Original
Research Article

Induced Topologies of Independent Domination in Helm Graphs

Abstract

Let G be a graph. (11) The independent domination topology (ID topology) of G, denoted by $\tau_I(G)$ is the topology generated by the family I_G of all independent dominating sets of G. In this paper, we introduce and characterize the independent domination topology through the context of the independent dominating sets of the helm graph H_n. This study highlights the significance of this topology in optimizing network designs and computational systems, offering a foundation for future research and practical applications.

Keywords: Helm graph; Independent Domination Topology.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

1 Preliminaries

The relationship between graph theory and topology can be made by defining a relation on a given graph (6), it is rooted in the ability to visualize a graph G as a topological space. This relation existed before and has been used many times by researchers to generate topology from a graph's vertex set and edge set, they studies graphs as a topologies and have been applied in almost every scientific field (1).

Previous studies have explored various topological methods applied to graphs. For instance, Lalithambigai and P. Gnanachandra in (6), developed a method for generating topologies based on adjacency and incidence relations on vertex sets, examining properties like closure and interior in graph adjacency topological spaces. Another method is independent topology. This topology is generated from the family of independent sets of each of the vertices in a given graph (4). In this method , a collection of a subset of a nonempty set (e.g. vertex set or edge set) is treated as a subbase to generate the desired topology (1). Another one is topological domination, introduced by Jabor and Omran (9), which generates the domination topology τ_d from minimal dominating sets of a graph. In τ_d, each minimal dominating set is open.

In (3), Duhaylungsod and Balingit expanded on the concept of topology by introducing a generalized topology formed by independent dominating sets. The study introduces a generalized topology generated by a basis consisting of all independent dominating sets of such a graph. Manla (11) further explored the expansion and modification of topological graph studies, including the examination of independent domination topology within different graph families. Independent domination in topology merges these ideas exploring how independent domination manifests in various topological configurations.

A graph G consists of a finite nonempty set $V(G)$ of vertices (or nodes), and a set $E(G)$ of edges (or arcs), denoted by $G = (V, E)$. If u and v are vertices and e is an edge such that $e = uv$, then e is said to join u and v, and each vertex u and v is adjacent and incident with the edge e (2; 7). A subset A of vertices of the vertex set $V(G)$ of a graph G is independent if no two vertices in A are adjacent. That is, $A \subseteq V(G)$ is independent if for all $x, y \in A$, x and y are not adjacent. On the other hand,
$A \subseteq V(G)$ is a dominating set if for all $x \in V(G) \setminus A$, there exists $y \in A$ such that x is adjacent to y. A subset A of vertices of the vertex set $V(G)$ of a graph G is an independent dominating set if for all $x, y \in A$, x and y are not adjacent and for all $u \in V(G) \setminus A$, there exists $w \in A$ such that u is adjacent to w.

A topology τ on a set X is a collection of subsets of X, called an open set that is closed under arbitrary union and finite intersection, and both X and $\emptyset$ are in τ. The topology containing all the subsets of X is called the discrete topology on X and the topology containing exactly X and $\emptyset$ is called the indiscrete topology on X (4).

From the above discussion, it is possible to further explore the independent domination topology in the context of independent domination of the helm graph. With this, we aim to introduce the construction of independent domination topology induced by the helm graph.

1.1 Some Known Result

Theorem 1.1. ***(Stars and Bars Theorem)*** *(12) The number of ways to place n indistinguishable balls into k labelled urns is*

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

2 Independent Domination Topology Induced by Helm Graph

This section contains the discussion about the independent domination induced by the helm graph H_n.

Definition 2.1. (11) Let G be a graph. The **independent domination topology** (ID topology) of G, denoted by $\tau_I(G)$ is the topology generated by the family I_G of all independent dominating sets of G.

Definition 2.2. (10) A **wheel graph** W_n is a graph with n vertices ($n \geq 4$), formed by joining a single vertex to all the vertices of a cycle with $n-1$ vertices. That is, $W_n = C_{n-1} + K_1$.

Definition 2.3. (5) The **helm graph** H_n is a graph obtained from a wheel by attaching a pendant edge at each vertex of the $n-$cycle.

For the helm graph H_n, let $V(H_n) = U \cup W$, where $W = \{w_0, w_1, ..., w_n\}$ are the vertices in the wheel such that w_0 is the center vertex, and $U = \{u_1, ..., u_n\}$ are the vertices of the pendant edge at each vertex of the $n-$cycle. And we let $[n] = 1, 2, ..., n$.

Example 2.1. *Consider the Helm graph H_3 in Figure 1 as shown below. Clearly, graph H_3 has order 7 and size 9.*

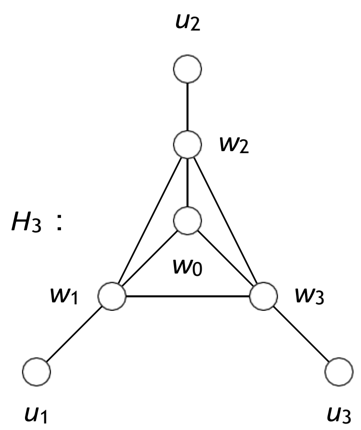

Figure 1: Helm Graph

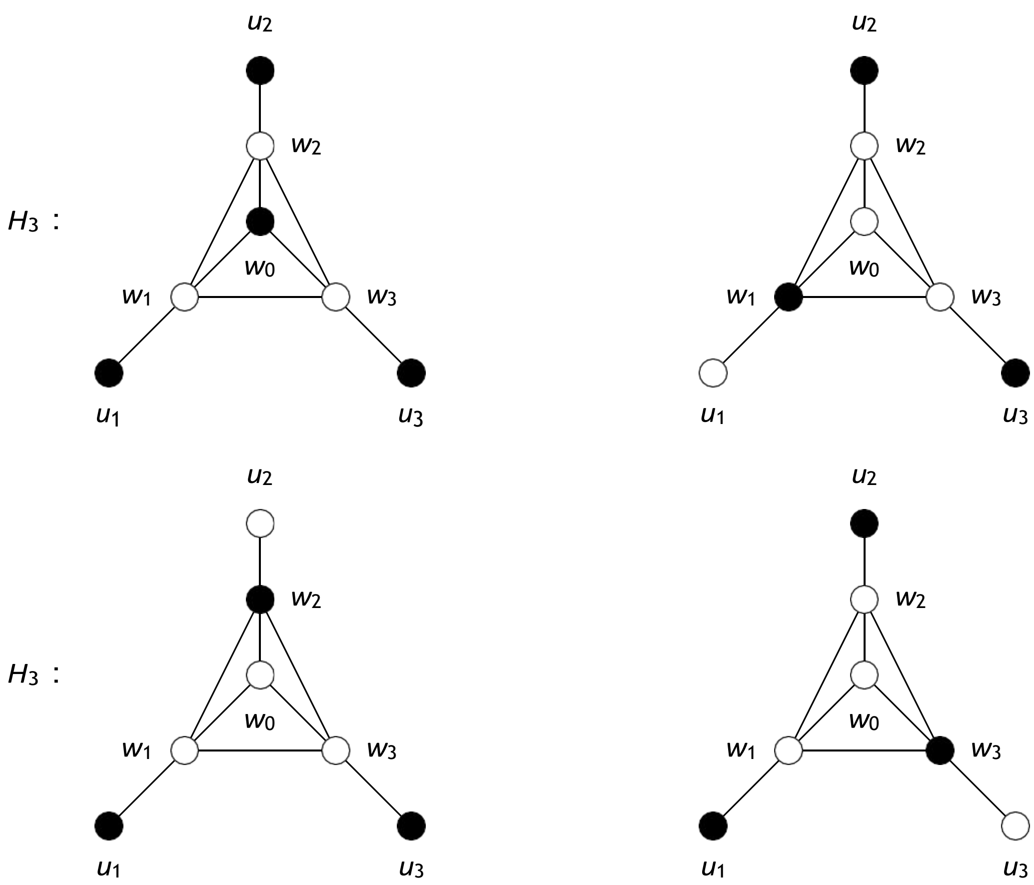

Figure 2: Independent Dominating Sets of Helm Graph G_3

The set $I_{H_3} = \{\{w_0, u_1, u_2, u_3\}, \{w_1, u_2, u_3\}, \{u_1, w_2, u_3\}, \{u_1, u_2, w_3\}\}$ is the set of all independent dominating sets of the graph H_3. By Definition 2.1, the topology generated by the family I_{H_3} is

$$\begin{aligned}\tau_I(H_3) = \{&\emptyset, V(H_3), \{w_0, u_1, u_2, u_3\}, \{w_1, u_2, u_3\}, \{u_1, w_2, u_3\}, \{u_1, u_2, w_3\},\\ &\{w_0, w_1, u_1, u_2, u_3\}, \{w_0, w_2, u_1, u_2, u_3\}, \{w_0, w_3, u_1, u_2, u_3\},\\ &\{w_1, w_2, u_1, u_2, u_3\}, \{w_1, w_3, u_1, u_2, u_3\}, \{w_2, w_3, u_1, u_2, u_3\},\\ &\{w_0, w_1, w_2, u_1, u_2, u_3\}, \{w_0, w_1, w_3, u_1, u_2, u_3\},\\ &\{w_0, w_2, w_3, u_1, u_2, u_3\}, \{w_1, w_2, w_3, u_1, u_2, u_3\},\\ &\{u_2, u_3\}, \{u_1, u_3\}, \{u_1, u_2\}, \{u_3\}, \{u_2\}, \{u_1\}\}\end{aligned}$$

which, therefore, is the independent domination topology of H_3.

Theorem 2.2. *Let $n \geq 3$ and consider the helm graph H_n. $B \subseteq V(H_n)$ is an independent dominating set of H_n, denoted by I_{H_n}, if and only if B takes one of the following forms:*

i. $\{w_0, u_1, ..., u_n\}$

ii. $\{w_i : i \in A\} \cup \{u_j : j \in A^c\}$ *where $A \subseteq [n]$ satisfies the following conditions;*

a_1. $1 \leq |A| \leq \lfloor \frac{n}{2} \rfloor$.

a_2. *If $i \in A$, then $i + 1 \notin A$.*

Proof. ($\Leftarrow$) Let $B = \{w_0, u_1, ..., u_n\}$. By definition of H_n, B is an independent set since for all $x, y \in B$, x and y are not adjacent. Also, $w_0 \in B$ dominates w_i for all $i = 1, ..., n$. Thus, B is an independent dominating set.

Now, let $B = \{w_i : i \in A\} \cup \{u_j : j \in A^c\}$. By the second condition (a_2) for A and by definition of H_n, $\{w_i : i \in A\}$ is an independent set. On the other hand, $\{u_j : j \in A^c\}$ is also an independent set by definition of H_n. Now, for each $j \in A^c$, u_j is only adjacent to $w_j \notin B$. Thus, B is an independent. Let $x \in V(H_n) \setminus B$. If $x = w_0$, then x is adjacent to $w_i \in B$ for all $i \in A$, given that $|A| \geq 1$ in condition(a_1). If $x = w_i$ for some $i \in A^c$, then x is adjacent to $u_i \in B$. If $x = u_j$ for some $j \in A$, then x is adjacent to $w_j \in B$. Hence, B is a dominating set. Consequently, B is an independent dominating set.

($\Rightarrow$) Conversely, suppose that $S \subseteq V(H_n)$ such that $S \notin I_{H_n}$.
Case 1: $w_0 \in S$.

If $w_0 \in S$, then there exist $i \in [n]$ such that $w_i \in S$. But w_i is adjacent to w_0, by definition of H_n. Thus, S is not an independent dominating set.

Case 2: $w_0 \notin S$.

If $w_0 \notin S$ and $S \notin I_{H_n}$, then either of the following holds;

i) There exists $i \in [n]$ such that $w_i, u_i \notin S$.

ii) There exists $i \in [n]$ such that $w_i, u_i \in S$.

In (i), S is not dominating set since w_0 and u_i are the only vertices adjacent to w_i. Also, in (ii), S is not an independent set since w_i and u_i are adjacent. In both cases, S is not an independent dominating set. Thus, $I_{H_n} = \{\{w_0, u_1, ..., u_n\}, \{\{w_i : i \in A\} \cup \{u_j : j \in A^c\}\}\}$. □

Lemma 2.3. *For each $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$, the number of independent dominating sets of a helm graph of the form $\{w_i : i \in A, |A| = r\} \cup \{u_j : j \notin A\}$ is $\binom{n-r-1}{r-1}\frac{n}{r}$.*

Proof. Here, we count the number of ways to choose r vertices from n w_is such that no two consecutive w_is are chosen (otherwise the collection will not be independent). Let $A \subseteq [n]$ such that $|A| = r$ and the vertices w_is, $i \in A$ be represented by bars and the w_is where $i \notin A$ be represented by stars. The desired sequences should be a string of bars and stars that contain exactly r bars (the chosen points) and $n - r$ stars (the unchosen points) such that no two bars are consecutive, and the first and last points cannot be both bars.

Case 1: First and last characters are stars.

$$\underbrace{* \; * \cdots}_{x_1} \underset{1^{st}\text{ bar}}{|} \underbrace{* \; * \cdots}_{x_2} \underset{2^{nd}\text{ bar}}{|} \underbrace{\cdots}_{x_3} \underset{r^{th}\text{ bar}}{|} \underbrace{* \; * \cdots}_{x_{r+1}}$$

We find $r + 1$ positive integers $x_1, x_2, ..., x_{r+1}$ such that $x_1 + x_2 + ... + x_{r+1} = n - r$. Let $x'_i = x_i - 1$. Since $x_1 \geq 1$ for all i, one has $x_1 \geq 0$.

Substituting $x_i = x'_i + 1$ into the equation, we have

$$(x'_1 + 1) + (x'_2 + 1) + ... + (x'_r + 1) + (x'_{r+1} + 1) = n - r.$$

Simplifying, we get:

$$\begin{aligned} x'_1 + x'_2 + \cdots + x'_r + x'_{r+1} + (r + 1) &= n - r \\ x'_1 + x'_2 + \cdots + x'_r + x'_{r+1} &= n - r - (r + 1) \\ x'_1 + x'_2 + \cdots + x'_r + x'_{r+1} &= n - 2r - 1 \end{aligned} \tag{2.1}$$

By (2.1) and the Stars and Bars Theorem 1.1, the number of ways to distribute $n - 2r - 1$ stars into $r + 1$ bars is

$$\binom{(n - 2r - 1) + (r + 1 - 1)}{r + 1 - 1} = \binom{n - 2r - 1 + r}{r} = \binom{n - r - 1}{r}, \tag{2.2}$$

which gives the number of ways to choose r w_is such that w_i and w_n are not chosen.

Case 2. The first character is a star and the last character is a bar, or the opposite.

$$\underset{1^{st}\text{ bar}}{|} \underbrace{* \; * \cdots}_{x_1} \underset{2^{nd}\text{ bar}}{|} \underbrace{* \; * \cdots}_{x_2} \underset{3^{rd}\text{ bar}}{|} \underbrace{\cdots}_{\cdots} \underset{r^{th}\text{ bar}}{|} \underbrace{* \; * \cdots}_{x_r}$$

We find r positive integers $x_1, x_2, ..., x_r$ such that $x_1 + x_2 + ... + x_r = n - r$. If we set $x'_i = x_i - 1$, then $x_1 \geq 0$.
Substituting $x_i = x'_i$ into the equation, we have

$$(x'_1 + 1) + (x'_1 + 1) + ... + (x'_r + 1) = n - r.$$

Simplifying, we get:

$$\begin{aligned} x'_1 + x'_2 + ... + x'_r + (r) &= n - r \\ x'_1 + x'_2 + ... + x'_r &= n - 2r \end{aligned} \tag{2.3}$$

By (2.3) and the Stars and Bars Theorem 1.1, the number of ways to distribute $n-2r$ stars into r bars is

$$\binom{(n-2r)+(r-1)}{r-1} = \binom{n-r-1}{r-1}, \tag{2.4}$$

which gives the number of ways to choose r w_is such that w_i is chosen, while w_n are not chosen. Note that, this is the same number of ways to choose r w_is such that w_i is not chosen, while w_n is chosen.

Thus, the number of ways to choose r w_is such that no two consecutive w_is are chosen is the sum of (2.2) and (2.4). That is,

$$\begin{aligned}
\binom{n-r-1}{r} + 2\binom{n-r-1}{r-1} &= \frac{(n-r-1)!}{r!(n-r-1-r)!} + 2\,\frac{(n-r-1)!}{(r-1)!(n-r-1-(r-1))!}\\
&= \frac{(n-r-1)!}{r!(n-2r-1)!} + 2\,\frac{(n-r-1)!}{(r-1)!(n-2r)!}\\
&= \left(\frac{r}{r}\right)\left(\frac{n-2r}{n-2r}\right)\frac{(n-r-1)!}{r!(n-2r-1)!} + 2\,\frac{(n-r-1)!}{(r-1)!(n-2r)!}\\
&= \frac{(n-2r)(n-r-1)!}{r(r-1)!(n-2r)!} + 2\,\frac{(n-r-1)!}{(r-1)!(n-2r)!}\\
&= \left(2 + \frac{n-2r}{r}\right)\binom{n-r-1}{r-1}\\
&= \frac{2r+n-2r}{r}\binom{n-r-1}{r-1}\\
&= \frac{n}{r}\binom{n-r-1}{r-1}.
\end{aligned}$$

□

Theorem 2.4. *For* $n \geq 3$, $|I_{H_n}| = 1 + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{r}\binom{n-r-1}{r-1}$.

Proof. In view of the previous Theorem 2.2 and Lemma 2.3, for $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$, there are $\frac{n}{r}\binom{n-r-1}{r-1}$ sets of the form $S = \{w_i : i \in A, |A| = r\} \cup \{u_i : i \notin A\}$ and a set of the form $\{w_0, u_1, u_2, ..., u_n\}$. If $r > \lfloor \frac{n}{2} \rfloor$, then there exist two vertices $w_i, w_j \in S$ such that either $j = i+1$ or $i = 1$ and $j = n$. This would have w_i and w_j to be adjacent which makes the set not independent. Thus, $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$.

Hence, $|I_{H_n}| = 1 + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{r}\binom{n-r-1}{r-1}$.

□

Remark 2.1. The independent dominating topology of the helm graph H_n is not the discrete topology on $V(H_n)$. To see this, $\{w_0\}$ cannot be $\tau_I(H_n)$- open since it is not the union nor intersection of independent dominating set, and the only independent dominating set containing w_0 is $\{w_0, u_1, ..., u_n\}$.

Theorem 2.5. *For each* $i = 1, ..., n$, $\{u_i\} \in \tau_I(H_n)$.

Proof. Let $i \in [n]$. Consider the sets $S_{A_k} = \{w_k : k \in A, 1 \leq |A| \leq \lfloor \frac{n}{2} \rfloor\} \cup \{u_j : j \neq k\}$ such that if $k \in A$, then $k+1 \notin A$ for all $k \neq i$. Then $u_i \in S_{A_k}$ for all $k \neq i$, $u_i \in \bigcap_{k \neq i} S_{A_k}$. Now, for $k, k^* \neq p$, w_k is distinct from w_{k^*}. It follows that $w_k \notin \bigcap S_{A_k}$ for all $k \neq i$. Consider w_j, $j = k$. So,

$u_k \notin S_{A_k} = \{w_k : k \in A, 1 \leq |A| \leq \lfloor \frac{n}{2} \rfloor\} \cup \{u_j : j \neq k\}$ for all $k \neq i$. Thus, $u_k \notin \bigcap_{k \neq i} S_{A_k}$. Since k is arbitrary, $\bigcap_{k \neq i} S_{A_k} = \{u_i\}$, so that $\{u_i\} \in \tau_I(H_n)$ for all $i \in [n]$.

Corollary 2.6. *For every subset $A \subseteq \{u_1, u_2, ..., u_n\}$, is $\tau_I(H_n)$- open.*

Proof. The proof follows immediately from Theorem 2.5 since u_i is $\tau_I(H_n)-$open for all i, and $A = \bigcup_{u_i \in A} \{u_i\}$. □

3 CONCLUSIONS

This paper introduces the independent domination topology induced by the helm graph, along with some of its characterizations and the construction of its independent dominating set. However, the study is limited to helm graphs and might not generalize to other graph types. Future research could explore independent domination topologies for various graph families and employ more flexible methods for analysis. Additionally, validating results through simulations and examining the behavior of these topologies under unary operations could further enhance our understanding.

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