Original Research Article

ON FUZZY SOBOLEV SPACES $\widetilde{W}^{1,p}(\Omega)$: ANALYSIS OF DOT PRODUCT AND FUZZY NORM

Abstract

This paper proposes a study of fuzzy Sobolev spaces $\widetilde{W}^{1,p}(\Omega)$, integrating functions with triangular fuzzy coefficients. These fuzzy functional spaces aim to better model fuzzy functions, thus generalizing classical Sobolev spaces.

We establish the theoretical foundations of these spaces by analyzing the fuzzy scalar product and the fuzzy norm. We focus on verifying essential properties such as bilinearity, symmetry, positivity, homogeneity, and triangle inequality, using the Dubois and Prade α-cut approach to formalize the notion of uncertainty.

This paper addresses observations identified in the literature, where the lack of suitable fuzzy functional spaces for solving fuzzy differential equations, in particular fuzzy Sobolev spaces$\widetilde{W}^{1,p}(\Omega)$, is often not taken into account. Moreover, the analysis of fuzzy scalar product and norm properties is often presented vaguely. We thus propose a detailed approach to the functional properties of these spaces, extending classical Sobolev spaces to a fuzzy framework.

This research opens up prospects for practical applications in areas such as fuzzy differential equations and decision-making in medicine, economics, artificial intelligence, information processing and various other fields.

Keywords: Fuzzy $\widetilde{W}^{1,p}(\Omega)$Sobolev spaces, fuzzy spaces $\tilde{L}^p(\Omega)$, fuzzy scalar products, fuzzy norms, fuzzy derivatives, fuzzy integrations, α-cuts and fuzzy functions .

1. Introduction

Sobolev spaces classiques $W^{1,p}(\Omega)$, defined by

$W^{1,p}(\Omega)$= $\left\{f: \Omega \to \mathbb{R} \;/\; f \in L^p(\Omega) \text{ and } \frac{\partial f}{\partial x_i} \in L^p(\Omega), 1 \leq i \leq n\right\}$, where $\frac{\partial f}{\partial x_i}$ denotes the partial derivatives of the function f in the sense of distributions, are fundamental tools in functional analysis and in the theory of partial differential equations. These spaces allow the formalization of notions of regularity and integrability, facilitating the establishment of existence and uniqueness results for various classes of mathematical problems. However, they have significant limitations in the modeling of phenomena when the functions are fuzzy.

Fuzzy $\widetilde{W}^{1,p}(\Omega)$ Sobolev spaces, which incorporate functions with triangular fuzzy coefficients. These spaces emerge as a natural extension of classical Sobolev spaces, offering a more flexible approach to capture the uncertainty inherent in many systems. As such, fuzzy Sobolev spaces prove particularly relevant for practical applications.

In the existing literature, several notable contributions have been observed regarding fuzzy scalar products, fuzzy norm, fuzzy derivatives, and fuzzy differential equations. [1]In 2013, Chamkha Fatima Zohra examined the scalar product and norm of triangular fuzzy numbers

using the α-cut approach, but did not extend these concepts to fuzzy functions. The author also studied fuzzy differential equations without resorting to the appropriate fuzzy functional space for their resolution, which limited the impact of his contribution. His study paved the way for subsequent works, including Muslim Malik et al. who, [2]in 2021, addressed the existence, uniqueness, and controllability of a Sobolev-type fuzzy differential equation, without defining the fuzzy Sobolev spaces or the type of fuzzy functions used. Their work relies on techniques from classical functional analysis and fuzzy theory, establishing sufficient conditions to guarantee the controllability of the system. However, this approach does not directly apply to fuzzy Sobolev spaces $\widetilde{W}^{1,p}(\Omega)$ when the functions have triangular fuzzy coefficients.
Saima Rashid et al. proposed in 2021 [3]innovations in the field of generalized fuzzy transformations to solve fractional partial differential equations in oceanography. They developed semi-analytical methods that improve the accuracy and efficiency of the solutions. However, this approach has a shortcoming in that it does not take into account fuzzy Sobolev spaces, which are essential to ensure the mathematical rigor of the solutions.
Shreya Mukherjee et al. in 2024 examined the analytical solution of fuzzy fractional differential equations (FDEs) using the HATM method with Caputo's fractional derivative. They identified upper and lower fuzzy solutions for two FDEs, highlighting a [4]symmetry between them. Their method is presented as reliable and efficient for applications in applied mathematics and engineering. However, the lack of an analysis of fuzzy Sobolev spaces limits the rigor of their study.
In the face of these observations, the objective of this article is to define the Sobolev spaces $\widetilde{W}^{1,p}(\Omega)$ fuzzy spaces in the context of triangular fuzzy coefficient functions, using the Dubois and Prade α-cut approach. We aim to deepen the theoretical understanding of these spaces and provide a comprehensive analysis of their associated functional properties, thus contributing to a better modeling of uncertain phenomena.

2. Some notions of fuzzy logic [11, 12, 13, 18,19]

Fuzzy logic is an extension of classical or Boolean logic, introduced by Lofti Zadeh in 1965. In classical logic, there are only two states: true or false (1 or 0), with no possibility of intermediate states. Unlike fuzzy logic which accepts intermediate states between 1 and 0.

In other words, fuzzy logic is a set of mathematical concepts designed to model and represent ill-defined data, that is, data that is vague, uncertain, or imprecise.

2.1. Characteristic function and membership function

Let A be a non-empty set, A^C its complement and x an element of the universe $X = A \cup A^C$
(i) We call characteristic functions $\mu_A(x)$ the function defined by:

$$\mu_A(x) = \begin{cases} 1 & \text{si } x \in A \\ 0 & \text{si } x \notin A \text{ or } x \in A^C \end{cases}$$

(ii) We call the membership function $\mu_A(x)$ the function $\mu_A : A \to [0,1]$, such that:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \text{ (totally)} \\ 0 < \mu_A(x) < 1 & \text{if } x \in A \text{ (partially)} \\ 0 & \text{if } x \notin A \text{ (totally)} \end{cases}$$

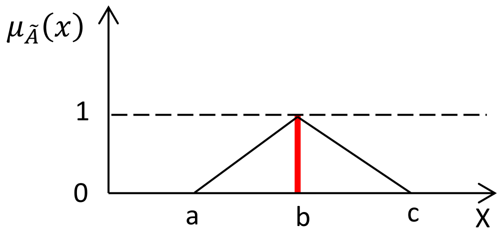
2.2.Fuzzy numbers

2.2.1. Triangular fuzzy number and membership function

A fuzzy subset $\tilde{A}$ with membership function $\mu_{\tilde{A}}(x)$ is called a triangular fuzzy number if there exist three real numbers $a < b < c$ such that:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ \dfrac{c-x}{c-b} & \text{if } b < x \le c \\ 0 & \text{elsewhere} \end{cases}$$

Graphically, this is represented as follows:

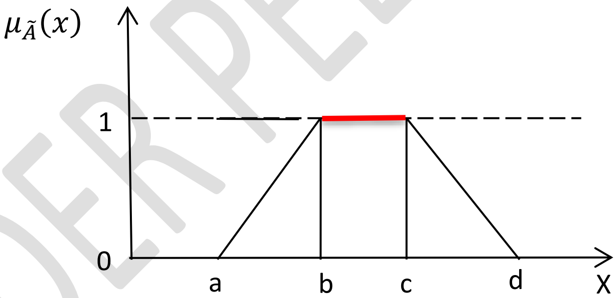


2.2.2. Trapezoidal fuzzy number and membership function

A fuzzy subset $\tilde{A}$ with membership function $\mu_{\tilde{A}}(x)$ is called a trapezoidal fuzzy number if there exist four real numbers $a < b < c < d$ such that:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } b < x \le c \\ \dfrac{d-x}{d-c} & \text{if } c < x \le d \\ 0 & \text{elsewhere} \end{cases}$$

Graphically, this is represented as follows:



2.3.Fuzzy arithmetic of α − coupes

2.3.1. α − coupe, kernel and support of a triangular fuzzy number

Definition 2.3.1.1. Let be $\tilde{A} = (a, b, c)$ a triangular fuzzy number such that $a < b < c$. The α − coupesare $\tilde{A}$ defined by the relation:

$\tilde{A} = [A_\alpha^-, A_\alpha^+] = [(b-a)\alpha + a, (b-c)\alpha + c], \qquad \alpha \in [0,1].$

Definition 2.3.1.2. We call the core of $\tilde{A}$ and denote it by $N(\tilde{A})$, the set:

$$N(\tilde{A}) = \{x \in X : \mu_{\tilde{A}}(x) = 1\}$$

Definition 2.3.1.3. We call support of $\tilde{A}$ and denote it by $supp(\tilde{A})$, the set:

$$supp(\tilde{A}) = \{x \in X : 0 \le \mu_{\tilde{A}}(x) \le 1\}$$

Or $\mu_{\tilde{A}}(x)$ represents the degree of membership of x the subset $\tilde{A}$.

Note 2.3.1.4. In practice, for a triangular fuzzy number $\tilde{A} = (a, b, c)$:

(i) if $\alpha = 0$ so $[A_0^-, A_0^+] = [a, c] = supp(\tilde{A})$.

(ii) if $\alpha = 1$ so $[A_1^-, A_1^+] = \{b\} = N(\tilde{A})$.

2.3.1. Operations on the$\alpha -$ coupes

Let $\tilde{A} = (a_1, c_1, b_1)$ and $\tilde{B} = (a_2, c_2, b_2)$ be two triangular fuzzy numbers, defined by their $\alpha -$ coupes respective: $\tilde{A} = [a_1, b_1]$ and $\tilde{B} = [a_2, b_2]$. We can then perform the following operations:

(i)Addition

$$\tilde{A} \oplus \tilde{B} = [a_1, b_1] \oplus [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$$

(ii)Subtraction

$$\tilde{A} \ominus \tilde{B} = [a_1, b_1] \ominus [a_2, b_2] = [a_1 - b_2, b_1 - a_2]$$

(iii)Multiplication

$$\tilde{A} \otimes \tilde{B} = [a_1, b_1] \otimes [a_2, b_2] = [Min\ G, Max\ G]$$

Where G is defined by $G = \{a_1a_2, a_1b_2, b_1a_2, b_1b_2\}$

(iv)Multiplication by a scalar
Let $\lambda \in \mathbb{R}$ and $\tilde{A} = [a_1, b_1]$
If $\lambda > 0, \lambda \otimes [a_1, b_1] = [\lambda a_1, \lambda b_1]$
If $\lambda < 0, \lambda \otimes [a_1, b_1] = [\lambda b_1, \lambda a_1]$
(v)Division

$$\frac{\tilde{A}}{\tilde{B}} = \frac{[a_1, b_1]}{[a_2, b_2]} = \left[\text{Min}\left(\frac{a_1}{a_2}, \frac{a_1}{b_2}, \frac{b_1}{a_2}, \frac{b_1}{b_2}\right), \text{Max}\left(\frac{a_1}{a_2}, \frac{a_1}{b_2}, \frac{b_1}{a_2}, \frac{b_1}{b_2}\right)\right] (\tilde{B} \neq 0).$$

2.4. Fuzzy functions, fuzzy differentiation and integration [5,14, 15, 16, 17,20]

2.4.1. Functions with triangular fuzzy coefficients

Definition 2.4.1.1. Let be $\mathbb{R}_{\mathcal{F}}$ the set of fuzzy numbers. We call a function of a variable with triangular fuzzy $\tilde{f}$coefficients any function denoted by:

$$\tilde{f}: \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$$

Definition 2.4.1.2. Let be $\mathbb{R}_{\mathcal{F}}$ the set of fuzzy numbers. We call a function of several variables with triangular fuzzy $\tilde{f}$ coefficients any function denoted by:

$$\tilde{f}: \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}}$$

The representation of $\alpha -$ coupesa fuzzy function is given by:

$$\tilde{f}_\alpha(x) = \left[\left(\tilde{f}_\alpha(x)\right)^-, \left(\tilde{f}_\alpha(x)\right)^+\right], \quad \alpha \in [0,1].$$

2.4.2. Fuzzy derivation

Definition 2.4.2.1. Let be $\tilde{f}: \mathbb{R} \longrightarrow \mathbb{R}_{\mathcal{F}}$ a function of a variable with triangular fuzzy coefficients. We call fuzzy derivative of $\tilde{f}$ in x the sense of Hukuhara and we denote $\tilde{f}'(x)$ the expression:

$$\tilde{f}'(x) = \lim_{h \to 0} \frac{\tilde{f}(x+h) \ominus \tilde{f}(x)}{h} < +\infty.$$

The approach of the $\alpha -$ coupesde $\tilde{f}'(x)$ is as follows:

$$\tilde{f}_\alpha(x) = \left[\left(\tilde{f}_\alpha{}'(x)\right)^-, \left(\tilde{f}_\alpha{}'(x)\right)^+\right], \quad \alpha \in [0,1].$$

Definition 2.4.2.2. Let $\tilde{f}: \mathbb{R}^2 \longrightarrow \mathbb{R}_{\mathcal{F}}$ a function of two variables with triangular fuzzy coefficients be given. We call partial derivative of order 1 of $\tilde{f}(x, y)$in the sense of Hukuhara the expressions denoted $\frac{\partial \tilde{f}}{\partial x}$ and $\frac{\partial \tilde{f}}{\partial y}$ by, defined by:

$$\frac{\partial \tilde{f}}{\partial x} = \lim_{h \to 0} \frac{\tilde{f}(x+h,y) \ominus \tilde{f}(x,y)}{h} < +\infty$$

And

$$\frac{\partial \tilde{f}}{\partial y} = \lim_{h \to 0} \frac{\tilde{f}(x,y+h) \ominus \tilde{f}(x,y)}{h} < +\infty$$

The approaches α − coupesare as follows:

$$\frac{\partial \tilde{f}_\alpha}{\partial x} = \left[\left(\frac{\partial \tilde{f}_\alpha}{\partial x}\right)^-, \left(\frac{\partial \tilde{f}_\alpha}{\partial x}\right)^+\right], \qquad \alpha \in [0,1].$$

$$\frac{\partial \tilde{f}_\alpha}{\partial y} = \left[\left(\frac{\partial \tilde{f}_\alpha}{\partial y}\right)^-, \left(\frac{\partial \tilde{f}_\alpha}{\partial y}\right)^+\right], \qquad \alpha \in [0,1].$$

2.4.3. Fuzzy integration

Definition 2.4.3.1 . Let be $\tilde{f}$ a function with triangular fuzzy coefficients, and let be $\Omega \subset \mathbb{R}$ its domain of integration. The fuzzy integral I of $\tilde{f}$ sur Ω is expressed by the following relation:

$$I = \int_\Omega \tilde{f}(x)dx.$$

Using the approach α − coupes, we can formulate:

$$\int_\Omega \tilde{f}(x)dx = \left[\int_\Omega \left(\tilde{f}_\alpha(x)\right)^- dx, \left(\tilde{f}_\alpha(x)\right)^+ dx\right], \qquad \alpha \in [0,1].$$

Properties 2.4.3.2. Consider $\tilde{f}$ and $\tilde{g}: [a,b] \to \mathbb{R}_\mathcal{F}$ two one-variable functions with triangular fuzzy coefficients, with $\ell, k \in \mathbb{R}$. The following properties are established:

$(i) \int_a^b \left[\ell \tilde{f}(x) + k\tilde{g}(x)\right]dx = \ell \int_a^b \tilde{f}(x)dx + k \int_a^b \tilde{g}(x)dx$

$(ii) \int_a^b \tilde{f}(x)dx = \int_a^c \tilde{f}(x)dx + \int_c^b \tilde{f}(x)dx, c \in [a,b].$

3. Sobolev spaces $W^{1,p}(\Omega)$ [6, 7, 8, 9, 10, 21, 22]

Definition 3.1. Let an open number and a real number $1 \leq p \leq +\infty$ be $\Omega \subset \mathbb{R}^n$. The Sobolev space of order 1, denoted $W^{1,p}(\Omega)$, is the set of functions $f: \Omega \to \mathbb{R}$ such that:

$W^{1,p}(\Omega) = \left\{f: \Omega \to \mathbb{R} \,/\, f \in L^p(\Omega) \text{ and } \frac{\partial f}{\partial x_i} \in L^p(\Omega), 1 \leq i \leq n\right\}$

where $\frac{\partial f}{\partial x_i}$ denotes the partial derivative of $f$with respect to the variable x_i, in the sense of distributions. These derivatives are defined by the following equality for all $\varphi \in \mathcal{D}(\Omega)$(the set of test functions with compact support in Ω):

$\langle \frac{\partial f}{\partial x_i}, \varphi \rangle = -\langle f, \frac{\partial \varphi}{\partial x_i} \rangle$, or $\langle .,. \rangle$ denotes the duality between distributions and test function.

When $p = 2$,space $W^{1,2}(\Omega)$is usually denoted $H^1(\Omega)$, and it is defined by:

$$H^1(\Omega) = \left\{f: \Omega \to \mathbb{R} \,/\, f \in L^2(\Omega) \text{ and } \frac{\partial f}{\partial x_i} \in L^2(\Omega), 1 \leq i \leq n\right\}.$$

Definition 3.2 . The space $W^{1,p}(\Omega)$ is equipped with the standard $\|.\|_{W^{1,p}(\Omega)}$defined, for any function $f \in W^{1,p}(\Omega)$, by:

$$\|f\|_{W^{1,p}(\Omega)} = \begin{cases} \left(\|f\|_{L^p(\Omega)}^p + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{L^p(\Omega)}^p \right)^{1/p} & \text{if } 1 \leq p < +\infty \\ \max\left(\|f\|_{L^p(\Omega)}, \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{L^p(\Omega)} \right) & \text{if } p = +\infty \end{cases}$$

Definition 3.3. The space $H^1(\Omega)$ is equipped with a scalar product $\langle .,. \rangle_{H^1(\Omega)}$defined, for all functions $f, g \in H^1(\Omega)$, by :
$\langle f, g \rangle_{H^1(\Omega)} = \int_\Omega \left(fg + \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} \right) dx.$
Proposition 3.4. Space $\left(H^1(\Omega), \langle .,. \rangle_{H^1(\Omega)}\right)$ is a pre-Hilbertian space.
Proposition 3.5. Space $\left(W^{1,p}(\Omega), \|.\|_{W^{1,p}(\Omega)}\right)$ is a normed space for $1 \leq p \leq +\infty$.

4. Fuzzy Sobolev spaces $\widetilde{W}^{1,p}(\Omega)$ [5, 14, 15, 16,17]

4.1. The $\tilde{L}^p(\Omega)$ fuzzy space

Definition 4.1.1 . Let Ω an open set of $\mathbb{R}^n$and p a real number such that $1 \leq p < +\infty$.We define the $\tilde{L}^p(\Omega)$fuzzy space as the set of functions with triangular fuzzy coefficients $\tilde{f}: \Omega \to \mathbb{R}_{\mathcal{F}}$ such that $\tilde{f}$ is integrable on Ω in the fuzzy sense, and that the p-integrable norm is defined. Let us formulate this more precisely:

$$\tilde{L}^p(\Omega) = \{\tilde{f}: \Omega \to \mathbb{R}_{\mathcal{F}} \ / \ \tilde{f} \text{ is integrable}, \int_\Omega |\tilde{f}(x)|^p dx < +\infty\}.$$

This space is equipped with the fuzzy standard $\|.\|_{\tilde{L}^p(\Omega)}$, defined for any function $\tilde{f} \in \tilde{L}^p(\Omega)$ by:

$$\|\tilde{f}\|_{\tilde{L}^p(\Omega)} = \left(\int_\Omega |\tilde{f}(x)|^p dx\right)^{1/p}.$$

The approach $\alpha -$ coupesfor the norm $\|\tilde{f}\|_{\tilde{L}^p(\Omega)}$is expressed as follows: for each $\alpha \in [0,1]$, the fuzzy norm is estimated as a function of the $\alpha -$ coupes lower $(\tilde{f}_\alpha^-)$ and upper bounds $(\tilde{f}_\alpha^+)$ of the function $\tilde{f}$:

$$\|\tilde{f}\|_{\tilde{L}^p(\Omega)} = \left(\left[\int_\Omega |\tilde{f}_\alpha^-(x)|^p dx, \int_\Omega |\tilde{f}_\alpha^+(x)|^p dx\right]\right)^{1/p}, \ \alpha \in [0,1].$$

Definition 4.1.2. For $p = 2$,us, we define space $\tilde{L}^2(\Omega)$as the set of functions $\tilde{f}: \Omega \to \mathbb{R}_{\mathcal{F}}$that are integrable and verify:

$$\tilde{L}^2(\Omega) = \left\{\tilde{f}: \Omega \to \mathbb{R}_{\mathcal{F}} \ / \ \tilde{f} \text{ is integrable}, \int_\Omega |\tilde{f}(x)|^2 dx < +\infty\right\}.$$

This space is equipped with a fuzzy scalar product $\langle .,. \rangle_{\tilde{L}^2(\Omega)}$, defined, for all functions $\tilde{f}, \tilde{g} \in \tilde{L}^2(\Omega)$,by:

$$\langle \tilde{f}, \tilde{g} \rangle_{\tilde{L}^2(\Omega)} = \int_\Omega \tilde{f}(x) \tilde{g}(x) dx.$$

The approach $\alpha -$ coupesfor the scalar product $\langle \tilde{f}, \tilde{g} \rangle_{\tilde{L}^2(\Omega)}$is expressed as follows. For each $\alpha \in [0,1]$, we consider the $\alpha -$ coupes lower $(\tilde{f}_\alpha^-, \tilde{g}_\alpha^-)$and upper $(\tilde{f}_\alpha^+, \tilde{g}_\alpha^+)$ of the functions $\tilde{f}$ and $\tilde{g}$:

$$\langle \tilde{f}, \tilde{g} \rangle_{\tilde{L}^2(\Omega)} = \left[\int_\Omega [\tilde{f}_\alpha(x) \tilde{g}_\alpha(x)]^- dx, \int_\Omega [\tilde{f}_\alpha(x) \tilde{g}_\alpha(x)]^+ dx\right], \ \alpha \in [0,1].$$

Proposition 4.1.3. Space $\left(\tilde{L}^2(\Omega), \langle .,. \rangle_{\tilde{L}^2(\Omega)}\right)$ is a fuzzy pre-Hilbertian space.
Proof. See [5], Proposition 5.4. ■

Proposition 4.1.4. The space $\left(\tilde{L}^p(\Omega), \|.\|_{\tilde{L}^p(\Omega)}\right)$ is a fuzzy normed space for $p \geq 1$.
Proof. See [5], Proposition 5.5. ■

4.2. Fuzzy Sobolev space $\widetilde{W}^{1,p}(\Omega)$

Definition 4.2.1. Let Ω an open set of $\mathbb{R}^n$ and p a real number such that $1 \le p \le +\infty$.We denote by $\widetilde{W}^{1,p}(\Omega)$the first-order fuzzy Sobolev space , which is made up of the functions with triangular fuzzy coefficients $\tilde{f}: \Omega \to \mathbb{R}_{\mathcal{F}}$ for which $\tilde{f}$and its partial derivatives belong to $\tilde{L}^p(\Omega)$. We therefore define:

$$\widetilde{W}^{1,p}(\Omega) = \left\{\tilde{f}: \Omega \to \mathbb{R}_{\mathcal{F}} \ / \ \tilde{f} \in \tilde{L}^p(\Omega) \text{ and } \frac{\partial \tilde{f}}{\partial x_i} \in \tilde{L}^p(\Omega), 1 \le i \le n\right\}$$

or $\frac{\partial \tilde{f}}{\partial x_i}$ denotes the fuzzy partial derivatives of $\tilde{f}$ in the sense of distributions, defined by:

$$\langle \frac{\partial \tilde{f}}{\partial x_i}, \tilde{\varphi} \rangle = -\langle \tilde{f}, \frac{\partial \tilde{\varphi}}{\partial x_i} \rangle, \ \forall \tilde{\varphi} \in \widetilde{\mathcal{D}}(\Omega).$$

For $p = 2$,we identify the fuzzy Sobolev space $\widetilde{W}^{1,2}(\Omega)$ with $\widetilde{H}^1(\Omega)$. Thus, we formulate:

$$\widetilde{H}^1(\Omega) = \left\{\tilde{f}: \Omega \to \mathbb{R}_{\mathcal{F}} \ / \ \tilde{f} \in \tilde{L}^2(\Omega) \text{ and } \frac{\partial \tilde{f}}{\partial x_i} \in \tilde{L}^2(\Omega), 1 \le i \le n\right\}.$$

Definition 4.2.2. Fuzzy dot product on $\widetilde{H}^1(\Omega)$

and $\tilde{g}$ be $\tilde{f}$ two functions with triangular fuzzy coefficients, and Ω an open set of $\mathbb{R}^n$.
We define the fuzzy scalar product of $\tilde{f}$and $\tilde{g}$ by the following relation:

$$\langle \tilde{f}, \tilde{g} \rangle_{\widetilde{H}^1(\Omega)} = \int_\Omega \left(\tilde{f} \cdot \tilde{g} + \sum_{i=1}^n \frac{\partial \tilde{f}}{\partial x_i} \cdot \frac{\partial \tilde{g}}{\partial x_i}\right) dx.$$

The approach $\alpha -$ coupesfor the scalar product $\langle \tilde{f}, \tilde{g} \rangle_{\widetilde{H}^1(\Omega)}$is formulated as follows:

$$\langle \tilde{f}, \tilde{g} \rangle_{\widetilde{H}^1(\Omega)} = \left[\int_\Omega \left(\widetilde{f_\alpha} \cdot \widetilde{g_\alpha} + \sum_{i=1}^n \frac{\partial \widetilde{f_\alpha}}{\partial x_i} \cdot \frac{\partial \widetilde{g_\alpha}}{\partial x_i}\right)^- dx, \int_\Omega \left(\widetilde{f_\alpha} \cdot \widetilde{g_\alpha} + \sum_{i=1}^n \frac{\partial \widetilde{f_\alpha}}{\partial x_i} \cdot \frac{\partial \widetilde{g_\alpha}}{\partial x_i}\right)^+ dx\right], \alpha \in [0,1].$$

Definition 4.2.3. Fuzzy standards in $\widetilde{W}^{1,p}(\Omega)$

Let us consider Ω as an open set of $\mathbb{R}^n$ with $1 \le p \le +\infty$. Let $\tilde{f}$ a function whose coefficients are triangular fuzzy numbers. We define:

$$(i) \qquad \|\tilde{f}\|_{\widetilde{H}^1(\Omega)} = \sqrt{\langle \tilde{f}, \tilde{f} \rangle_{\tilde{L}^2(\Omega)}} = \left(\|\tilde{f}\|^2_{\tilde{L}^2(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \tilde{f}}{\partial x_i}\right\|^2_{\tilde{L}^2(\Omega)}\right)^{1/2}$$

The approach $\alpha -$ coupesfor the standard $\|\tilde{f}\|_{\widetilde{H}^1(\Omega)}$is formulated as follows:

$$\|\tilde{f}\|_{\widetilde{H}^1(\Omega)} = \left(\left[\left(\|\widetilde{f_\alpha}\|^2_{\tilde{L}^2(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \widetilde{f_\alpha}}{\partial x_i}\right\|^2_{\tilde{L}^2(\Omega)}\right)^-, \left(\|\widetilde{f_\alpha}\|^2_{\tilde{L}^2(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \widetilde{f_\alpha}}{\partial x_i}\right\|^2_{\tilde{L}^2(\Omega)}\right)^+\right]\right)^{1/2}, \alpha \in [0,1]$$

$$(ii) \qquad \|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} = \begin{cases} \left(\|\tilde{f}\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \tilde{f}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^{1/p} & \text{if } 1 \le p < +\infty \\ \max\left(\|\tilde{f}\|_{\tilde{L}^p(\Omega)}, \sum_{i=1}^n \left\|\frac{\partial \tilde{f}}{\partial x_i}\right\|_{\tilde{L}^p(\Omega)}\right) & \text{if } p = +\infty \end{cases}$$

The approach $\alpha -$ coupesfor the standard $\|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)}$is expressed as follows:

For $1 \le p < +\infty$:

$$\|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} = \left(\left[\left(\|\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^-, \left(\|\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+\right]\right)^{1/p}.$$

For $p = +\infty$:

$$\|\tilde{f}\|_{\widetilde{W}^{1,+\infty}(\Omega)} = \left[\min\left(\|\widetilde{f_\alpha}\|_{\tilde{L}^p(\Omega)}, \sum_{i=1}^n \left\|\frac{\partial \widetilde{f_\alpha}}{\partial x_i}\right\|_{\tilde{L}^p(\Omega)}\right), \max\left(\|\widetilde{f_\alpha}\|_{\tilde{L}^p(\Omega)}, \sum_{i=1}^n \left\|\frac{\partial \widetilde{f_\alpha}}{\partial x_i}\right\|_{\tilde{L}^p(\Omega)}\right)\right], \alpha \in [0,1].$$

Proposition 4.2.4. Space $\left(\widetilde{H}^1(\Omega),\langle .,.\rangle_{\widetilde{H}^1(\Omega)}\right)$ is a fuzzy pre-hilbertian space.

Proof.
It suffices to show that $\langle .,.\rangle_{\widetilde{H}^1(\Omega)}$constitutes a fuzzy scalar product on $\widetilde{H}^1(\Omega)$ using the approach of $\alpha -$ coupes.We must therefore check the following properties: bilinearity, symmetry and definite positivity.

$(\boldsymbol{i})$ **Bilinearity**
For everyone $\tilde{f},\tilde{g},\tilde{p},\tilde{q}\in\widetilde{H}^1(\Omega)$and for all $a,b,c,d\in\mathbb{R}$, we have:
$\langle a\tilde{f}+b\tilde{g},c\tilde{p}+d\tilde{q}\rangle_{\widetilde{H}^1(\Omega)}=ac\langle\tilde{f},\tilde{p}\rangle_{\widetilde{H}^1(\Omega)}+ad\langle\tilde{f},\tilde{q}\rangle_{\widetilde{H}^1(\Omega)}+bc\langle\tilde{g},\tilde{p}\rangle_{\widetilde{H}^1(\Omega)}+bd\langle\tilde{g},\tilde{q}\rangle_{\widetilde{H}^1(\Omega)}$
Let's calculate $\langle a\tilde{f}+b\tilde{g},c\tilde{p}+d\tilde{q}\rangle_{\widetilde{H}^1(\Omega)}$using the approach of $\alpha -$ coupes :

$$\langle a\tilde{f}+b\tilde{g},c\tilde{p}+d\tilde{q}\rangle_{\widetilde{H}^1(\Omega)}=\left[\int_\Omega\left((a\tilde{f}_\alpha+b\tilde{g}_\alpha)(c\tilde{p}_\alpha+d\tilde{q}_\alpha)+\sum_{i=1}^n\frac{\partial}{\partial x_i}(a\tilde{f}_\alpha+b\tilde{g}_\alpha)\frac{\partial}{\partial x_i}(c\tilde{p}_\alpha+d\tilde{q}_\alpha)\right)^- dx,\int_\Omega\left((a\tilde{f}_\alpha+b\tilde{g}_\alpha)(c\tilde{p}_\alpha+d\tilde{q}_\alpha)+\sum_{i=1}^n\frac{\partial}{\partial x_i}(a\tilde{f}_\alpha+b\tilde{g}_\alpha)\frac{\partial}{\partial x_i}(c\tilde{p}_\alpha+d\tilde{q}_\alpha)\right)^+ dx\right],\alpha\in[0,1]$$

Expanding the derivatives term, we obtain:

$$\frac{\partial}{\partial x_i}(a\tilde{f}_\alpha+b\tilde{g}_\alpha)=a\frac{\partial\tilde{f}_\alpha}{\partial x_i}+b\frac{\partial\tilde{g}_\alpha}{\partial x_i}\ \text{et}\ \frac{\partial}{\partial x_i}(c\tilde{p}_\alpha+d\tilde{q}_\alpha)=c\frac{\partial\tilde{p}_\alpha}{\partial x_i}+d\frac{\partial\tilde{q}_\alpha}{\partial x_i}$$

$$\sum_{i=1}^n\frac{\partial}{\partial x_i}(a\tilde{f}_\alpha+b\tilde{g}_\alpha)\frac{\partial}{\partial x_i}(c\tilde{p}_\alpha+d\tilde{q}_\alpha)=\sum_{i=1}^n ac\ \frac{\partial\tilde{f}_\alpha}{\partial x_i}\frac{\partial\tilde{p}_\alpha}{\partial x_i}+\sum_{i=1}^n ad\ \frac{\partial\tilde{f}_\alpha}{\partial x_i}\frac{\partial\tilde{q}_\alpha}{\partial x_i}+\sum_{i=1}^n bc\ \frac{\partial\tilde{g}_\alpha}{\partial x_i}\frac{\partial\tilde{p}_\alpha}{\partial x_i}+\sum_{i=1}^n bd\ \frac{\partial\tilde{g}_\alpha}{\partial x_i}\frac{\partial\tilde{q}_\alpha}{\partial x_i}$$

By developing the product term, we have:

$$(a\tilde{f}_\alpha+b\tilde{g}_\alpha)(c\tilde{p}_\alpha+d\tilde{q}_\alpha)=ac\ \tilde{f}_\alpha\tilde{p}_\alpha+ad\ \tilde{f}_\alpha\tilde{q}_\alpha+bc\tilde{g}_\alpha\tilde{p}_\alpha+bd\tilde{g}_\alpha\tilde{q}_\alpha\ .$$

Applying the properties of sum, multiplication and linearity of integrals, we obtain:

$$\begin{aligned}\langle a\tilde{f}+b\tilde{g},c\tilde{p}+d\tilde{q}\rangle_{\widetilde{H}^1(\Omega)}&=ac\left[\int_\Omega\left(\tilde{f}_\alpha\tilde{p}_\alpha+\sum_{i=1}^n\frac{\partial\tilde{f}_\alpha}{\partial x_i}\frac{\partial\tilde{p}_\alpha}{\partial x_i}\right)^- dx,\left(\tilde{f}_\alpha.\tilde{p}_\alpha+\sum_{i=1}^n\frac{\partial\tilde{f}_\alpha}{\partial x_i}\frac{\partial\tilde{p}_\alpha}{\partial x_i}\right)^+ dx\right]\\&+ad\left[\int_\Omega\left(\tilde{f}_\alpha\tilde{q}_\alpha+\sum_{i=1}^n\frac{\partial\tilde{f}_\alpha}{\partial x_i}\frac{\partial\tilde{q}_\alpha}{\partial x_i}\right)^- dx,\left(\tilde{f}_\alpha\tilde{q}_\alpha+\sum_{i=1}^n\frac{\partial\tilde{f}_\alpha}{\partial x_i}\frac{\partial\tilde{q}_\alpha}{\partial x_i}\right)^+ dx\right]\\&+bc\left[\int_\Omega\left(\tilde{g}_\alpha\tilde{p}_\alpha+\sum_{i=1}^n\frac{\partial\tilde{g}_\alpha}{\partial x_i}\frac{\partial\tilde{p}_\alpha}{\partial x_i}\right)^- dx,\left(\tilde{g}_\alpha\tilde{p}_\alpha+\sum_{i=1}^n\frac{\partial\tilde{g}_\alpha}{\partial x_i}\frac{\partial\tilde{p}_\alpha}{\partial x_i}\right)^+ dx\right]\\&+bd\left[\int_\Omega\left(\tilde{g}_\alpha\tilde{q}_\alpha+\sum_{i=1}^n\frac{\partial\tilde{g}_\alpha}{\partial x_i}\frac{\partial\tilde{q}_\alpha}{\partial x_i}\right)^- dx,\left(\tilde{g}_\alpha\tilde{q}_\alpha+\sum_{i=1}^n\frac{\partial\tilde{g}_\alpha}{\partial x_i}\frac{\partial\tilde{q}_\alpha}{\partial x_i}\right)^+ dx\right]\\&=ac\langle\tilde{f},\tilde{p}\rangle_{\widetilde{H}^1(\Omega)}+ad\langle\tilde{f},\tilde{q}\rangle_{\widetilde{H}^1(\Omega)}+bc\langle\tilde{g},\tilde{p}\rangle_{\widetilde{H}^1(\Omega)}+bd\langle\tilde{g},\tilde{q}\rangle_{\widetilde{H}^1(\Omega)}.\end{aligned}$$

This proves the bilinearity of $\langle .,.\rangle_{\widetilde{H}^1(\Omega)}$.

$(\boldsymbol{ii})$ **Symmetry**

To show symmetry, we need to show that:
For all $\tilde{f},\tilde{g}\in\widetilde{H}^1(\Omega)$, we have:

$$\langle\tilde{f},\tilde{g}\rangle_{\widetilde{H}^1(\Omega)}=\langle\tilde{g},\tilde{f}\rangle_{\widetilde{H}^1(\Omega)}.$$

Let's define the integrals on each side:

$$\langle \tilde{f},\tilde{g}\rangle_{\widetilde{H}^1(\Omega)}=\left[\int_\Omega\left(\tilde{f}_\alpha\tilde{g}_\alpha+\sum_{i=1}^{n}\frac{\partial\tilde{f}_\alpha}{\partial x_i}\frac{\partial\tilde{g}_\alpha}{\partial x_i}\right)^{-}dx,\int_\Omega\left(\tilde{f}_\alpha\tilde{g}_\alpha+\sum_{i=1}^{n}\frac{\partial\tilde{f}_\alpha}{\partial x_i}\frac{\partial\tilde{g}_\alpha}{\partial x_i}\right)^{+}dx\right],\alpha\in[0,1].$$

$$\langle \tilde{g},\tilde{f}\rangle_{\widetilde{H}^1(\Omega)}=\left[\int_\Omega\left(\tilde{g}_\alpha\tilde{f}_\alpha+\sum_{i=1}^{n}\frac{\partial\tilde{g}_\alpha}{\partial x_i}\frac{\partial\tilde{f}_\alpha}{\partial x_i}\right)^{-}dx,\int_\Omega\left(\tilde{g}_\alpha\tilde{f}_\alpha+\sum_{i=1}^{n}\frac{\partial\tilde{g}_\alpha}{\partial x_i}\frac{\partial\tilde{f}_\alpha}{\partial x_i}\right)^{+}dx\right],\alpha\in[0,1].$$

Given the commutativity of multiplication and addition of functions with triangular fuzzy coefficients, we can deduce:

$\tilde{f}_\alpha\tilde{g}_\alpha=\tilde{g}_\alpha\tilde{f}_\alpha$, and $\sum_{i=1}^{n}\frac{\partial\tilde{f}_\alpha}{\partial x_i}\frac{\partial\tilde{g}_\alpha}{\partial x_i}=\sum_{i=1}^{n}\frac{\partial\tilde{g}_\alpha}{\partial x_i}\frac{\partial\tilde{f}_\alpha}{\partial x_i}$

Thus, it follows that:

$$\int_\Omega\left(\tilde{f}_\alpha\tilde{g}_\alpha+\sum_{i=1}^{n}\frac{\partial\tilde{f}_\alpha}{\partial x_i}\frac{\partial\tilde{g}_\alpha}{\partial x_i}\right)^{-}dx=\int_\Omega\left(\tilde{g}_\alpha\tilde{f}_\alpha+\sum_{i=1}^{n}\frac{\partial\tilde{g}_\alpha}{\partial x_i}\frac{\partial\tilde{f}_\alpha}{\partial x_i}\right)^{-}dx,\ \alpha\in[0,1]$$

$$\int_\Omega\left(\tilde{f}_\alpha\tilde{g}_\alpha+\sum_{i=1}^{n}\frac{\partial\tilde{f}_\alpha}{\partial x_i}\frac{\partial\tilde{g}_\alpha}{\partial x_i}\right)^{+}dx=\int_\Omega\left(\tilde{g}_\alpha\tilde{f}_\alpha+\sum_{i=1}^{n}\frac{\partial\tilde{g}_\alpha}{\partial x_i}\frac{\partial\tilde{f}_\alpha}{\partial x_i}\right)^{+}dx,\ \alpha\in[0,1]$$

So we get $\langle \tilde{f},\tilde{g}\rangle_{\widetilde{H}^1(\Omega)}=\langle \tilde{g},\tilde{f}\rangle_{\widetilde{H}^1(\Omega)}$, which demonstrates the symmetry.

$(\boldsymbol{iii})$ **Positivity Defined**

To show definite positivity, we have:

$$\langle \tilde{f},\tilde{f}\rangle_{\widetilde{H}^1(\Omega)}=\left[\int_\Omega\left([\tilde{f}_\alpha]^2+\sum_{i=1}^{n}\left(\frac{\partial\tilde{f}_\alpha}{\partial x_i}\right)^2\right)^{-}dx,\int_\Omega\left([\tilde{f}_\alpha]^2+\sum_{i=1}^{n}\left(\frac{\partial\tilde{f}_\alpha}{\partial x_i}\right)^2\right)^{+}dx\right],\alpha\in[0,1].$$

The term $[\tilde{f}_\alpha]^2+\sum_{i=1}^{n}\left(\frac{\partial\tilde{f}_\alpha}{\partial x_i}\right)^2$ is positive or zero, which implies that the integral is also positive or zero.

So we have:$\langle \tilde{f},\tilde{f}\rangle_{\widetilde{H}^1(\Omega)}\geq 0$.

It is null if and only if:

$[\tilde{f}_\alpha]^2+\sum_{i=1}^{n}\left(\frac{\partial\tilde{f}_\alpha}{\partial x_i}\right)^2=0$, for almost everything $\alpha\in[0,1]$.

This implies that $\tilde{f}_\alpha=0$ and $\frac{\partial\tilde{f}_\alpha}{\partial x_i}=0$ for $\alpha\in[0,1]$.Thus, we have $\tilde{f}=0$ on Ω in $\widetilde{H}^1(\Omega)$.Therefore, the positivity is clearly established.

By virtue of the points $(i),(ii)$ et (iii), $\langle .,.\rangle_{\widetilde{H}^1(\Omega)}$constitutes a fuzzy scalar product on $\widetilde{H}^1(\Omega)$.

Therefore, the space $\left(\widetilde{H}^1(\Omega),\langle .,.\rangle_{\widetilde{H}^1(\Omega)}\right)$ is a fuzzy pre-hilbertian space.■

Proposition 4.2.5. The space $\left(\widetilde{W}^{1,p}(\Omega),\|.\|_{\widetilde{W}^{1,p}(\Omega)}\right)$ is a fuzzy normed space for $1\leq p<+\infty$.

Proof.

It suffices to show that $\|.\|_{\widetilde{W}^{1,p}(\Omega)}$is a fuzzy norm on the fuzzy Sobolev space $\widetilde{W}^{1,p}(\Omega)$.

We must check the three characteristic properties of a norm: positivity, homogeneity and triangle inequality.

The fuzzy norm $\|.\|_{\widetilde{W}^{1,p}(\Omega)}$is defined according to the approach $\alpha-$coupes as follows:

For all $\tilde{f} \in \widetilde{W}^{1,p}(\Omega)$, we have :

$$\|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} = \left(\left[\left(\|\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^-, \left(\|\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+\right]\right)^{\frac{1}{p}}, \alpha \in [0,1].$$

(i) Positivity

We must verify that for all $\tilde{f} \in \widetilde{W}^{1,p}(\Omega)$:
$\|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} \geq 0$ and $\|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} = 0$ if and only if $\tilde{f} = 0$ on Ω.

$$\|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} = \left(\left[\left(\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^-, \left(\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+\right]\right)^{1/p}, \quad \alpha \in [0,1].$$

To verify that this expression is positive, we note that:
$\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} \geq 0$, $\left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)} \geq 0$ and $\sum_{i=1}^n \left\|\frac{\partial f_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)} \geq 0$.
Therefore :

$$\left(\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^- \geq 0 \text{ et } \left(\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+ \geq 0, \ \alpha \in [0,1].$$

Which implies that $\|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} \geq 0$.
Furthermore, $\|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} = 0$ if and only if

$$\left(\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^- = 0 \text{ and } \left(\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+ = 0 \quad ,\alpha \in [0,1].$$

$\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} = 0$ and $\sum_{i=1}^n \left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)} = 0$, which means that $\tilde{f} = 0$ almost everywhere on Ω.
Positivity is thus established.

(ii) Homogeneity

For any $\lambda \in \mathbb{R}$ and a fuzzy function $\tilde{f} \in \widetilde{W}^{1,p}(\Omega)$,we must check that:
$\|\lambda\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} = |\lambda|\|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)}$.
Applying the property of homogeneity of norms $\tilde{L}^p(\Omega)$, we obtain:

$$\|\lambda\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} = \left(\left[\left(\|(\lambda\widetilde{f_\alpha})\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial(\lambda\widetilde{f_\alpha})}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^-, \left(\|(\lambda\widetilde{f_\alpha})\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial(\lambda\widetilde{f_\alpha})}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+\right]\right)^{1/p}, \alpha \in [0,1]$$

$$= \left(\left[\left(\|\lambda\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\lambda\partial\widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^-, \left(\|\lambda\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\lambda\partial\widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+\right]\right)^{1/p}$$

$$= \left(\left[\left(|\lambda|^p\|\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + |\lambda|^p\sum_{i=1}^n \left\|\frac{\partial\widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^-, \left(|\lambda|^p\|\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + |\lambda|^p\sum_{i=1}^n \left\|\frac{\partial\widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+\right]\right)^{1/p}$$

$$= \left(|\lambda|^p\left[\left(\|\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial\widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^-, \left(\|\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial\widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+\right]\right)^{1/p}$$

$$= |\lambda|\left(\left[\left(\|\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial\widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^-, \left(\|\widetilde{f_\alpha}\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^n \left\|\frac{\partial\widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+\right]\right)^{1/p}$$

$$= |\lambda|.\|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)}.$$

Homogeneity is indeed verified.

$(\boldsymbol{iii})$ **Triangular inequality**

For two fuzzy functions $\tilde{f}, \tilde{g} \in \widetilde{W}^{1,p}(\Omega)$, we must show that:
$\|\tilde{f} + \tilde{g}\|_{\widetilde{W}^{1,p}(\Omega)} \leq \|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} + \|\tilde{g}\|_{\widetilde{W}^{1,p}(\Omega)}$.
Let's calculate $\|\tilde{f} + \tilde{g}\|_{\widetilde{W}^{1,p}(\Omega)}$.
Applying the triangle inequality for the norms $\|.\|_{\tilde{L}^p(\Omega)}$,we obtain:

$$\|\tilde{f} + \tilde{g}\|^p_{\tilde{L}^p(\Omega)} \leq \|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \|\widetilde{g_\alpha}\|^p_{\tilde{L}^p(\Omega)}$$

and for the derivatives:

$$\left\|\frac{\partial}{\partial x_i}(\tilde{f}_\alpha + \tilde{g}_\alpha)\right\|^p_{\tilde{L}^p(\Omega)} \leq \left\|\frac{\partial \widetilde{f_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)} + \left\|\frac{\partial \tilde{g}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}.$$

Which means that:

$$\left(\|\tilde{f}_\alpha + \tilde{g}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}(\tilde{f}_\alpha + \tilde{g}_\alpha)\right\|^p_{\tilde{L}^p(\Omega)}\right)^- \leq \left(\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^- + \left(\|\tilde{g}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial \widetilde{g_\alpha}}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^-.$$

$$\left(\|\tilde{f}_\alpha + \tilde{g}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}(\tilde{f}_\alpha + \tilde{g}_\alpha)\right\|^p_{\tilde{L}^p(\Omega)}\right)^+ \leq \left(\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+ + \left(\|\tilde{g}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial \tilde{g}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+.$$

Combining these inequalities, we obtain:

$$\|\tilde{f} + \tilde{g}\|_{\widetilde{W}^{1,p}(\Omega)} = \left(\left[\left(\|\tilde{f}_\alpha + \tilde{g}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}(\tilde{f}_\alpha + \tilde{g}_\alpha)\right\|^p_{\tilde{L}^p(\Omega)}\right)^-, \left(\|\tilde{f}_\alpha + \tilde{g}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial}{\partial x_i}(\tilde{f}_\alpha + \tilde{g}_\alpha)\right\|^p_{\tilde{L}^p(\Omega)}\right)^+\right]\right)^{1/p}$$

$$\leq \left(\left[\left(\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^-, \left(\|\tilde{f}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial \tilde{f}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+\right]\right)^{1/p}$$
$$+ \left(\left[\left(\|\tilde{g}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial \tilde{g}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^-, \left(\|\tilde{g}_\alpha\|^p_{\tilde{L}^p(\Omega)} + \sum_{i=1}^{n}\left\|\frac{\partial \tilde{g}_\alpha}{\partial x_i}\right\|^p_{\tilde{L}^p(\Omega)}\right)^+\right]\right)^{1/p}$$
$$\leq \|\tilde{f}\|_{\widetilde{W}^{1,p}(\Omega)} + \|\tilde{g}\|_{\widetilde{W}^{1,p}(\Omega)}.$$

This establishes the triangle inequality.
Thus, we have just shown that $\|.\|_{\widetilde{W}^{1,p}(\Omega)}$satisfies all the properties of a norm, which proves that it is indeed a fuzzy norm on the fuzzy Sobolev space$\widetilde{W}^{1,p}(\Omega)$. Therefore, $\left(\widetilde{W}^{1,p}(\Omega), \|.\|_{\widetilde{W}^{1,p}(\Omega)}\right)$ is a fuzzy normed space.■

5. Conclusion

In this paper, we have studied fuzzy $\widetilde{W}^{1,p}(\Omega)$Sobolev spaces, focusing on the fuzzy scalar product and the fuzzy norm, which are based on functions with triangular fuzzy coefficients. We demonstrated that the fuzzy scalar product defines a symmetric and positive definite bilinear map in the space $\widetilde{H}^1(\Omega)$, thus giving this space a pre-hilbertian structure, essential for the development of analytical and numerical methods adapted to fuzzy contexts. Furthermore, the fuzzy norm turns out to be a fundamental tool to quantify the properties of fuzzy functions in $\widetilde{W}^{1,p}(\Omega)$. α-cut based approaches have allowed to deepen our understanding of the continuity and convergence behaviors of these functions. Finally, this study highlights the growing importance of fuzzy Sobolev spaces $\widetilde{W}^{1,p}(\Omega)$ in fields such as fuzzy differential equations, engineering, data science, image processing, medicine and economics, where uncertainty management is essential. The results obtained open the way to future research work including:

- Optimization of structures using fuzzy Sobolev spaces to evaluate uncertainties.
- Development of machine learning algorithms exploiting fuzzy Sobolev spaces to process uncertain data.
- Improving image restoration techniques using fuzzy Sobolev space norms.
- Using fuzzy Sobolev spaces to model individual variations in biomedical data.
- Analysis of uncertain economic models via fuzzy Sobolev spaces to refine forecasts.
- Study of solutions of partial differential equations using fuzzy Sobolev spaces to deal with coefficient uncertainty.

By integrating these research avenues, we hope to contribute to the advancement of fuzzy Sobolev spaces and their applications in practical contexts, thus enriching the field of fuzzy mathematical analysis.

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