	29.2
0.001	0.23	2	5.0	24.5
0.001	0.29	2	5.0	20.7
0.001	0.31	2	5.0	19.6
0.001	0.36	2	5.0	17.5
0.001	0.40	2	5.0	16.1

Table 2: Negative shift in the value of gamma

γ_0	γ_1	α	λ	θ
0.1	0.2	2	5.0	7.6
0.1	0.13	2	5.0	9.5
0.1	0.09	2	5.0	11.5
0.1	0.05	2	5.0	14.9
0.1	0.02	2	5.0	21.2
0.1	0.015	2	5.0	23.2
0.1	0.007	2	5.0	28.8
0.1	0.005	2	5.0	31.2
0.1	0.003	2	5.0	34.8
0.1	0.001	2	5.0	41.8

5.2 Effects of positive and negative shift in gamma on the lead distance

The lead distance of the V-mask is the distance between the last point plotted and the angle of the mask. It represents the maximum allowable deviations from the center line before a point is considered out of control. A larger lead distance shows: wider control limits, greater tolerance for variation, and reduced sensitivity to small shifts in the process. On the other hand, a smaller lead distance indicates a narrow control limit, less tolerance for variation, and increased sensitivity to small shifts in the process. It is by

$$|d| = \frac{-1/\alpha\left(\log\left(\frac{1}{\theta}\right)\right)}{-\left[\lambda^{\alpha}\left(\gamma_0-\gamma_1\right)-\left(\gamma_0-\gamma_1\right)\right]^{1/\alpha}}, \text{ where } \lambda_1 > \lambda_0 \text{ and } \delta_1 > \delta_0 .$$

When there is a positive shift in the value of gamma, the lead distance decreases as shown in Table 3. However, a positive increase in gamma leads to an increase in the value of the lead distance as indicated in Table 4.

Table 3: Positive shift in gamma and its effect on the lead distance

γ_0	γ_1	λ	α	θ	d
0.1	0.4	0.01	0.2	0.03	97.1
0.1	0.7	0.01	0.2	0.03	48.5
0.1	1.0	0.01	0.2	0.03	32.4
0.1	1.3	0.01	0.2	0.03	24.3
0.1	1.8	0.01	0.2	0.03	17.1
0.1	2.3	0.01	0.2	0.03	13.2
0.1	2.9	0.01	0.2	0.03	10.4
0.1	3.4	0.01	0.2	0.03	8.8
0.1	3.6	0.01	0.2	0.03	8.3
0.1	4.0	0.01	0.2	0.03	7.5

Table 4: negative shift in gamma and its effect on the lead distance

γ_0	γ_1	λ	α	θ	d
4	3.9	12	0.1	2	- 245.7
4	3.4	12	0.1	2	- 41.0
4	3.0	12	0.1	2	- 24.6
4	2.6	12	0.1	2	- 17.6
4	2.1	12	0.1	2	- 12.9
4	1.7	12	0.1	2	- 10.7
4	1.3	12	0.1	2	- 9.1
4	0.8	12	0.1	2	- 7.7
4	0.3	12	0.1	2	- 6.6
4	0.09	12	0.1	2	- 6.3

An increase in the value of lambda with a positive shift in gamma increases the value of the lead distance as shown in Table 5. However,an increase in the value of lambda under a positive shift in gamma decreases the value of the lead distance as displayed in Table 6.

Table 5: Increase in the value of lambda with a positive shift in gamma

γ_0	γ_1	λ	α	θ	d
0.1	0.4	10.0	0.2	0.03	- 99.9
0.1	0.7	10.0	0.2	0.03	- 50.0
0.1	1.0	10.0	0.2	0.03	- 33.3
0.1	1.3	10.0	0.2	0.03	- 25.0
0.1	1.8	10.0	0.2	0.03	- 17.6
0.1	2.3	10.0	0.2	0.03	- 13.6
0.1	2.9	10.0	0.2	0.03	- 10.7
0.1	3.4	10.0	0.2	0.03	- 9.1
0.1	3.6	10.0	0.2	0.03	- 8.6
0.1	4.0	10.0	0.2	0.03	- 7.7

Table 6: Decrease in the value of lambda with a positive shift in gamma

γ_0	γ_1	λ	α	θ	d
0.1	0.4	0.008	0.2	0.03	94.4
0.1	0.7	0.008	0.2	0.03	47.2
0.1	1.0	0.008	0.2	0.03	31.5
0.1	1.3	0.008	0.2	0.03	23.6
0.1	1.8	0.008	0.2	0.03	16.7
0.1	2.3	0.008	0.2	0.03	12.9
0.1	2.9	0.008	0.2	0.03	10.1
0.1	3.4	0.008	0.2	0.03	8.6
0.1	3.6	0.008	0.2	0.03	8.1
0.1	4.0	0.008	0.2	0.03	7.3

Not all but also, when lambda increases in value with a negative shift in gamma increases the value of the lead distance as shown in Table 7. On the other hand, adecrease in the value of lambda with a negative shift in gamma also increases the value of the lead distance as shown in Table 8.

Table 7: Increase in the value of lambda with a negative shift in gamma

γ_0	γ_1	λ	α	θ	d
4	3.9	20	0.1	2	- 198.4
4	3.4	20	0.1	2	- 33.1
4	3.0	20	0.1	2	- 19.8
4	2.6	20	0.1	2	- 14.2
4	2.1	20	0.1	2	- 10.4
4	1.7	20	0.1	2	- 8.6
4	1.3	20	0.1	2	- 7.3
4	0.8	20	0.1	2	- 6.2
4	0.3	20	0.1	2	- 5.4
4	0.09	20	0.1	2	- 5.0

Table 8: Decrease in the value of lambda with a negative shift in gamma

γ_0	γ_1	λ	α	θ	d
4	3.9	5	0.1	2	- 396.9
4	3.4	5	0.1	2	- 66.2
4	3.0	5	0.1	2	- 39.7
4	2.6	5	0.1	2	- 28.4
4	2.1	5	0.1	2	- 20.9
4	1.7	5	0.1	2	- 17.3
4	1.3	5	0.1	2	- 14.7
4	0.8	5	0.1	2	- 12.4
4	0.3	5	0.1	2	- 10.7
4	0.09	5	0.1	2	- 10.1

Furthermore, an increase in the value of alpha with a positive shift in gamma decreases the value of the lead distance whereas a decrease in the value of alpha with a positive shift in gamma decreases the value of the lead distance. The details are shown in Tables 9 and 10 respectively.

Table 9: Increase in alpha with a positive shift in gamma

γ_0	γ_1	λ	α	θ	d
0.1	0.4	0.01	0.9	0.03	13.2
0.1	0.7	0.01	0.9	0.03	6.6
0.1	1.0	0.01	0.9	0.03	4.4
0.1	1.3	0.01	0.9	0.03	3.3
0.1	1.8	0.01	0.9	0.03	2.3
0.1	2.3	0.01	0.9	0.03	1.8
0.1	2.9	0.01	0.9	0.03	1.4
0.1	3.4	0.01	0.9	0.03	1.2
0.1	3.6	0.01	0.9	0.03	1.1
0.1	4.0	0.01	0.9	0.03	1.0

Table 10: Decrease in alpha with a positive shift in gamma

γ_0	γ_1	λ	α	θ	d
0.1	0.4	0.01	0.05	0.03	1136.6
0.1	0.7	0.01	0.05	0.03	568.3
0.1	1.0	0.01	0.05	0.03	378.9
0.1	1.3	0.01	0.05	0.03	284.2
0.1	1.8	0.01	0.05	0.03	200.6
0.1	2.3	0.01	0.05	0.03	155.0
0.1	2.9	0.01	0.05	0.03	121.8
0.1	3.4	0.01	0.05	0.03	103.3
0.1	3.6	0.01	0.05	0.03	97.4
0.1	4.0	0.01	0.05	0.03	87.4

Again, when alpha increases in value with a negative shift in gamma increases the value of the lead distance whilst a decrease in the value of alpha with a negative shift in gamma also increases the value of the lead distance. The details are presented in Tables 11 and 12.

Table 11: Increase in the value of alpha with a negative shift in gamma

γ_0	γ_1	λ	α	θ	d
4	3.9	20	0.9	2	- 0.92
4	3.4	20	0.9	2	- 0.15
4	3.0	20	0.9	2	- 0.09
4	2.6	20	0.9	2	- 0.07
4	2.1	20	0.9	2	- 0.05
4	1.7	20	0.9	2	- 0.04
4	1.3	20	0.9	2	- 0.03
4	0.8	20	0.9	2	- 0.03
4	0.3	20	0.9	2	- 0.02
4	0.09	20	0.9	2	- 0.02

Table 12: Decrease in the value of alpha with a negative shift in gamma

γ_0	γ_1	λ	α	θ	d
4	3.9	12	0.01	2	- 27549.2
4	3.4	12	0.01	2	- 4591.5
4	3.0	12	0.01	2	- 2754.9
4	2.6	12	0.01	2	- 1967.8
4	2.1	12	0.01	2	- 1450.0
4	1.7	12	0.01	2	- 1197.8
4	1.3	12	0.01	2	- 1020.3
4	0.8	12	0.01	2	- 860.9
4	0.3	12	0.01	2	- 744.6
4	0.09	12	0.01	2	- 704.6

Moreover, an increase in the value of theta with a positive shift in gamma decreases the lead distance and a decrease in the value of theta also decreases the value of the lead distance when there is a positive shift in gamma. The details can be found in Tables 13 and 14.

Table 13: Increase in theta with a positive shift in gamma

γ_0	γ_1	λ	α	θ	d
0.1	0.4	0.01	0.2	0.3	33.3
0.1	0.7	0.01	0.2	0.3	16.7
0.1	1.0	0.01	0.2	0.3	11.1
0.1	1.3	0.01	0.2	0.3	8.3
0.1	1.8	0.01	0.2	0.3	5.9
0.1	2.3	0.01	0.2	0.3	4.5
0.1	2.9	0.01	0.2	0.3	3.6
0.1	3.4	0.01	0.2	0.3	3.0
0.1	3.6	0.01	0.2	0.3	2.9
0.1	4.0	0.01	0.2	0.3	2.6

Table 14: Decrease in theta with a negative shift in gamma

γ_0	γ_1	λ	α	θ	d
0.1	0.4	0.01	0.2	0.01	127.5
0.1	0.7	0.01	0.2	0.01	63.8
0.1	1.0	0.01	0.2	0.01	42.5
0.1	1.3	0.01	0.2	0.01	31.9
0.1	1.8	0.01	0.2	0.01	22.5
0.1	2.3	0.01	0.2	0.01	17.4
0.1	2.9	0.01	0.2	0.01	13.7
0.1	3.4	0.01	0.2	0.01	11.6
0.1	3.6	0.01	0.2	0.01	10.9
0.1	4.0	0.01	0.2	0.01	9.8

Another observation is that an increase in the value of theta with a negative shift in the value of gamma increases the value of the lead distance as shown in Table 15. On the other hand, a decrease in the value of theta with a negative shift in the value of gamma decreases the value of the lead distance as indicated in Table 16.

Table 15: Increase in theta with a negative shift in gamma

γ_0	γ_1	λ	α	θ	d
4	3.9	12	0.1	9	- 778.9
4	3.4	12	0.1	9	- 129.8
4	3.0	12	0.1	9	- 77.9
4	2.6	12	0.1	9	- 55.6
4	2.1	12	0.1	9	- 41.0
4	1.7	12	0.1	9	- 33.9
4	1.3	12	0.1	9	- 28.8
4	0.8	12	0.1	9	- 24.3
4	0.3	12	0.1	9	- 21.1
4	0.09	12	0.1	9	- 19.9

Table 16: Decrease in theta with a negative shift in gamma

γ_0	γ_1	λ	α	θ	d
4	3.9	12	0.1	0.5	245.7
4	3.4	12	0.1	0.5	41.0
4	3.0	12	0.1	0.5	24.6
4	2.6	12	0.1	0.5	17.6
4	2.1	12	0.1	0.5	12.9
4	1.7	12	0.1	0.5	10.7
4	1.3	12	0.1	0.5	9.1
4	0.8	12	0.1	0.5	7.7
4	0.3	12	0.1	0.5	6.6

4	0.09	12	0.1	0.5	6.3

6. AVERAGE RUN LENGTH

Average Run length (ARL) is the average number of samples or observations taken before a control chart signals that an out-of-control condition. It measures the effectiveness of a control chart in detecting process shifts. A smaller ARL indicates faster detection of a process shift and also shows a reduced time to signal an out-of-control condition. The ARL is affected by: sample size, control limits, process variability, shift size, and type of chart. The ARL is given by

$$ARL = \frac{-\ln\alpha}{E[Z]}$$

Where $Z = \dfrac{f_1(x,\alpha,\gamma_1,\lambda)}{f_0(x,\alpha,\gamma_0,\lambda)}$

Using the PDF given in equation (1) then

$$Z = \frac{\alpha\gamma_1(1+\lambda^\alpha)x^{\alpha-1}e^{-\left(\gamma_1 x^\alpha+\gamma_1(\lambda x)^\alpha\right)}}{\alpha\gamma_0(1+\lambda^\alpha)x^{\alpha-1}e^{-\left(\gamma_0 x^\alpha+\gamma_0(\lambda x)^\alpha\right)}} \tag{15}$$

This is simplified as

$$Z = \frac{\gamma_1}{\gamma_0}e^{\left(\gamma_0 x^\alpha+\gamma_0(\lambda x)^\alpha\right)-\left(\gamma_1 x^\alpha+\gamma_1(\lambda x)^\alpha\right)} \tag{16}$$

Taking natural logarithm of equation (16) yields

$$\ln Z = \ln\left[\frac{\gamma_1}{\gamma_0}e^{\left(\gamma_0 x^\alpha+\gamma_0(\lambda x)^\alpha\right)-\left(\gamma_1 x^\alpha+\gamma_1(\lambda x)^\alpha\right)}\right] \tag{17}$$

Taking expectation of both sides gives

$$E[\ln Z] = \ln\left(\frac{\gamma_1}{\gamma_0}\right)\left(\gamma_0 E[x]^\alpha+\gamma_0\left(\lambda E[x]\right)^\alpha\right)-\left(\gamma_1 E[x]^\alpha+\gamma_1\left(\lambda E[x]\right)^\alpha\right) \tag{18}$$

Simplifying further gives

$$E[\ln Z] = \ln\left(\frac{\gamma_1}{\gamma_0}\right)+\gamma_0-\gamma_1+\left(\gamma_0\lambda^\alpha-\gamma_1\lambda^\alpha\right)E[x]^\alpha \tag{20}$$

By definition,

$$E[x] = \int_\gamma^\infty xf(x,\alpha,\gamma,\lambda)dx$$

Substituting equation (2) into the integral yields

$$E\left[x\right]=\alpha\gamma\left(1+\lambda^{\alpha}\right)\int_{\gamma}^{\infty}x^{\alpha}e^{-\left(\gamma x^{\alpha}+\gamma\left(\lambda x\right)^{\alpha}\right)}dx.$$

Let $u=\gamma x^{\alpha}+\gamma\left(\lambda x\right)^{\alpha}$ then $x=\left(\frac{u}{\gamma+\gamma\lambda^{\alpha}}\right)^{1/\alpha}$, if $x=1$ then $u=\gamma\left(1+\lambda^{\alpha}\right)$ and if $x=\infty$ then $u=\infty$.

Also $\frac{du}{dx}=\alpha\gamma x^{\alpha-1}+\alpha\gamma\lambda^{\alpha}x^{\alpha-1}$, and $dx=\frac{du}{\alpha\gamma x^{\alpha-1}+\alpha\gamma\lambda^{\alpha}x^{\alpha-1}}$.

This implies

$$E\left[x\right]=\alpha\gamma\left(1+\lambda^{\alpha}\right)\int_{\gamma\left(1+\lambda^{\alpha}\right)}^{\infty}ue^{-u}\frac{du}{\alpha\gamma\left(1+\lambda^{\alpha}\right)x^{\alpha-1}}.$$

Simplifying gives

$$E\left[x\right]=\left(\gamma+\gamma\lambda^{\alpha}\right)^{-1/\alpha}\int_{\gamma\left(1+\lambda^{\alpha}\right)}^{\infty}u^{\frac{1+\alpha}{\alpha}-1}e^{-u}du,$$

and

$$E\left[x\right]=\left(\gamma+\gamma\lambda^{\alpha}\right)^{-1/\alpha}\Gamma\left(\tfrac{1+\alpha}{\alpha},\gamma\left(1+\lambda^{\alpha}\right)\right).$$

Substituting the expression for $E\left[x\right]$ into equation (20) produces

$$E\left[\ln Z\right]=\ln\left(\frac{\gamma_{1}}{\gamma_{0}}\right)+\gamma_{0}-\gamma_{1}+\left(\gamma_{0}\lambda^{\alpha}-\gamma_{1}\lambda^{\alpha}\right)\left[\left(\gamma+\gamma\lambda^{\alpha}\right)^{-1/\alpha}\Gamma\left(\tfrac{1+\alpha}{\alpha},\gamma\left(1+\lambda^{\alpha}\right)\right)\right]^{\alpha}. \tag{21}$$

Simplify further gives

$$E\left[\ln Z\right]=\ln\left(\frac{\gamma_{1}}{\gamma_{0}}\right)+\gamma_{0}-\gamma_{1}+\left(\gamma_{0}\lambda^{\alpha}-\gamma_{1}\lambda^{\alpha}\right)\frac{\Gamma\left(\tfrac{1+\alpha}{\alpha},\gamma\left(1+\lambda^{\alpha}\right)\right)^{\alpha}}{\gamma+\gamma\lambda^{\alpha}}.$$

The ARL is finally expressed as

$$ARL=\frac{-\ln\alpha}{\ln\left(\frac{\gamma_{1}}{\gamma_{0}}\right)+\gamma_{0}-\gamma_{1}+\left(\gamma_{0}\lambda^{\alpha}-\gamma_{1}\lambda^{\alpha}\right)\frac{\Gamma\left(\tfrac{1+\alpha}{\alpha},\gamma\left(1+\lambda^{\alpha}\right)\right)^{\alpha}}{\gamma+\gamma\lambda^{\alpha}}}. \tag{22}$$

When there is a positive change in gamma, the ARL increases in value whereas a negative shift in gamma decreases ARL. The details are shown in Tables 17 and 18.

Table 17: effect of positive shift in gamma on the ARL

γ_0	γ_1	γ	λ	α	ARL
7	12	3	2	0.1	- 1.9891
7	13.5	3	2	0.1	- 1.5248
7	14.6	3	2	0.1	- 1.3013
7	16.5	3	2	0.1	- 1.0376
7	17.5	3	2	0.1	- 0.9374
7	19.0	3	2	0.1	- 0.8185
7	21.4	3	2	0.1	- 0.6802
7	22.5	3	2	0.1	- 0.6312
7	25.8	3	2	0.1	- 0.5189
7	27.1	3	2	0.1	- 0.4660

Table 18: effect of a negative shift in gamma on the ARL

γ_0	γ_1	γ	λ	α	ARL
7	6.85	3	2	0.1	67.5629
7	6.2	3	2	0.1	12.7162
7	5.9	3	2	0.1	9.2660
7	5.4	3	2	0.1	6.3930
7	5.1	3	2	0.1	4.4731
7	4.7	3	2	0.1	3.8251
7	4.3	3	2	0.1	3.3461
7	3.9	3	2	0.1	2.9784
7	3.5	3	2	0.1	2.6254
7	3.0	3	2	0.1	2.2648
7	2.3	3	2	0.1	2.1089

The ARL also increases in value when lambda is increased with a positive shift in gamma as shown in Table 19. However, the ARL decreases in value as lambda increases with a negative shift in gamma as displayed in Table 20.

Table 19: Increase in the value of lambda with a positive shift in gamma

γ_0	γ_1	γ	λ	α	ARL
7	12	3	4.1	0.1	- 1.9850
7	13.5	3	4.1	0.1	- 1.5219
7	14.6	3	4.1	0.1	- 1.2989
7	16.5	3	4.1	0.1	- 1.0359
7	17.5	3	4.1	0.1	- 0.9359
7	19.0	3	4.1	0.1	- 0.8173
7	21.4	3	4.1	0.1	- 0.6792
7	22.5	3	4.1	0.1	- 0.6303
7	25.8	3	4.1	0.1	- 0.5182
7	27.1	3	4.1	0.1	- 0.4842

Table 20: Increase in lambda with a negative shift in gamma

γ_0	γ_1	γ	λ	α	ARL
7	6.85	3	4.1	0.1	67.3758
7	6.2	3	4.1	0.1	12.6790
7	5.9	3	4.1	0.1	9.2383
7	5.4	3	4.1	0.1	6.3729
7	5.1	3	4.1	0.1	5.3789
7	4.7	3	4.1	0.1	4.4580
7	4.3	3	4.1	0.1	3.8116
7	3.9	3	4.1	0.1	3.3337
7	3.5	3	4.1	0.1	2.9668
7	3.0	3	4.1	0.1	2.6144
7	2.3	3	4.1	0.1	2.2541

A decrease in the value of lambda with a positive shift in the value of gamma increases the value of the ALR whereas the same decrease in lambda with a negative shift in the value of gamma produces a decreasing ARL. The details are shown in Tables 21 and 22.

Table 21: Decrease in lambda with a positive shift in gamma

γ_0	γ_1	γ	λ	α	ARL
7	12	3	0.5	0.1	- 1.9970
7	13.5	3	0.5	0.1	- 1.5305
7	14.6	3	0.5	0.1	- 1.3059
7	16.5	3	0.5	0.1	- 1.0410
7	17.5	3	0.5	0.1	- 0.9404
7	19.0	3	0.5	0.1	- 0.8210
7	21.4	3	0.5	0.1	- 0.6821
7	22.5	3	0.5	0.1	- 0.6329
7	25.8	3	0.5	0.1	- 0.5202
7	27.1	3	0.5	0.1	- 0.4860

Table 22: Decrease in the value of lambda with a negative shift in gamma

γ_0	γ_1	γ	λ	α	ARL
7	6.85	3	0.5	0.1	67.9282
7	6.2	3	0.5	0.1	12.7887
7	5.9	3	0.5	0.1	9.3203
7	5.4	3	0.5	0.1	6.4322
7	5.1	3	0.5	0.1	5.4303
7	4.7	3	0.5	0.1	4.5026
7	4.3	3	0.5	0.1	3.8515
7	3.9	3	0.5	0.1	3.3703
7	3.5	3	0.5	0.1	3.0012
7	3.0	3	0.5	0.1	2.6471
7	2.3	3	0.5	0.1	2.2860

Increasing the value of alpha with a positive shift in gamma slowly increases the value of the ARL and the same positive shift in gamma with a decrease in alpha also increases the ARL. The details are displayed in Tables 23 and 24.

Table 23: Increase in alpha with a positive shift in gamma

γ_0	γ_1	γ	λ	α	ARL
7	12.0	3	2	0.9	- 1.0795
7	13.5	3	2	0.9	- 0.8284
7	14.6	3	2	0.9	- 0.7074
7	16.5	3	2	0.9	- 0.5646
7	17.5	3	2	0.9	- 0.5103
7	19.0	3	2	0.9	- 0.4458
7	21.4	3	2	0.9	- 0.3708
7	22.5	3	2	0.9	- 0.3442
7	25.8	3	2	0.9	- 0.2832
7	27.1	3	2	0.9	- 0.2647
7	27.9	3	2	0.9	- 0.2544

Table 24: Decrease in alpha with a positive shift in gamma

γ_0	γ_1	γ	λ	α	ARL
7	12.0	3	2	0.01	- 3.0122
7	13.5	3	2	0.01	- 2.3089
7	14.6	3	2	0.01	- 1.9702
7	16.5	3	2	0.01	- 1.5709
7	17.5	3	2	0.01	- 1.4190
7	19.0	3	2	0.01	- 1.2390
7	21.4	3	2	0.01	- 1.0295
7	22.5	3	2	0.01	- 0.9553
7	25.8	3	2	0.01	- 0.7852
7	27.1	3	2	0.01	- 0.7337
7	27.9	3	2	0.01	- 0.7052

The ARL also decreases in value when alpha is increased with a negative shift in gamma as shown in Table 25. Similarly,a decrease in alpha with a negative shift in gamma produces a decreasing ARL value as indicated in Table 26.

Table 25: Increase in alpha with a negative shift in gamma

γ_0	γ_1	γ	λ	α	ARL
7	6.85	3	2	0.9	36.4686
7	6.2	3	2	0.9	6.8562
7	5.9	3	2	0.9	4.9932
7	5.4	3	2	0.9	3.4414
7	5.1	3	2	0.9	2.9028
7	4.7	3	2	0.9	2.4038
7	4.3	3	2	0.9	2.0532
7	3.9	3	2	0.9	1.7938
7	3.5	3	2	0.9	1.5944

7	3.0	3	2	0.9	1.4023
7	2.3	3	2	0.9	1.2048

Table 26: Decrease in alpha with a negative shift in gamma

γ_0	γ_1	γ	λ	α	ARL
7	6.85	3	2	0.01	102.3800
7	6.2	3	2	0.01	19.2717
7	5.9	3	2	0.01	14.0439
7	5.4	3	2	0.01	9.6907
7	5.1	3	2	0.01	8.1804
7	4.7	3	2	0.01	6.7818
7	4.3	3	2	0.01	5.8001
7	3.9	3	2	0.01	5.0745
7	3.5	3	2	0.01	4.5178
7	3.0	3	2	0.01	3.9834
7	2.3	3	2	0.01	3.4380

7. Practical Demonstration

In this section the developed CUSUM chart is evaluated with random numbers generated from the distribution and real dataset from the world of work

7.1. Random Sample from the WW Distribution

A random sample of size of twenty (20) was drawn from the WW distribution using the quantile function given in equation (4) and the values are shown in Table 27.The CUSUM plot of these random numbers isillustrated in Figure 2. It can be deduced that only three points are in control and the rest of the plotted points are found outside the left area of the V-mask showing the sensitivity of the WW CUSM control chart.

Table 27: Practical illustration with random numbers

x	Log (x)	CUSUM
5.1259	1.6343	1.6
2816.6450	7.9433	9.5
227.9300	5.4249	14.9
17.6351	2.8699	17.8
296.9666	5.6902	23.5
145.8691	4.9827	28.5
681.4453	6.5242	35.0
986.7509	6.8944	41.9
338.4893	5.8245	47.7
107.6000	4.6784	52.4
1441.1070	7.2732	59.7
18.5188	2.9187	62.6

33.1530	3.5011	66.1
216.8271	5.3791	71.5
435.9000	6.0774	77.6
11.9293	2.4790	80.1
102.3200	4.6281	84.7
43.4645	3.7719	88.5
9.5314	2.2544	90.8
77462.2400	11.2575	102.1

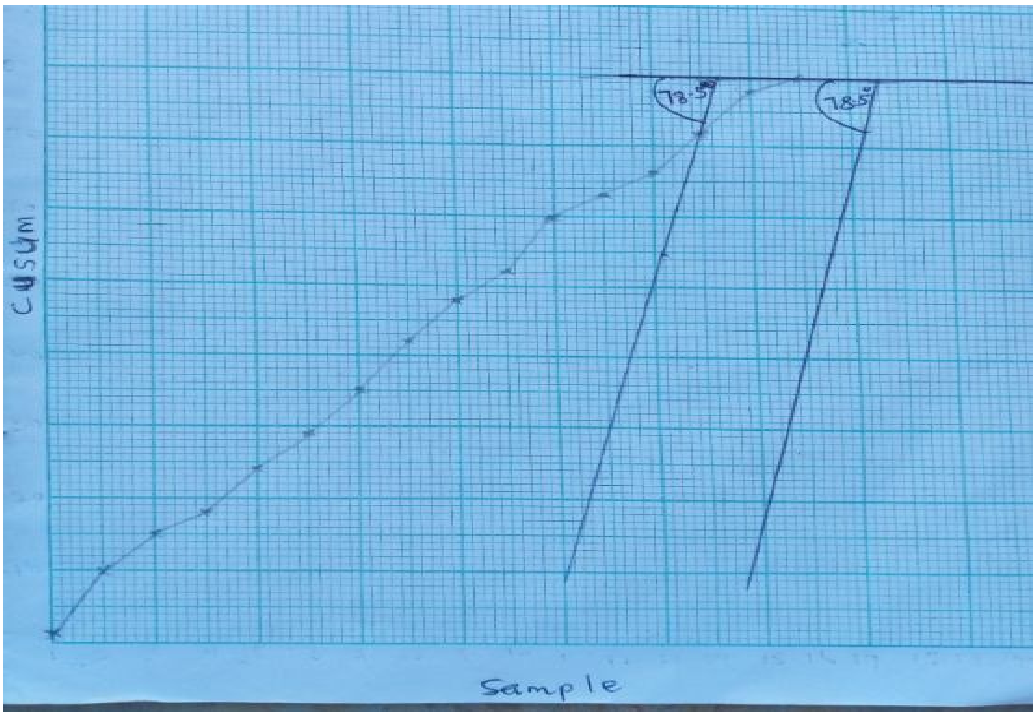

Figure 2: V-mask CUSUM of the Weighted Weibull distribution

7.2 Temperature dataset

The second dataset is obtained from the Kabsad Scientific Hospital in the Tamale Metropolis in the Northern region of Ghana. The data represents the temperature values of some thirty (30) selected patients in the outpatient department for September 2024.Figure 3 shows the mean plot of the dataset and shows the system is in control since no value is plotted outside the control limits. The V-mask plot of the same temperature dataset is shown in Figure 4. It shows only one point is in control and the rest of the plotted points are located outside the control limits. This shows the sensitivity of the CUSUM in detecting smaller shifts in a process system than the Shewhart control charts. The details are shown in Figure 4. The observations are as follows:

37.1, 36.9, 36.4, 36.1, 36.4, 36.6, 35.6, 37.0, 36.7, 36.7, 36.2, 36.0, 36.8, 36.6,36.6, 36.0, 36.7, 36.7, 37.1, 36.4, 36.2, 36.8, 36.5, 37.3, 37.5, 36.3, 36.4, 36.7, 36.2, 36.6.

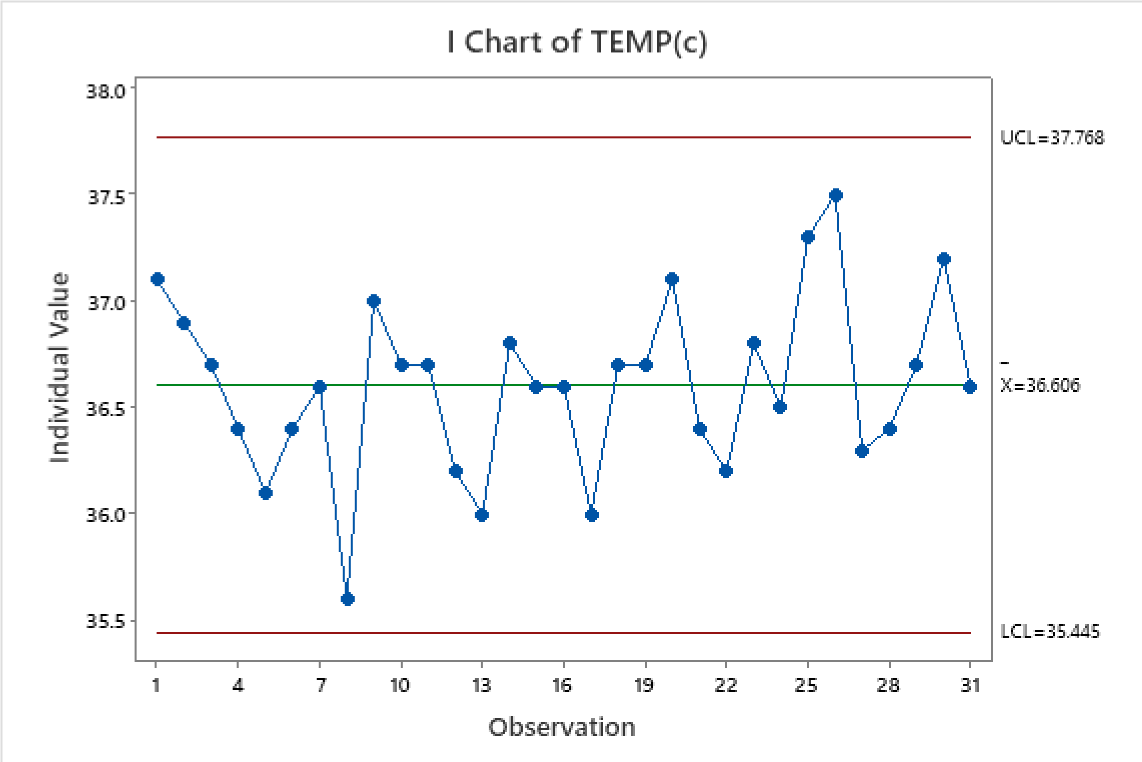

Figure 3: Shewhart control Chart

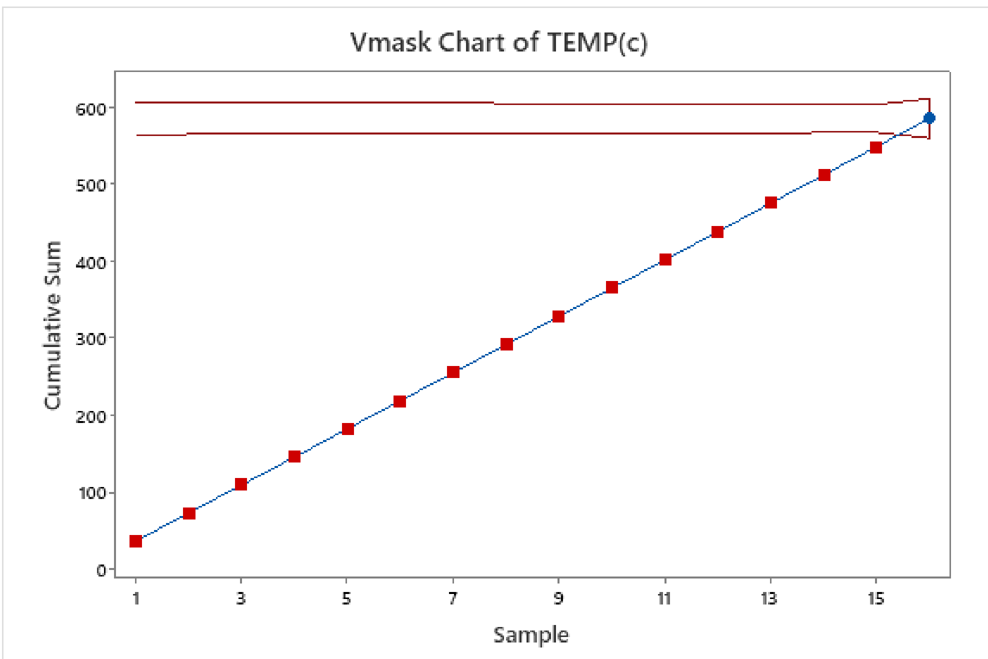

Figure 4: V-mast plot of the temperature dataset

8. Conclusion

The positive shift in the size of gamma decreases the size of the mask angle whereas the negative shift in gamma increases the size of the mask angle. The positive shift in the value of gamma reduces the lead distance and the negative shift in gamma increases the size of the lead distance. If there is a positive shift in gamma and lambda, the lead distance increases and a positive shift in gamma with a decrease in the value of lambda decreases the lead distance. Again, a negative shift in gamma with an increase or decrease in the value of lambda increases the value of the lead distance. Also, both an increase or decrease in the value of alpha decreases the value of lead distance when there is a positive shift in gamma. On the other hand, an increase or decrease in the value of alpha with a negative shift in gamma increases the value of the lead distance. Both increase or decrease in the value of theta with a positive shift decreases the value of the lead distance. An increase in the value of theta with a negative shift in gamma increases the value of the lead distance and a decrease in theta with a negative shift in gamma decreases the value of the lead distance. Not all but also a positive shift in gamma increases the value of the average run length anda decrease in the shift of gamma decreases the value of the average run length. An increase ora decrease in lambda with a positive shift in gamma increases the value of the average run length. Similarly,an increase or decrease in the value of lambda with a negative shift decreases the value of the average run length. Both increase or decrease in the value of alpha increases in the average run length. Last but not least, an increase or a decrease in the alpha with a negative shift in gamma decreases the average run length. The practical demonstration indicates that the proposed constructed cumulative sum control chart is sensitive to detecting a slight shift in a process system.

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