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Research Article

Independent Domination Topology induced by Helm Graph

Abstract

Let G be a graph. (11) The independent domination topology (ID topology) of G, denoted by $\tau_I(G)$ is the topology generated by the family I_G of all independent dominating sets of G. In this paper, we introduce the independent domination topology induced by the helm graph H_n. Moreover, we characterize and describe the independent domination topology through the context of the independent dominating sets of the helm graph H_n.

Keywords: Helm graph; Independent Domination Topology.

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1 Preliminaries

The relationship between graph theory and topology can be made by defining a relation on a given graph (6), it is rooted from the ability to visualize a graph G as a topological space. This relation existed before and have been used many times by researchers to generate topology from a graph's vertex set and edge set, they studies graphs as a topologies and have been applied in almost every scientific field (1).

K. Lalithambigai and P. Gnanachandra in (6), described a method of generating topology using adjacency, incidence relations on vertex set of graphs and studied the properties of closure and interior of vertex set of subgraphs in the graph adjacency topological space. One of the method to generate topology from a graph is independent topology. Independent topology is generated from the family of independent sets of each of the vertices in a given graph (4). In this method , a collection of subset of a nonempty set (e.g. vertex set or edge set) is treated as a subbase to generate the desired topology (1). Another method is topological domination. As mentioned in (3; 11), Jabor and Omran introduced topological space rooted in the concept of minimal dominating sets (9), that is, the

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domination topology τ_d has been generated from the set of minimal dominating sets of a graph G, and each minimal dominating set (MDS) is open in τ_d.

In (3), Duhaylungsod and Balingit connect the concept of topology in a general sense with the collection of all independent dominating sets of a finite simple graph. The study introduces a generalized topology generated by a basis consisting of all independent dominating sets of such a graph. Manla (11)further explored the expansion and modification of topological graph studies, including the examination of independent domination topology within different graph families. Independent domination in topology merges these ideas exploring how independent domination manifests in various topological configuration.

A graph G consists of a finite nonempty set $V(G)$ of vertices (or nodes), and a set $E(G)$ of edges (or arcs), denoted by $G = (V, E)$. If u and v are vertices and e is an edge such that $e = uv$, then e is said to join u and v, and each vertex u and v is adjacent and incident with the edge e (2; 7). A subset A of vertices of the vertex set $V(G)$ of a graph G is independent if no two vertices in A are adjacent. That is, $A \subseteq V(G)$ is independent if for all $x, y \in A$, x and y are not adjacent. On the other hand, $A \subseteq V(G)$ is a dominating set if for all $x \in V(G) \setminus A$, there exists $y \in A$ such that x is adjacent to y. A subset A of vertices of the vertex set $V(G)$ of a graph G is an independent dominating set if for all $x, y \in A$, x and y are not adjacent and for all $u \in V(G) \setminus A$, there exists $w \in A$ such that u is adjacent to w.

A topology τ on a set X is a collection of subsets of X, called open set that is closed under arbitrary union and finite intersection, and both X and $\emptyset$ are in τ. The topology containing all the subset of X is called the discrete topology on X and the topology containing exactly X and $\emptyset$ is called the indiscrete topology on X (4).

From above discussion, it is possible to further explore the independent domination topology in the context of independent domination of the helm graph. With this, we aim to introduce the construction of independent domination topology induced by the helm graph.

1.1 Some Known Result

Theorem 1.1. ***(Stars and Bars Theorem)*** *(12) The number of ways to place n indistinguishable balls into k labelled urns is*

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

2 Independent Domination Topology Induced by Helm Graph

This section contains the discussion about the independent domination induced by the helm graph H_n.

Definition 2.1. (11) Let G be a graph. The **independent domination topology** (ID topology) of G, denoted by $\tau_I(G)$ is the topology generated by the family I_G of all independent dominating sets of G.

Definition 2.2. (10) A **wheel graph** W_n is a graph with n vertices ($n \geq 4$), formed by joining a single vertex to all the vertices of a cycle with $n-1$ vertices. That is, $W_n = C_{n-1} + K_1$.

Definition 2.3. (5) The **helm graph** H_n is a graph obtained from a wheel by attaching a pendant edge at each vertex of the $n-$cycle.

For the helm graph H_n, let $V(H_n) = U \cup W$, where $W = \{w_0, w_1, ..., w_n\}$ are the vertices in the wheel such that w_0 is the center vertex, and $U = \{u_1, ..., u_n\}$ are the vertices of the pendant edge at each vertex of the $n-$cycle. And we let $[n] = 1, 2, ..., n$.

Example 2.1. *Consider the Helm graph H_3 in Figure 1 as shown below. Clearly, graph H_3 has order 7 and size 9.*

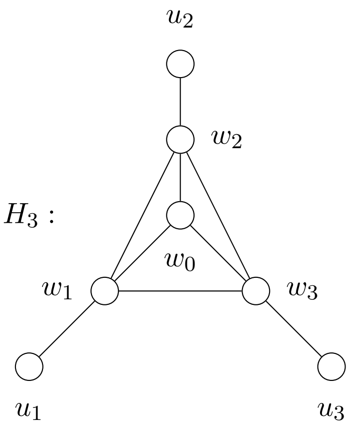

Figure 1: Helm Graph

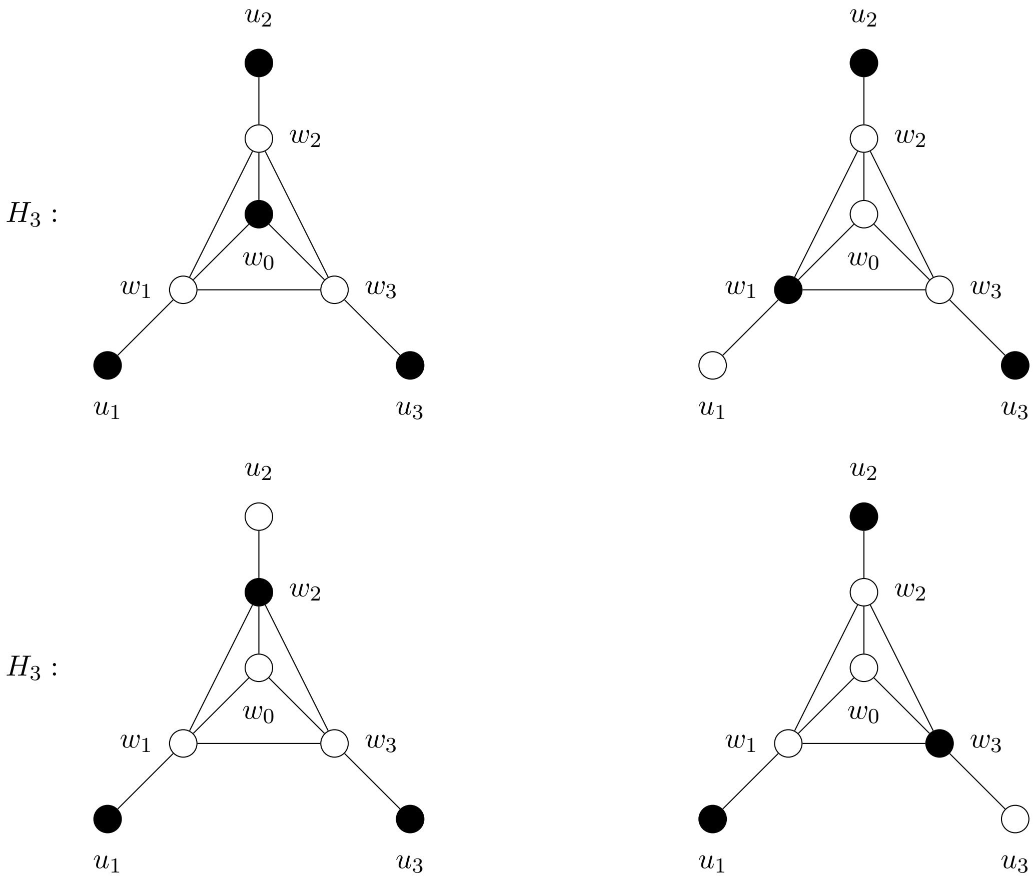

Figure 2: Independent Dominating Sets of Helm Graph G_3

The set $I_{H_3} = \{\{w_0, u_1, u_2, u_3\}, \{w_1, u_2, u_3\}, \{u_1, w_2, u_3\}, \{u_1, u_2, w_3\}\}$ is the set of all independent dominating sets of the graph H_3. By Definition 2.1, the topology generated by the family I_{H_3} is

$$\begin{aligned}\tau_I(H_3) = \{&\emptyset, V(H_3), \{w_0, u_1, u_2, u_3\}, \{w_1, u_2, u_3\}, \{u_1, w_2, u_3\}, \{u_1, u_2, w_3\},\\ &\{w_0, w_1, u_1, u_2, u_3\}, \{w_0, w_2, u_1, u_2, u_3\}, \{w_0, w_3, u_1, u_2, u_3\},\\ &\{w_1, w_2, u_1, u_2, u_3\}, \{w_1, w_3, u_1, u_2, u_3\}, \{w_2, w_3, u_1, u_2, u_3\},\\ &\{w_0, w_1, w_2, u_1, u_2, u_3\}, \{w_0, w_1, w_3, u_1, u_2, u_3\},\\ &\{w_0, w_2, w_3, u_1, u_2, u_3\}, \{w_1, w_2, w_3, u_1, u_2, u_3\},\\ &\{u_2, u_3\}, \{u_1, u_3\}, \{u_1, u_2\}, \{u_3\}, \{u_2\}, \{u_1\}\}\end{aligned}$$

which, therefore, is the independent domination topology of H_3.

Theorem 2.2. *Let $n \geq 3$ and consider the helm graph H_n. $B \subseteq V(H_n)$ is an independent dominating set of H_n, denoted by I_{H_n}, if and only if B takes one of the following forms:*

i. $\{w_0, u_1, ..., u_n\}$

ii. $\{w_i : i \in A\} \cup \{u_j : j \in A^c\}$ *where $A \subseteq [n]$ satisfies the following conditions;*

  a_1*.* $1 \leq |A| \leq \lfloor \frac{n}{2} \rfloor$*.*
  a_2*. If $i \in A$, then $i + 1 \notin A$.*

Proof. ($\Leftarrow$) Let $B = \{w_0, u_1, ..., u_n\}$. By definition of H_n, B is an independent set since for all $x, y \in B$, x and y are not adjacent. Also, $w_0 \in B$ dominates w_i for all $i = 1, ..., n$. Thus, B is an independent dominating set.

Now, let $B = \{w_i : i \in A\} \cup \{u_j : j \in A^c\}$. By the second condition (a_2) for A and by definition of H_n, $\{w_i : i \in A\}$ is an independent set. On the other hand, $\{u_j : j \in A^c\}$ is also an independent set by definition of H_n. Now, for each $j \in A^c$, u_j is only adjacent to $w_j \notin B$. Thus, B is an independent. Let $x \in V(H_n) \setminus B$. If $x = w_0$, then x is adjacent to $w_i \in B$ for all $i \in A$, given that $|A| \geq 1$ in condition(a_1). If $x = w_i$ for some $i \in A^c$, then x is adjacent to $u_i \in B$. If $x = u_j$ for some $j \in A$, then x is adjacent to $w_j \in B$. Hence, B is a dominating set. Consequently, B is an independent dominating set.

($\Rightarrow$) Conversely, suppose that $S \subseteq V(H_n)$ such that $S \notin I_{H_n}$.
Case 1: $w_0 \in S$.

If $w_0 \in S$, then there exist $i \in [n]$ such that $w_i \in S$. But w_i is adjacent to w_0, by definition of H_n. Thus, S is not an independent dominating set.

Case 2: $w_0 \notin S$.

If $w_0 \notin S$ and $S \notin I_{H_n}$, then either of the following holds;

i) There exists $i \in [n]$ such that $w_i, u_i \notin S$.

ii) There exists $i \in [n]$ such that $w_i, u_i \in S$.

In (i), S is not dominating set since w_0 and u_i are the only vertices adjacent to w_i. Also, in (ii), S is not an independent set since w_i and u_i are adjacent. In both cases, S is not an independent dominating set. Thus, $I_{H_n} = \{\{w_0, u_1, ..., u_n\}, \{\{w_i : i \in A\} \cup \{u_j : j \in A^c\}\}\}$. ☐

Lemma 2.3. *For each $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$, the number of independent dominating sets of a helm graph of the form $\{w_i : i \in A, |A| = r\} \cup \{u_j : j \notin A\}$ is $\binom{n-r-1}{r-1}\binom{n}{r}$.*

Proof. Here, we count the number of ways to choose r vertices from n w_is such that no two consecutive w_is are chosen (otherwise the collection will not be independent). Let $A \subseteq [n]$ such that $|A| = r$ and the vertices w_is, $i \in A$ be represented by bars and the w_is where $i \notin A$ be represented by stars. The desired sequences should be a string of bars and stars that contain exactly r bars (the chosen points) and $n - r$ stars (the unchosen points) such that no two bars are consecutive, and the first and last points cannot be both bars.

Case 1: First and last characters are stars.

$$\underbrace{*\ \ *\ \cdots}_{x_1}\ \underset{1^{st}\text{ bar}}{|}\ \underbrace{*\ \ *\ \cdots}_{x_2}\ \underset{2^{nd}\text{ bar}}{|}\ \underbrace{\ \ \cdots\ \ }_{x_3}\ \underset{r^{th}\text{ bar}}{|}\ \underbrace{*\ \ *\ \cdots}_{x_{r+1}}$$

We find $r + 1$ positive integers $x_1, x_2, ..., x_{r+1}$ such that $x_1 + x_2 + ... + x_{r+1} = n - r$. Let $x_i' = x_i - 1$. Since $x_1 \geq 1$ for all i, one has $x_1 \geq 0$.

Substituting $x_i = x_i' + 1$ into the equation, we have

$$(x_1' + 1) + (x_2' + 1) + ... + (x_r' + 1) + (x_{r+1}' + 1) = n - r.$$

Simplifying, we get:

$$\begin{aligned} x_1' + x_2' + \cdots + x_r' + x_{r+1}' + (r + 1) &= n - r \\ x_1' + x_2' + \cdots + x_r' + x_{r+1}' &= n - r - (r + 1) \\ x_1' + x_2' + \cdots + x_r' + x_{r+1}' &= n - 2r - 1 \end{aligned} \tag{2.1}$$

By (2.1) and the Stars and Bars Theorem 1.1, the number of ways to distribute $n - 2r - 1$ stars into $r + 1$ bars is

$$\binom{(n - 2r - 1) + (r + 1 - 1)}{r + 1 - 1} = \binom{n - 2r - 1 + r}{r} = \binom{n - r - 1}{r}, \tag{2.2}$$

which gives the number of ways to choose r w_is such that w_i and w_n are not chosen.

Case 2. The first character is a star and the last character is a bar, or the opposite.

$$\underset{1^{st}\text{ bar}}{|}\ \underbrace{*\ \ *\ \cdots}_{x_1}\ \underset{2^{nd}\text{ bar}}{|}\ \underbrace{*\ \ *\ \cdots}_{x_2}\ \underset{3^{rd}\text{ bar}}{|}\ \underbrace{\ \ \cdots\ \ }_{\cdots}\ \underset{r^{th}\text{ bar}}{|}\ \underbrace{*\ \ *\ \cdots}_{x_r}$$

We find r positive integers $x_1, x_2, ..., x_r$ such that $x_1 + x_2 + ... + x_r = n - r$. If we set $x_i' = x_i - 1$, then $x_1 \geq 0$.

Substituting $x_i = x_i'$ into the equation, we have

$$(x_1' + 1) + (x_1' + 1) + ... + (x_r' + 1) = n - r.$$

Simplifying, we get:

$$\begin{aligned} x_1' + x_2' + ... + x_r' + (r) &= n - r \\ x_1' + x_2' + ... + x_r' &= n - 2r \end{aligned} \tag{2.3}$$

By (2.3) and the Stars and Bars Theorem 1.1, the number of ways to distribute $n-2r$ stars into r bars is

$$\binom{(n-2r)+(r-1)}{r-1} = \binom{n-r-1}{r-1}, \tag{2.4}$$

which gives the number of ways to choose r w_is such that w_i is chosen, while w_n are not chosen. Note that, this is the same number of ways to choose r w_is such that w_i is not chosen, while w_n is chosen.

Thus, the number of ways to choose r w_is such that no two consecutive w_is are chosen is the sum of (2.2) and (2.4). That is,

$$\begin{aligned}
\binom{n-r-1}{r} + 2\binom{n-r-1}{r-1} &= \frac{(n-r-1)!}{r!(n-r-1-r)!} + 2\left[\frac{(n-r-1)!}{(r-1)!(n-r-1-(r-1))!}\right] \\
&= \frac{(n-r-1)!}{r!(n-2r-1)!} + 2\left[\frac{(n-r-1)!}{(r-1)!(n-2r)!}\right] \\
&= \left(\frac{r}{r}\right)\left(\frac{n-2r}{n-2r}\right)\left(\frac{(n-r-1)!}{r!(n-2r-1)!} + 2\left[\frac{(n-r-1)!}{(r-1)!(n-2r)!}\right]\right) \\
&= 2\left[\frac{(n-2r)(n-r-1)!}{r(r-1)!(n-2r)!} + \frac{(n-r-1)!}{(r-1)!(n-2r)!}\right] \\
&= \left(2 + \frac{n-2r}{r}\right)\binom{n-r-1}{r-1} \\
&= \frac{2r+n-2r}{r}\binom{n-r-1}{r-1} \\
&= \frac{n}{r}\binom{n-r-1}{r-1}.
\end{aligned}$$

□

Theorem 2.4. *For* $n \geq 3$, $|I_{H_n}| = 1 + \sum\limits_{r=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{r}\binom{n-r-1}{r-1}$.

Proof. In view of the previous Theorem 2.2 and Lemma 2.3, for $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$, there are $\frac{n}{r}\binom{n-r-1}{r-1}$ sets of the form $S = \{w_i : i \in A, |A| = r\} \cup \{u_i : i \notin A\}$ and a set of the form $\{w_0, u_1, u_2, ..., u_n\}$. If $r > \lfloor \frac{n}{2} \rfloor$, then there exist two vertices $w_i, w_j \in S$ such that either $j = i+1$ or $i = 1$ and $j = n$. This would have w_i and w_j to be adjacent which makes the set not independent. Thus, $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Hence, $|I_{H_n}| = 1 + \sum\limits_{r=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{r}\binom{n-r-1}{r-1}$.

□

Remark 2.1. The independent dominating topology of the helm graph H_n is not the discrete topology on $V(H_n)$. To see this, $\{w_0\}$ cannot be $\tau_I(H_n)$- open since it is not the union nor intersection of independent dominating set, and the only independent dominating set containing w_0 is $\{w_0, u_1, ..., u_n\}$.

Theorem 2.5. *For each* $i = 1, ..., n$, $\{u_i\} \in \tau_I(H_n)$.

Proof. Let $i \in [n]$. Consider the sets $S_{A_k} = \{w_k : k \in A, 1 \leq |A| \leq \lfloor \frac{n}{2} \rfloor\} \cup \{u_j : j \neq k\}$ such that if $k \in A$, then $k+1 \notin A$ for all $k \neq i$. Then $u_i \in S_{A_k}$ for all $k \neq i$, $u_i \in \bigcap\limits_{k \neq i} S_{A_k}$. Now, for $k, k^* \neq p$, w_k is distinct from w_{k^*}. It follows that $w_k \notin \bigcap S_{A_k}$ for all $k \neq i$. Consider w_j, $j = k$. So,

$u_k \notin S_{A_k} = \{w_k : k \in A, 1 \leq |A| \leq \lfloor \frac{n}{2} \rfloor\} \cup \{u_j : j \neq k\}$ for all $k \neq i$. Thus, $u_k \notin \bigcap\limits_{k \neq i} S_{A_k}$. Since k is arbitrary, $\bigcap\limits_{k \neq i} S_{A_k} = \{u_i\}$, so that $\{u_i\} \in \tau_I(H_n)$ for all $i \in [n]$.

Corollary 2.6. *For every subset* $A \subseteq \{u_1, u_2, ..., u_n\}$*, is* $\tau_I(H_n)$*- open.*

Proof. The proof follows immediately from Theorem 2.5 since u_i is $\tau_I(H_n)-$open for all i, and $A = \bigcup\limits_{u_i \in A} \{u_i\}$. □

3 CONCLUSIONS

This paper introduces the independent domination topology induced by the helm graph, along with some of its characterizations and the construction of its independent dominating set. Further study could focus on constructing independent domination topological graphs for other graph families within the context of independent domination topology. Additionally, research could extend to exploring how these graphs behave under unary operations on graphs.

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