Existence and uniqueness of solution of Hadamard-type temporal fractional equations and numerical application using the SBA plus method

Original Research Article

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Abstract

The work addressed in this article consists in guaranteeing the existence and uniqueness of solutions for Cauchy problems involving fractional Hadamard differential equations. Our results are based on the classical Weissinger fixed-point theorem. Finally, we present three examples of solutions using the SBA plus method.

Keywords: SOME BLAISE ABBO (SBA)plus method, Hadamard fractional integral, Hadamard fractional derivative, Hadamard fractional equation

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1 Introduction

The growing interest of fractional calculus in the field of modeling is well established. A number of authors have developed theories on fractional integration and derivation operators, which have led to the understanding, popularization and use of these notions in various fields of applied science and engineering. In the literature, the most commonly used operators are the Riemann-Liouville integral, the Riemann-Liouville derivative, the Caputo derivative and the Grünwald-Letnikov derivative. But alongside these operators we have Hadamard's fractional integral, Hadamard's derivative, Caputo-Hadamard's derivative, etc., which are less widely used. In all cases, the equations resulting from the modeling of a given phenomenon are, among others, algebraic, differential, integro-differential and integral, whose solutions are sometimes difficult to find, or even inaccessible, hence the need to ensure the existence and/or uniqueness of such solutions before any solution-seeking investigation.

In this article we focus on the existence and uniqueness of fractional equations using fractional derivation and integration in Hadamard's sense. Work on this question can be found in the literature.

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For example, in (1; 7; 17; 19; 25; 27; 30; 32; 36) various results on the existence and uniqueness of Hadamard-type equations are presented. The major innovation in our work is that we apply Weissinger's fixed-point theorem to prove the existence and uniqueness of fractional Hadamard equations of Cauchy type. In addition, we use the SBA plus method to illustrate the solutions of some equations.

After recalling some basic notions of fractional calculus in section 2 and studying the existence and uniqueness of Hadamard's fractional time differential equations in section 3, the section 4 is devoted to the application of the SBA plus method on a few equations. Section 5 is the conclusion.

2 Preliminaires

The essential definitions, properties and theorems we present for our work can be found in (1; 13; 16; 18; 30; 33; 36). We invite the reader to refer to them for further details.

2.1 Fractional integral in Hadamard's sense

Let $a, b \in \mathbb{R}$ $0 < a < t < b < +\infty$, $\alpha \in \mathbb{R}^+$ and $f \in L^1[a, b]$. The fractional Hadamard integral of order α of f is defined by

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$$ {}^{H}\mathcal{I}_a^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} f(\tau)\frac{d\tau}{\tau}, \; for\ \alpha > 0 \\ f(\tau), for\ \alpha = 0 \end{cases} \tag{2.1} $$

2.2 Fractional derivation in Hadamard's sense

Let $a, b \in \mathbb{R}$, $0 < a < t < b < +\infty$ and $\alpha \in \mathbb{R}^+$ with $n = [\alpha]+1$, $\delta = t\dfrac{d}{dt}$, let $AC[a; b]$ and $f \in AC_{\delta}^{n}[a, b]$ a space of absolutely continuous functions and $AC_{\delta}^{n} = \{f : [a; b] \to \mathbb{R}; f, \delta^{n-1} f(t) \in AC[a; b]$. The Hadamard fractional derivative of order α of f is defined by

$$ {}^{H}\mathcal{D}_a^{\alpha} f(t) = \begin{cases} \delta^n \; {}^{H}\mathcal{I}_a^{n-\alpha} f(t) = \dfrac{1}{\Gamma(n-\alpha)} \left(t\dfrac{d}{dt}\right)^n \displaystyle\int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{n-\alpha-1} f(\tau)\frac{d\tau}{\tau}, \; for\ \alpha > 0 \\ f(\tau), for\ \alpha = 0 \end{cases} \tag{2.2} $$

Let $a, b \in \mathbb{R}$ $0 < a < t < b < +\infty$ and $\alpha \in \mathbb{R}^+$ with $n = [\alpha] + 1$. For $f \in L^1[a; b]$, we have the following properties:

$$ {}^{H}\mathcal{D}_a^{\alpha} \; {}^{H}\mathcal{I}_a^{\alpha} f(t) = f(t); \tag{2.3} $$

$$ {}^{H}\mathcal{I}_a^{\alpha} \; {}^{H}\mathcal{D}_a^{\alpha} f(t) = f(t) - \sum_{j=1}^{n} \frac{\left(\delta^{n-j} \; {}^{H}\mathcal{I}_a^{n-\alpha} f\right)(a)}{\Gamma(\alpha - j + 1)} \left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j} \tag{2.4} $$

such that ${}^{H}\mathcal{I}_a^{n-\alpha} f \in AC_{\delta}^{n}[a, b]$.

2.3 Fractional time differential equations in Hadamard's sense

Let $a, b \in \mathbb{R}$, $0 < a < t < b < +\infty$ and $\alpha \in \mathbb{R}^+$ with $n = [\alpha] + 1$ and $\delta = t\dfrac{d}{dt}$ and let $AC[a; b]$ a space of absolutely continuous functions and $AC_{\delta}^{n} = \left\{u : [a; b] \to \mathbb{R}; u, \delta^{n-1} u(t) \in AC[a; b]\right\}$. Then, for all

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$u \in AC^n_\delta[a,b]$, the equality

$$ {}^H\mathcal{D}^\alpha_a u(t) = f(t,u(t)) \tag{2.5}$$

is called a Hadamard fractional differential equation. The initial conditions for this Fractional differential equation (FDE) are :

$$(\delta^{n-j}\ {}^H\mathcal{I}^{n-\alpha}_a u)(a) = b_j, j = 1,2,...,n. \tag{2.6}$$

2.4 Weissinger's fixed point theorem

Suppose (U,d) is a complete non-empty metric space, and let $\alpha_j \geq 0$ for all $j \in \mathbb{N}$ and such that the series $\sum_{j=0}^{+\infty} \alpha_j$ converges. Let the mapping $A : U \to U$ which verifies the inequality $d(A^j u, A^j v) \leq \alpha_j d(u,v)$ for all $j \in \mathbb{N}$ and all $u,v \in U$. Then A has a unique fixed point u^*. Moreover, the sequence $(A^j u_0)_{j=1}^{+\infty}$ converges to this fixed point u^* (16).

3 Existence and uniqueness of fractional time differential equations in Hadamard's sense

Here we discuss the existence and uniqueness properties of solutions of fractional-order differential equations in Hadamard's sense. We will restrict ourselves to Cauchy problems.

:Let $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $n-1 < \alpha < n$. In addition, either $K > 0$, $h^* > 0$ and $b_j \in \mathbb{R}$, $j = 1,2,...,n$. The set E is defined by :

$$E = \left\{ (t,u) \in \mathbb{R}^2, 0 < a < t < h^*, u \in \mathbb{R}\ if\ t = a\ and \left| \left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha} u - \sum_{j=1}^{n} \frac{b_j \left[ln\left(\frac{t}{a}\right)\right]^{n-j}}{\Gamma(n-j+1)} \right| < K\ otherwise \right\}.$$

Suppose that the function $f : E \to \mathbb{R}$ is continuous and bounded on E and verifies the Lipschitz condition with respect to the second variable, that is, there exists a constant $L > 0$ such that, for all (t,u_1) and (t,u_2) of E, we have :

$$|f(t,u_1) - f(t,u_2)| < L|u_1 - u_2|. \tag{3.1}$$

Then the Hadamard fractional differential equation (2.5) with initial condition (2.6) has a unique solution $u \in C(a,h]$ where

$$h = min\left\{ h^*, \tilde{h}, \left(\frac{\Gamma(\alpha+1)K}{M}\right)^{\frac{1}{n}} \right\}\ with\ M = \sup_{t,z \in E} |f(t,z)|, \tag{3.2}$$

and $\tilde{h}$ is a positive real that satisfies:

$$\tilde{h} < \left(\frac{\Gamma(2\alpha-n+1)}{\Gamma(\alpha-n+1)L}\right)^{\frac{1}{\alpha}}. \tag{3.3}$$

This result is analogous to the theorem known for fractional differential equations in the Riemann-Liouville sense. We'll start by transforming the initial-value problem into an equivalent Volterra integral equation. Next, we'll prove the existence and uniqueness of the solution to the integral equation using Weissinger's fixed-point theorem.

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Under the assumptions of the theorem (3) with $h > 0$, the function $u \in C(a, h]$ is a solution of the differential equation

$$\begin{cases} {}^H\mathcal{D}_a^\alpha u(t) = f(t, u(t)) \\ (\delta^{n-j}\ {}^H\mathcal{I}_a^{n-\alpha} u)(a) = b_j, j = 1, 2, ..., n \end{cases} \tag{3.4}$$

if and only if it is a solution of the Volterra integral equation

$$u(t) = \sum_{j=1}^{n} \frac{b_j \left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j}}{\Gamma(\alpha - j + 1)} + \frac{1}{\Gamma(\alpha)} \int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} f(\tau, u(\tau))\frac{d\tau}{\tau}. \tag{3.5}$$

: **Let's show that u solution of (3.5) implies that u is solution of (3.4).**
Suppose that u is a solution of the Volterra equation (3.5). This equation can be written as.

$$u(t) = \sum_{j=1}^{n} \frac{b_j \left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j}}{\Gamma(\alpha - j + 1)} + {}^H\mathcal{I}_a^\alpha f(., u(.))(t). \tag{3.6}$$

When we apply the differential operator ${}^H\mathcal{D}_a^\alpha$ to the two members of equality (3.6), we immediately obtain that u is also a solution of the equation (3.4).
Indeed :

$$\begin{aligned} {}^H\mathcal{D}_a^\alpha u(t) &= {}^H\mathcal{D}_a^\alpha \sum_{j=1}^{n} \frac{b_j \left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j}}{\Gamma(\alpha - j + 1)} + {}^H\mathcal{D}_a^\alpha\ {}^H\mathcal{I}_a^\alpha f(., u(.))(t) \\ &= \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)}\ \delta^n \left(\frac{1}{\Gamma(\alpha - n)} \int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{n-\alpha-1} \left[ln\left(\frac{\tau}{a}\right)\right]^{\alpha-j} \frac{d\tau}{\tau} \right) + f(t, u(t)). \end{aligned}$$

Posing $y = \dfrac{ln\left(\frac{\tau}{a}\right)}{ln\left(\frac{t}{a}\right)}$, we obtain:

$$\begin{aligned} {}^H\mathcal{D}_a^\alpha u(t) &= \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)\Gamma(\alpha - n)}\ \delta^n \left(\int_0^1 (1-y)^{n-\alpha-1} y^{\alpha-j} \left[ln\left(\frac{t}{a}\right)\right]^{n-j} dy \right) + f(t, u(t)) \\ &= \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)\Gamma(\alpha - n)} \beta(n - \alpha, \alpha - j + 1)\ \delta^n \left(\left[ln\left(\frac{t}{a}\right)\right]^{n-j} \right) + f(t, u(t)) \\ &= \sum_{j=1}^{n} \frac{b_j}{\Gamma(n - j + 1)}\ \delta^n \left(\left[ln\left(\frac{t}{a}\right)\right]^{n-j} \right) + f(t, u(t)). \end{aligned}$$

As $1 \leq j \leq n$ when $n = j$ we have

$$\delta^n \left(\left[ln\left(\frac{t}{a}\right)\right]^{n-j} \right) = 0. \tag{3.7}$$

Thus

$${}^H\mathcal{D}_a^\alpha u(t) = f(t, u(t)). \tag{3.8}$$

For the initial conditions, let's apply $\delta^{n-j}\ {}^H\mathcal{I}_a^{n-\alpha}$ to both members of equation (3.5), we have:

$$\delta^{n-j}\ {}^H\mathcal{I}_a^{n-\alpha} u(t) = \delta^{n-j}\ {}^H\mathcal{I}_a^{n-\alpha} \sum_{j=1}^{n} \frac{b_j \left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j}}{\Gamma(\alpha - j + 1)} + \delta^{n-j}\ {}^H\mathcal{I}_a^{n-\alpha} \frac{1}{\Gamma(\alpha)} \int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} f(\tau, u(\tau))\frac{d\tau}{\tau}.$$

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Or

$$\begin{aligned}
\delta^{n-j}\ {}^{H}\mathcal{I}_a^{n-\alpha}\sum_{j=1}^{n}\frac{b_j\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j}}{\Gamma(\alpha-j+1)} &= \sum_{j=1}^{n}\frac{b_j}{\Gamma(\alpha-j+1)}\delta^{n-j}\ {}^{H}\mathcal{I}_a^{n-\alpha}\left(\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j}\right)\\
&= \sum_{j=1}^{n}\frac{b_j}{\Gamma(\alpha-j+1)\Gamma(n-\alpha)}\delta^{n-j}\int_a^t\left(\left[ln\left(\frac{t}{\tau}\right)\right]^{n-\alpha-1}\left[ln\left(\frac{\tau}{a}\right)\right]^{\alpha-j}\frac{d\tau}{\tau}\right)\\
&= \sum_{j=1}^{n}\frac{b_j}{\Gamma(n-j+1)}\delta^{n-j}\left(\left[ln\left(\frac{t}{a}\right)\right]^{n-j}\right)\\
&= \sum_{j=1}^{n}\frac{b_j}{\Gamma(n-j+1)}\times\Gamma(n-j+1)\\
&= \sum_{j=1}^{n}b_j\ (*)
\end{aligned}$$

and

$$\begin{aligned}
\delta^{n-j}\ {}^{H}\mathcal{I}_a^{n-\alpha}\frac{1}{\Gamma(\alpha)}\int_a^t\left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1}f(\tau,u(\tau))d\tau &= \delta^{n-j}\ {}^{H}\mathcal{I}_a^{n}f(t,u(t))\\
&= \delta^{n-j}\frac{1}{\Gamma(n)}\int_a^t\left[ln\left(\frac{t}{\tau}\right)\right]^{n-1}f(\tau,u(\tau))\frac{d\tau}{\tau}\\
&= 0\ for\ t=a\\
&= 0\ (**)
\end{aligned}$$

From (*) and (**) we obtain the initial conditions.

Let us now show that u solution of (3.4) implies that u is solution of (3.5).
Assume u is a continuous solution of the problem (3.4), and posit $P(t) = f(t, u(t))$. Next, assume that P is a continuous function. As

$$P(t) = f(t,u(t)) = {}^{H}\mathcal{D}_a^{\alpha}u(t) = \delta^{n}\ {}^{H}\mathcal{I}_a^{n-\alpha}u(t) \tag{3.9}$$

then $\delta^{n}\ {}^{H}\mathcal{I}_a^{n-\alpha}u(t)$ is continuous, i.e. ${}^{H}\mathcal{I}_a^{n-\alpha}u \in C(a,h]$. By applying ${}^{H}\mathcal{I}_a^{\alpha}(.)$ to the two members of the first equality of (3.4), we obtain from proposition (2.2):

$$u(t) = {}^{H}\mathcal{I}_a^{\alpha}f(.,u(.))(t) + \sum_{j=1}^{n}d_j\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j}. \tag{3.10}$$

By introducing the initial conditions defined by the second line of (3.4), we can determine the constants $d_j, j=\{1,2,...n\}$ where $d_j = \frac{b_j}{\Gamma(\alpha-j+1)}$.
This completes the demonstration. Under the assumptions of the theorem (3), the Volterra equation

$$u(t) = \sum_{j=1}^{n}\frac{b_j\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j}}{\Gamma(\alpha-j+1)} + \frac{1}{\Gamma(\alpha)}\int_a^t\left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1}f(\tau,u(\tau))\frac{d\tau}{\tau}$$

has a unique solution $u \in C(a,h]$. : Consider the set

$$U = \left\{u \in C(a,h] : \sup_{a<t\le h}\left|\left[ln\left(\frac{t}{\tau}\right)\right]^{n-\alpha}u(t) - \sum_{j=1}^{n}\frac{b_j\left[ln\left(\frac{t}{\tau}\right)\right]^{n-j}}{\Gamma(\alpha-j+1)}\right| \le K\right\} \tag{3.11}$$

on which we define the operator A by:

$$Au(t) = \sum_{j=1}^{n} \frac{b_j \left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j}}{\Gamma(\alpha-j+1)} + \frac{1}{\Gamma(\alpha)} \int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} f(\tau, u(\tau)) \frac{d\tau}{\tau}. \tag{3.12}$$

Let's show that if $u \in U$ then $Au \in U$ i.e. $A : U \to U$.
First, let's note that for all $u \in U$, Au is also a continuous function on $(a, h]$.
From the relation (3.12), the continuity of Au and the definition of M we have:

$$\begin{aligned}
\left| \left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha} Au(t) - \sum_{j=1}^{n} \frac{b_j \left[ln\left(\frac{t}{a}\right)\right]^{n-j}}{\Gamma(\alpha-j+1)} \right| &= \left| \frac{\left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha}}{\Gamma(\alpha)} \int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} f(\tau, u(\tau)) \frac{d\tau}{\tau} \right| \\
&= \frac{\left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha}}{\Gamma(\alpha)} \int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} \left| f(\tau, u(\tau)) \right| \frac{d\tau}{\tau} \\
&\leq \frac{\left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha}}{\Gamma(\alpha)} M \int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} \frac{d\tau}{\tau} \\
&\leq \frac{\left[ln\left(\frac{t}{a}\right)\right]^{n} M}{\Gamma(\alpha+1)} \\
&\leq K.
\end{aligned}$$

The last inequality is linked to the definition of h. Indeed, for $h = \left[ln\left(\frac{t}{a}\right)\right]$,

$$h < \left(\frac{\Gamma(\alpha+1)K}{M}\right)^{\frac{1}{n}} \Rightarrow \frac{\left[ln\left(\frac{t}{a}\right)\right]^{n} M}{\Gamma(\alpha+1)} < K. \tag{3.13}$$

We have thus shown that if $u \in U$ then $Au \in U$.
Let

$$\mathcal{V} = \left\{ u \in C(a, h] : \sup_{a<t\leq h} \left| \left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha} u(t) \right| < \infty \right\} \tag{3.14}$$

on which we define a norm $\|.\|$ by:

$$\|u\|_{\mathcal{V}} = \sup_{a<t\leq h} \left| \left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha} u(t) \right|. \tag{3.15}$$

Using the definition of A, we can rewrite the Volterra equation in a more compact form

$$u = Au. \tag{3.16}$$

So, to prove the desired relationship, all we need to do is show that the operator A has a unique fixed point. To do this, we'll use Weissinger's fixed point theorem. We'll prove that for $u_1, u_2 \in U$,

$$\| A^j u_1 - A^j u_2 \|_{\mathcal{V}} \leq \left(\frac{Lh^{\alpha}\Gamma(\alpha-n+1)}{\Gamma(2\alpha-n+1)}\right)^{j} \| u_1 - u_2 \|_{\mathcal{V}} . \tag{3.17}$$

We'll show this result by recurrence.
For $j = 0$, it is trivial.
Suppose that

$$\| A^{j-1} u_1 - A^{j-1} u_2 \|_{\mathcal{V}} \leq \left(\frac{Lh^{\alpha}\Gamma(\alpha-n+1)}{\Gamma(2\alpha-n+1)}\right)^{j-1} \| u_1 - u_2 \|_{\mathcal{V}} . \tag{3.18}$$

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Then,

$$\begin{aligned}
\| A^j u_1 - A^j u_2 \|_{\mathcal{V}} &= \sup_{a<t\leq h} \left| \left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha} \left(A^j u_1(t) - A^j u_2(t)\right) \right| \\
&= \sup_{a<t\leq h} \left| \left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha} \left(AA^{j-1} u_1(t) - AA^{j-1} u_2(t)\right) \right| \\
&= \sup_{a<t\leq h} \left| \frac{\left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha}}{\Gamma(\alpha)} \int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} \left(f\left(\tau, A^{j-1}u_1(\tau)\right) - f\left(\tau, A^{j-1}u_2(\tau)\right)\right) \frac{d\tau}{\tau} \right|
\end{aligned}$$

thanks to the definition of A.

Using the Lipschitz condition of f and the definition of the norm $\|.\|_{\mathcal{V}}$, we obtain:

$$\begin{aligned}
&\| A^j u_1 - A^j u_2 \|_{\mathcal{V}} \\
= \; & \sup_{a<t\leq h} \frac{\left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha}}{\Gamma(\alpha)} \int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} \left| f\left(\tau, A^{j-1}u_1(\tau)\right) - f\left(\tau, A^{j-1}u_2(\tau)\right)\right| \frac{d\tau}{\tau} \\
= \; & \frac{1}{\Gamma(\alpha)} \sup_{a<t\leq h} \left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha} \int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} \left| f\left(\tau, A^{j-1}u_1(\tau)\right) - f\left(\tau, A^{j-1}u_2(\tau)\right)\right| \frac{d\tau}{\tau} \\
\leq \; & \frac{L}{\Gamma(\alpha)} \| A^{j-1}u_1 - A^{j-1}u_2 \|_{\mathcal{V}} \sup_{a<t\leq h} \left[ln\left(\frac{t}{a}\right)\right]^{n-\alpha} \int_a^t \left[ln\left(\frac{t}{\tau}\right)\right]^{\alpha-1} \left[ln\left(\frac{t}{a}\right)\right]^{\alpha-n} \frac{d\tau}{\tau} \\
\leq \; & \frac{L}{\Gamma(\alpha)} \| A^{j-1}u_1 - A^{j-1}u_2 \|_{\mathcal{V}} \sup_{a<t\leq h} \frac{\Gamma(\alpha)\Gamma(\alpha-n+1)}{\Gamma(2\alpha-n+1)} \left[ln\left(\frac{t}{a}\right)\right]^{\alpha} \\
\leq \; & \frac{Lh^\alpha \Gamma(\alpha-n+1)}{\Gamma(2\alpha-n+1)} \| A^{j-1}u_1 - A^{j-1}u_2 \|_{\mathcal{V}} \\
\leq \; & \left(\frac{Lh^\alpha \Gamma(\alpha-n+1)}{\Gamma(2\alpha-n+1)}\right)^j \| u_1 - u_2 \|_{\mathcal{V}} \; (according\ to\ the\ recurrence\ hypothesis).
\end{aligned}$$

We can therefore use Weissinger's theorem with $\alpha_j = \lambda^j$ where

$$\lambda = \left(\frac{Lh^\alpha \Gamma(\alpha-n+1)}{\Gamma(2\alpha-n+1)}\right). \tag{3.19}$$

Now let's prove that the series $\sum_{j=0}^{+\infty} \alpha_j$ is convergent.

Using the definition of h, we have:

$$h < \tilde{h} < \left(\frac{\Gamma(2\alpha-n+1)}{\Gamma(\alpha-n+1)L}\right)^{\frac{1}{\alpha}} \Rightarrow \frac{Lh^\alpha \Gamma(\alpha-n+1)}{\Gamma(2\alpha-n+1)} < 1. \tag{3.20}$$

Therefore $\lambda < 1$. This justifies the convergence of the series $\sum_{j=0}^{+\infty} \alpha_j$.

Consequently, an application of the fixed-point theorem guarantees the existence and uniqueness of the solution of the Volterra integral equation (3.5).

4 Applications

In this section we propose the application of the SBA method to some fractional equations in the Hadamard sense. The method is described below, but for convergence we refer the reader to (10).

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4.1 Description in the general case

Solve the following fractional equation in Hadamard's sense:

$$\begin{cases} {}^{H}\mathcal{D}_a^{\alpha} u = -R(u) - N(u),\ 0 < a < t < T \\ (\delta^{m-j}\ {}^{H}\mathcal{I}_a^{n-\alpha} u)(a) = b_j, j = 1, 2..., m, \end{cases} \tag{4.1}$$

in a Banach space V where: R and N are linear and non-linear operators respectively in V; $u \in V$ the unknown function; ${}^{H}\mathcal{D}_a^{\alpha} u$ the fractional derivative of order α of u in Hadamard's sense; $m = [\alpha] + 1$, $[\alpha]$ is the integer part of α and $\delta = t\dfrac{d}{dt}$. Using the method of successive approximations, the above problem (4.1) can be approximated by the following iterative scheme:

$$\begin{cases} {}^{H}\mathcal{D}_a^{\alpha} u^k = -R(u^k) - N(u^{k-1}),\ 0 < a < t < T \\ (\delta^{m-j}\ {}^{H}\mathcal{I}_a^{n-\alpha} u^k)(a) = b_j, j = 1, 2..., m. \end{cases} , k \geq 1 \tag{4.2}$$

Solving the scheme (4.2) by the method of approximations consists in determining at each iteration ($k = 1, 2...$) approximate solutions $u^1, u^2..., u^k$ which are series. And the solution u sought for the problem (4.1), if it exists, is obtained by $u = \lim\limits_{k\to+\infty} u^k$, if the sequence $(u^k)_k$ is convergent in V. Indeed, with the choice of operators $L(.) = {}^{H}\mathcal{D}_a^{\alpha}(.),\ and\ L^{-1}(.) = {}^{H}\mathcal{I}_a^{\alpha}(.)$, where L^{-1} the inverse of L in the Adomian sense; the problem (4.2) can be written as:

$$u^k = \theta^k - L^{-1}\Big(R(u^k)\Big) - L^{-1}N(u^{k-1}), \tag{4.3}$$

with $\theta^k = \displaystyle\sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} \left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j}$ and the condition that the operator $L^{-1}R$ is contracting, because:

$$L^{-1}Lu^k(t) = u^k(t) - \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} \left[ln\left(\frac{t}{a}\right)\right]^{\alpha-j} = u^k(t) - \theta^k. \tag{4.4}$$

Then posing $u^k = \displaystyle\sum_{n=0}^{+\infty} u_n^k$, we derive the following SBA algorithm for a fixed k:

$$\begin{cases} u_0^k = \theta^k - L^{-1}(N(u^{k-1})) \\ u_{n+1}^k = -L^{-1}\left(R\left(u_n^k\right)\right), n \geq 0. \end{cases} , k \geq 1 \tag{4.5}$$

The development of the algorithm (4.5) consists in calculating the series $u^k, k \geq 1$.
Explicitly, for each fixed k, we calculate the terms of the series $(u_n^k)_n, n \in \mathbb{N}$ and deduce u^k. For the first iteration, $k = 1$, we choose u^0 such that $Nu^0 = 0$, calculate the terms $u_0^1, u_1^1, u_2^1, u_3^1...u_n^1$ of the series $(u_n^1)_n$ and deduce

$$u^1 = \sum_{n=0}^{+\infty} u_n^1; \tag{4.6}$$

Then we evaluate Nu^1.
If $Nu^1 = 0$ then u^1 is the general solution of the problem.
Otherwise, we have two alternatives:

- or, if possible, we replace the initial problem by an equivalent transformation, with $\overline{N}$ the new non-linear term, so that by repeating the algorithm, the new u^1 that we find can verify $\overline{N}u^1 = 0$;
- or we calculate $u^2, u^3, u^4, ..., u^k$ and deduce $\lim\limits_{k\longrightarrow+\infty} u^k$.

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4.2 Ordinary fractional non-linear differential equation in Hadamard's sense

Example 4.1. *: Consider the following fractional-time model:*

$$\begin{cases} \left({}^{H}\mathcal{D}_{a}^{\alpha}u(t)\right)^{2} + {}^{H}\mathcal{D}_{a}^{\alpha}u(t) - u^{2}(t) + u(t) = 0 \\ (\delta\ {}^{H}\mathcal{I}_{a}^{2-\alpha}u)(a) = \theta \\ ({}^{H}\mathcal{I}_{a}^{2-\alpha}u)(a) = \beta, \end{cases} \tag{4.7}$$

where $0 < a < t < b,\ 1 < \alpha \leq 2,\ \theta\ and\ \beta \in \mathbb{R},\ u \in C_{\delta}^{2}([a;b]),\ \delta = t\dfrac{d}{dt},\ {}^{H}\mathcal{D}_{a}^{\alpha}(.)$ derivative in Hadamard's sense, ${}^{H}\mathcal{I}_{a}^{\alpha}(.)$ the integral in Hadamard's sense.
Posing: $L(.) = {}^{H}\mathcal{D}_{a}^{\alpha}(.); L^{-1}(.) = {}^{H}\mathcal{I}_{a}^{\alpha}(.); Ru = -u; Nu = u^{2}(t) - \left({}^{H}\mathcal{D}_{a}^{\alpha}u(t)\right)^{2}$; where L^{-1} the inverse of L Adomian sense, we then derive the following SBA algorithm:

$$\begin{cases} u_{0}^{k} = \dfrac{\theta}{\Gamma(\alpha)}\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1} + \dfrac{\beta}{\Gamma(\alpha-1)}\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-2} + L^{-1}\left(N(u^{k-1})\right) \\ u_{n+1}^{k} = L^{-1}\left(R(u_{n}^{k})\right), n \geq 0 \end{cases} ,k \geq 1. \tag{4.8}$$

By developing the algorithm for $k = 1$, we obtain:

$$\begin{cases} u_{0}^{1}(t) = \dfrac{\theta}{\Gamma(\alpha)}\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1} + \dfrac{\beta}{\Gamma(\alpha-1)}\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-2} \\ u_{1}^{1}(t) = -\dfrac{\theta}{\Gamma(2\alpha)}\left[ln\left(\frac{t}{a}\right)\right]^{2\alpha-1} - \dfrac{\beta}{\Gamma(2\alpha-1)}\left[ln\left(\frac{t}{a}\right)\right]^{2\alpha-2} \\ u_{2}^{1}(t) = \dfrac{\theta}{\Gamma(3\alpha)}\left[ln\left(\frac{t}{a}\right)\right]^{3\alpha-1} + \dfrac{\beta}{\Gamma(3\alpha-1)}\left[ln\left(\frac{t}{a}\right)\right]^{3\alpha-2} \\ \vdots \\ u_{n}^{1}(t) = (-1)^{n}\dfrac{\theta}{\Gamma((n+1)\alpha)}\left[ln\left(\frac{t}{a}\right)\right]^{(n+1)\alpha-1} + (-1)^{n}\dfrac{\beta}{\Gamma((n+1)\alpha-1)}\left[ln\left(\frac{t}{a}\right)\right]^{(n+1)\alpha-2}. \end{cases} \tag{4.9}$$

Therefore the approximate solution of the problem in the first step is given by :

$$\begin{aligned} u^{1}(t) &= \sum_{n=0}^{+\infty} u_{n}^{1} \\ &= \sum_{n=0}^{+\infty}\left(\frac{\theta}{\Gamma((n+1)\alpha)}\left[ln\left(\frac{t}{a}\right)\right]^{(n+1)\alpha-1} + \frac{\beta}{\Gamma((n+1)\alpha-1)}\left[ln\left(\frac{t}{a}\right)\right]^{(n+1)\alpha-2}\right) \\ &= \theta\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\sum_{n=0}^{+\infty}\frac{\left(-\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right)^{n}}{\Gamma(n\alpha+\alpha)} + \beta\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-2}\sum_{n=0}^{+\infty}\frac{\left(-\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right)^{n}}{\Gamma(n\alpha+\alpha-1)} \\ &= \theta\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(-\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right) + \beta\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-2}\mathbb{E}_{\alpha,\alpha-1}\left(-\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right). \end{aligned}$$

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Let's evaluate $N(u^1)$. We have:

$$\begin{aligned} N(u^1(t)) &= \left(u^1(t)\right)^2 - \left(\frac{d^\alpha u^1(t)}{dt^\alpha}\right)^2 \\ &= \left(\theta\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(-\left[ln\left(\frac{t}{a}\right)\right]^\alpha\right)+\beta\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-2}\mathbb{E}_{\alpha,\alpha-1}\left(-\left[ln\left(\frac{t}{a}\right)\right]^\alpha\right)\right)^2 \\ &\quad -\left(\theta\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(-\left[ln\left(\frac{t}{a}\right)\right]^\alpha\right)+\beta\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-2}\mathbb{E}_{\alpha,\alpha-1}\left(-\left[ln\left(\frac{t}{a}\right)\right]^\alpha\right)\right)^2 \\ &= 0. \end{aligned}$$

Therefore, the exact solution of the problem is

$$u(t) = \theta\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(-\left[ln\left(\frac{t}{a}\right)\right]^\alpha\right)+\beta\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-2}\mathbb{E}_{\alpha,\alpha-1}\left(-\left[ln\left(\frac{t}{a}\right)\right]^\alpha\right).$$

4.3 Non-linear time fractional partial differential equations in Hadamard's sense

Example 4.2. *:*

Consider the following fractional-time model:

$$\begin{cases} {}^H\mathcal{D}_a^\alpha u = \lambda\dfrac{\partial^2 u}{\partial x^2} + u^3 + \left(x\dfrac{\partial^2 u}{dx^2}\right)^3 \\ ({}^H\mathcal{I}_a^{1-\alpha}u)(a,x) = cos(x), \end{cases} \tag{4.10}$$

with $0 < a < t < b$, $0 < \alpha \leq 1$, $\lambda \in \mathbb{R}$, $u \in C_\delta^1([a;b])$, ${}^H\mathcal{D}_a^\alpha(.)$ the derivative in Hadamard's sense, ${}^H\mathcal{I}_a^\alpha(.)$ the integral in Hadamard's sense.

Posing $L(.) ={}^H \mathcal{D}_a^\alpha(.); L^{-1}(.) = {}^H\mathcal{I}_a^\alpha(.); Ru = \lambda\dfrac{\partial^2 u}{\partial x^2}; Nu = u^3 + \left(x\dfrac{\partial^2 u}{dx^2}\right)^3$, where L^{-1} is the inverse of L Adomian sense, we obtain the following SBA algorithm:

$$\begin{cases} u_0^k = \dfrac{cos(x)}{\Gamma(\alpha)}\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1} + L^{-1}\left(N(u^{k-1})\right) \\ u_{n+1}^k = L^{-1}\left(R(u_n^k)\right), n \geq 0 \end{cases}, k \geq 1. \tag{4.11}$$

By developing the algorithm for $k = 1$, we obtain:

$$\begin{cases} u_0^1 = \dfrac{cos(x)}{\Gamma(\alpha)}\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1} \\ u_1^1 = -\dfrac{cos(x)\lambda}{\Gamma(2\alpha)}\left[ln\left(\frac{t}{a}\right)\right]^{2\alpha-1} \\ u_2^1 = \dfrac{cos(x)\lambda^2}{\Gamma(3\alpha)}\left[ln\left(\frac{t}{a}\right)\right]^{3\alpha-1} \\ \vdots \\ u_n^1 = (-1)^n\dfrac{cos(x)\lambda^n}{\Gamma((n+1)\alpha)}\left[ln\left(\frac{t}{a}\right)\right]^{(n+1)\alpha-1}. \end{cases} \tag{4.12}$$

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Therefore the approximate solution of the problem in the first step is given by :

$$\begin{aligned} u^1 &= \sum_{n=0}^{+\infty} u_n^1 \\ &= cos(x)\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\sum_{n=0}^{+\infty}\frac{\left(-\lambda\left(\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right)^n\right.}{\Gamma(n\alpha+\alpha)} \\ &= cos(x)\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(-\lambda\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right). \end{aligned}$$

Let's evaluate $N(u^1)$. We have:

$$\begin{aligned} N(u^1) &= \left(u^1\right)^3 + \left(\frac{\partial^2 u^1}{dx^2}\right)^3 \\ &= \left(cos(x)\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(-\lambda\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right)\right)^3 + \\ &\quad -\left(cos(x)\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(-\lambda\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right)\right)^3 \\ &= 0. \end{aligned}$$

So the exact solution to the problem is

$$u = cos(x)\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(-\lambda\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right).$$

Example 4.3. *:*

Consider the following fractional time model in dimension two (2) space:

$$\begin{cases} {}^H\mathcal{D}_a^{\alpha}u = \dfrac{\partial^2 u}{\partial x\partial y}(u_{xx}u_{yy}) - \dfrac{\partial^2 u}{\partial x\partial y}(xyu_xu_y) - u \\ (\delta^{m-\eta}\ {}^H\mathcal{I}_a^{m-\alpha}u)(a,x,y) = e^{xy}, \eta = \{1,...,m\} \end{cases} \tag{4.13}$$

with $0 < a < t < b$; $\alpha > 0$; $\grave{u}(x,y) \in \mathbb{R}^2$; $u \in C_\delta^m([a;b])$; $m-1 < \alpha \leq m$; $\delta = t\dfrac{d}{dt}$; ${}^H\mathcal{D}_a^{\alpha}(.)$ derivative in Hadamard's sense; ${}^H\mathcal{I}_a^{\alpha}(.)$ the integral in Hadamard's sense;
$u = u(t,x,y)$; $u_x = \dfrac{\partial u}{\partial x}$; $u_y = \dfrac{\partial u}{\partial y}$; $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$ et $u_{yy} = \dfrac{\partial^2 u}{\partial y^2}$.
Posing $L(.) = {}^H\mathcal{D}_a^{\alpha}(.); L^{-1}(.) = {}^H\mathcal{I}_a^{\alpha}(.); Ru = -u; Nu = \dfrac{\partial^2 u}{\partial x\partial y}(u_{xx}u_{yy}) - \dfrac{\partial^2 u}{\partial x\partial y}(xyu_xu_y)$, we obtain the following SBA plus algorithm:

$$\begin{cases} u_0^k = e^{xy}\displaystyle\sum_{\eta=1}^{m}\frac{\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-\eta}}{\Gamma(\alpha-\eta+1)} + L^{-1}\left(N(u^{k-1})\right) \\ u_{n+1}^k = L^{-1}\left(R(u_n^k)\right), n \geq 0 \end{cases}, k \geq 1. \tag{4.14}$$

Developing the algorithm for $k = 1$ gives.

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$$\begin{cases} u_0^1 = e^{xy}\sum_{\eta=1}^{m}\frac{\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-\eta}}{\Gamma(\alpha-\eta+1)} \\ u_1^1 = -e^{xy}\sum_{\eta=1}^{m}\frac{\left[ln\left(\frac{t}{a}\right)\right]^{2\alpha-\eta}}{\Gamma(2\alpha-\eta+1)} \\ u_2^1 = e^{xy}\sum_{\eta=1}^{m}\frac{\left[ln\left(\frac{t}{a}\right)\right]^{3\alpha-\eta}}{\Gamma(3\alpha-\eta+1)} \\ \vdots \\ u_n^1 = (-1)^n e^{xy}\sum_{\eta=1}^{m}\frac{\left[ln\left(\frac{t}{a}\right)\right]^{(n+1)\alpha-\eta}}{\Gamma((n+1)\alpha-\eta+1)}. \end{cases} \tag{4.15}$$

So the approximate solution at the first iteration is

$$\begin{aligned} u^1 &= \sum_{n=0}^{+\infty} u_n^1 \\ &= e^{xy}\sum_{\eta=1}^{m}\left(\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-\eta}\sum_{n=0}^{+\infty}\frac{(-\left[ln\left(\frac{t}{a}\right)\right]^{\alpha})^n}{\Gamma(n\alpha+\alpha-\eta+1)}\right) \\ &= e^{xy}\sum_{\eta=1}^{m}\left(\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-\eta}\mathbb{E}_{\alpha,\alpha-\eta+1}\left(-\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right)\right). \end{aligned}$$

Let's evaluate $N(u^1)$. We have:

$$\begin{aligned} N(u^1) &= \frac{\partial^2 u^1}{\partial x\partial y}(u_{xx}^1 u_{yy}^1) - \frac{\partial^2 u^1}{\partial x\partial y}(xyu_x^1 u_y^1) \\ &= x^3y^3\Bigg(xye^{xy}\sum_{\eta=1}^{m}\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-\eta}\mathbb{E}_{\alpha,\alpha-\eta+1}\left(-\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right)\Bigg)^3 - \\ &\quad +x^3y^3\Bigg(xye^{xy}\sum_{\eta=1}^{m}\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-\eta}\mathbb{E}_{\alpha,\alpha-\eta+1}\left(-\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right)\Bigg)^3. \\ &= 0 \end{aligned}$$

Therefore

$$u(t,x,y) = e^{xy}\sum_{\eta=1}^{m}\left(\left[ln\left(\frac{t}{a}\right)\right]^{\alpha-\eta}\mathbb{E}_{\alpha,\alpha-\eta+1}\left(-\left[ln\left(\frac{t}{a}\right)\right]^{\alpha}\right)\right).$$

5 Conclusion

In this article, we have studied the existence and uniqueness of solutions to fractional Hadamard differential equations of Cauchy type. The theoretical result for the existence and uniqueness of solutions was obtained by Weissinger's fixed point theorem. Finally, we have used the SBA plus method to determine the concrete solutions of some fractional time differential equations in the Hadamard sense, to illustrate this theory. The results obtained also confirm the effectiveness of the SBA plus method.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] **Abdelaziz OUALID;** *Équations Différentielles au Sens de Hadamard. Université Abdelhamid Ben Badis dde Mostaganem. Master soutenu le 8 juin 2018.*

[2] **Abdoul wassiha NEBIE, Frédéric BERE, Bakari ABBO, Youssouf PARE;** *Solving Some Derivative Equations Fractional Order Nonlinear Partials Using the Some Blaise Abbo Method, Journal of Mathematics Research; Vol. 13, No. 2; April 2021; doi:10.5539/jmr.v13n2p101*

[3] **A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo;** *Deory And Applications Of Fractional Differential Equations, Vol. 204, Elsevier Science Limited, Amsterdam, Netherlands, 2006.*

[4] **B.ABBO, O. SO, G. BARRO & B. SOME ;** *A new Numerical algorithm for solving nonlinear partial differential equations with initial and Boundary conditions. Far East J. Appl. Math. 28 (1) (2007), 37-52.*

[5] **Blaise SOME;** *Méthode SBA de résolution des modèles mathématiques en environnement. Éditions Universitaires Européennes, 2018*

[6] **B.Abbo;** *Nouvel algorithme numérique de résolution des équations différentielles ordinaires (EDO) et des équations aux dérivées partielles (EDP) non linéaires. Thèse de Doctorat unique. Université de Ouagadougou, Janvier 2007, UFR/ SEA, Département Mathématique et Informatique (Burkina Faso)*

[7] **Coskun M., , Çetin Yildiz and Luminita-Ioana Cotîrla;** *Novel Estimations of Hadamard-Type Integral Inequalities for Raina's Fractional Operators. Fractal Fract. 2024, 8, 302. https://doi.org/10.3390/fractalfract8050302*

[8] **C. Young, G. Farid, S. Ullah, W. Nazeer, K. Mahreen, and M. Shin;** *Inequalities for a unified integral operator and associated results in fractional calculus, IEEE Access, vol. 7, pp. 126283–126292, 2019.*

[9] **G.KABORE, W.SOME, M.KÉRÉ, O.SO, B.SOME;** *Solving Some Fractional Ordinary Differential Equations by SBA Method. Journal of Mathematics Research; Vol. 15, No. 1; February 2023. doi:10.5539/jmr.v15n1p47*

[10] **Germain KABORE, KÉRÉ Moumini, Windjiré SOME, Ousséni SO and Blaise SOME;** *Solving some fractional equations, in the sense of Riemann-Liouville, of Navier-Stokes by the numerical method SBA plus, International Journal of Numerical Methods and Applications 23(2) (2023), 209-228. http://dx.doi.org/10.17654/0975045223012 This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Published Online: August 24, 2023*

---

[11] **Germain KABORE, Bakari Abbo, Windjiré SOME, Ousséni SO and Blaise SOME;** *Solving a fractional evolution equation in the sense of Caputo-Hadamard with Cauchy and boundary conditions by SBA method, Advances in Differential Equations and Control Processes, 31(1) (2024), 1-14. http://dx.doi.org/10.17654/0974324324001*

[12] **Germain KABORE, Abakar Mahamat SEID, Bakari Abbo, Ousséni SO and Blaise SOME;** *Application of the SBA method to the solution of some nonlinear fractional equations in the sense of Caputo Hadamard, Universal Journal of Mathematics and Mathematical Sciences, 19(2) (2023), 103-117. http://dx.doi.org/10.17654/2277141723019*

[13] **F. Jarad, T. Abdeljawad and D. Baleanu;** *Caputo-type modification of the Hadamard fractional derivatives.Article in Advances in Difference Equations · December 2012. DOI:10.1186/1687-1847-2012-142*

[14] **Hadamarad, J;** *: Essai sur l'etude des fonctions données par leur développment de Taylor. J. Pure Appl. Math. 4(8), 101-186 (1892)*

[15] **Mountassir Hamdi Cherif;** *Résolution numérique des équations différentielles et aux dérivées partielles non linéaire et d'ordre fractionnaire par la méthode HPM. Thèse de Doctorat. Université d'ORAN 1, AHMED BEN BELLA*

[16] **Houmor Tarek;** *Analyse du chaos dans un système d'équations différentielles fractionnaires. Thèse soutenue le 30/09/2014. Université Constantine1*

[17] **Huijuan Zhu, Yuanfang Ru, and Fanglei Wang;** *Analysis of solutions for the fractional differential equation with Hadamard-type DE GRUYTER Open Mathematics 2023; 21: 20230131; doi.org/10.1515/math-2023-0131*

[18] **H.R.Marasi and H. Aydi;** *Existence and Uniqueness Results for Two-Term Nonlinear Fractional Differential Equations via a Fixed Point Technique. Hindawi, Journal of Mathematics Volume 2021, Article ID 6670176, 7 pages https://doi.org/10.1155/2021/6670176*

[19] **Istafa, G., ur Rehman, M. A Legendre;** *spectral method for Hadamard fractional partial differential equations. Math Sci 18, 239–255 (2024). https://doi.org/10.1007/s40096-022-00497-7*

[20] **I. Podlubny;** *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Deir Solution and Some of Deir Applications, Elsevier, Amsterdam, Netherlands, 1998.*

[21] **Kamate Adama, Bationo Jeremie Yiyureboula, Djibet Mbaiguesse, Youssouf Pare;** *Analytical Solutions of Classical and Fractional Navier-Stokes Equations by the SBA Method, Journal of Mathematics Research; Vol. 14, No. 4; August 2022, https://doi.org/10.5539/jmr.v14n4p20*

[22] **KHALOUTA ALI;** *Résolution des équations aux dérivées partielles linéaires et non-linéaires moyennant des approches analytiques. Extension aux cas d'EDP d'ordre fractionnaire. Thèse de doctorat de l'Université Ferhat ABBAS SETIF1, le 14/12/19*

[23] **K. Muthuselvan, B. Sundaravadivoo;** *Analyze existence, uniqueness and controllability of impulsive fractional functional differential equations, Advanced Studies: Euro-Tbilisi Mathematical Journal, 10 (2022) 171-190.*

[24] **K. Muthuselvan, B. Sundaravadivoo, K.S. Nisar, S. Alsaeed;** *;Discussion on iterative process of nonlocal controllability exploration for Hilfer neutral impulsive fractional integro-differential equation, AIMS Mathematics, 8(7) (2023) 16846–16863.*

[25] **BOHNER, MARTIN; DOMOSHNITSKY, ALEXANDER, PADHI, Seshadev; and SRIVASTAVA, Satyam Narayan;** *(2024)"Existence of solutions by coincidence degree theory for Hadamard fractionaldifferential equations at resonance," Turkish Journal of Mathematics: Vol. 48: No. 2, Article 13. https://doi.org/ 10.55730/1300-0098.3507*

[26] **Ma, Y., Maryam, M., Riaz, U. et al;** *Existence and Hyers–Ulam Stability of Jerk-Type Caputo and Hadamard Mixed Fractional Differential Equations. Qual. Theory Dyn. Syst. 23, 132 (2024). https://doi.org/10.1007/s12346-024-00971-8*

[27] **M.H. Heydari, M. Razzaghi;** *Jacobi polynomials method for a coupled system of Hadamard fractional Klein–Gordon–Schrödinger equations. Alexandria Engineering Journal Volume 107 , November 2024, Pages 73-86. https://doi.org/10.1016/j.aej.2024.07.016*

[28] **M. Shin, G. Farid, W. Nazeer, and S. Mehmood;** *(h - m)-convex functions and associated fractional hadamard and Fejer–Hadamard inequalities via an extended ´ generalized Mittag-Leffler function, Journal of Inequalities and Applications, vol. 2019, no. 1, pp. 1–10, 2019.*

[29] **M. Shin, G. Farid, W. Nazeer, and B. Tariq;** *Hadamard and Fejer–Hadamard inequalities for extended generalized fractional integrals involving special functions, Journal of Inequalities and Applications, vol. 2018, no. 1, p. 119, 2018.*

[30] **P.Thiramanus, S.K. Ntouyas and J. Tariboon;** *Existence and Uniqueness Results for Hadamard-Type Fractional Differential Equations with Nonlocal Fractional Integral Boundary Conditions. Hindawi, Abstract and Applied Analysis Volume 2014, Article ID 902054, 9 pages http://dx.doi.org/10.1155/2014/902054*

[31] **Samko, SG, Kilbas, AA, Marichev, OI;** *Fractional Integrals and Derivatives -Theory and Applications. Gordon & Breach, Langhorne (1993)*

[32] **Saud Fahad Aldosary, , Mohamed M. A. Metwali , Manochehr Kazemi and Ateq Alsaadi;** *On integrable and approximate solutions for Hadamard fractional quadratic integral equations; AIMS Mathematics, 9(3): 5746–5762. DOI: 10.3934/math.2024279*

[33] **S.N.Rao, A.H.Msmali,M.Singh and A.A.H.Ahmadini;** *Existence and Uniqueness for a System of Caputo-Hadamard Fractional Differential Equations with Multipoint Boundary Conditions. Hindawi,Journal of Function Spaces Volume 2020, Article ID 8821471, 10 pages https://doi.org/10.1155/2020/8821471*

[34] **S. Zhao, S. I. Butt, W. Nazeer, J. Nasir, M. Umar, and Y. Liu;** *Some Hermite–Jensen–mercer type inequalities for k-caputo-fractional derivatives and related results, Advances in Difference Equations, vol. 2020, no. 1, pp. 1–17, 2020.*

[35] **Umer Saeed, Mujeeb ur Rehman;** *A new scheme for the solution of the nonlinear Caputo–Hadamard fractional differential equations. Alexandria Engineering Journal Volume 105, October 2024, Pages 56-69; https://doi.org/10.1016/j.aej.2024.06.050*

[36] **Zaid Laadjal;** *ETUDE DE QUELQUES PROBLEMES AUX LIMITES NON LINEAIRES D'ORDRE FRACTIONNAIRE. Université Abbès Laghrour-Khenchela. Thèse soutenue le 25 Mars 2021.*

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