S-Prime Graph of S-meet Semilattice

Original Research Article

Abstract

In this paper, the S-prime ideal in a lattice $\mathfrak{L}$ is introduced where S is the meet subset of $\mathfrak{L}$. Also, it is shown that the prime ideal of $\mathfrak{L}$ is an S-prime ideal of $\mathfrak{L}$ and studied with suitable examples. Further, the S-prime ideal $\mathfrak{I}_s$ of S-meet semilattice L_s is introduced. Finally, a new graph called S-prime graph of S-meet semilattice is defined and their topological measures are generalized.

Keywords: Ideal of a lattice; Lattice; Partially ordered set; Prime ideal; Semilattice; S-prime ideal.

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1 Introduction

In 1961, Gratzer and Schmidt [1] defined a standard ideal in $\mathfrak{L}$ and Noor and Latif [2] introduced and discussed about the standard n-ideal of $\mathfrak{L}$. In 1994, the n-ideals in $\mathfrak{L}$ were introduced by Latif and Noor [3]. After that, they studied finitely generated n-ideals of $\mathfrak{L}$ [4]. In 2000, the properties of standard n-ideal of $\mathfrak{L}$ were discussed by Noor and Latif [5].

In 2015, Meenakshi P and Karuna T [6], introduced the 2-absorbing and weakly 2-absorbing ideals of $\mathfrak{L}$ which was from [7 - 8]. A proper ideal I of $\mathfrak{L}$ is called a 2-absorbing ideal if $a \wedge b \wedge c$ is in I for a, b, c is in $\mathfrak{L}$ then either $a \wedge b$ or $a \wedge c$ or $b \wedge c$ is in I. Also, they defined the triple zero in lattices and given some results related to triple zero. In 2021, Ali Akbar and Toktam Haghdadi [9], introduced the n-absorbing ideals in $\mathfrak{L}$ which is from [10]. Many authors have introduced and studied different ideals in a lattice, such as: semiprime n-ideal of $\mathfrak{L}$ [11], modular n-ideals of $\mathfrak{L}$ [12] and so on.

In 2019, Ahmed Hamed and Achraf Malek [13] defined S-prime ideals of $\mathfrak{R}$. A proper ideal I of $\mathfrak{R}$ is called an S-prime ideal I_s of $\mathfrak{R}$ if x, y is in $\mathfrak{R}$ and xy is in I_s then sx or sy is in I_s for some $s \in S$ where S is the multiplicative subset of $\mathfrak{R}$. The multiplicative subset is the complement of the prime ideal of a ring $\mathfrak{R}$.

Recently, Kalamani and Mythily [14] introduced a graph called S-prime ideal graph in which the vertices of the graph are elements of $\mathfrak{R}$ and they are connected iff $sa \in I_s$ or $sb \in I_s$ for some $s \in S$ whenever $ab \in I$ where $a, b \in \mathfrak{R}$ and the set S is disjoint from I_s. Some of the properties of the S-prime ideal I_s of $\mathfrak{R}$ are discussed in [15] and they [16] studied the interplay of the semilattice theoretic properties of a poset with the ring theoretic properties.

In this article, the S-prime ideal of $\mathfrak{R}$ is defined in a lattice $\mathfrak{L}$ and in a S-meet semilattice L_s and some results are discussed. Also, a new graph called the S-prime graph is defined and their topological measures are generalized. Refer [17 - 19] for background research related to the indices.

Throughout this paper, the first and second Zagreb indices and the Randi´c index of $\mathfrak{G}(\mathfrak{I}_s)$ are denoted $M_1(\mathfrak{G}(\mathfrak{I}_s)), M_2(\mathfrak{G}(\mathfrak{I}_s))$ and $R(\mathfrak{G}(\mathfrak{I}_s))$ respectively.

This article is organized as follows: Section 2 recalls some basic notions and definitions of lattice theory and topological indices of a graph. In section 3, the definitions of meet subset and S-prime ideal of a lattice are given with suitable examples. In section 4, the S-prime ideal of S-meet semilattice are introduced. Also, a new graph called the S-prime graph of S-meet semilattice is introduced with suitable examples. Some topological measures of the S-prime graph are discussed in sections 5, 6 and 7.

2 Preliminaries

In this section, the necessary definitions are recalled from [17 - 20].

Definition 2.1. *A relation $\mathcal{R}$ on a set A is said to be partial order relation if the relation $\mathcal{R}$ is reflexive, antisymmetric and transitive which may be described as follows: 1) Reflexivity: $a \sim a$ for all $a \in A$. 2) Antisymmetry: If $a \sim b$ and $b \sim a$ then $a = b$. 3) Transitivity: If $a \sim b; b \sim c$ then $a \sim c$. A set together with the partial order relation $\mathcal{R}$ is called poset.*

Definition 2.2. *A **lattice** $\mathfrak{L}$ is a poset in which every a, b in $\mathfrak{L}$ has meet $(\wedge)$ and join $(\vee)$. It is denoted as $(\mathfrak{L}, \wedge, \vee)$.*

Definition 2.3. *Let $(\mathfrak{L}, \wedge, \vee)$ be a lattice and $M \subseteq \mathfrak{L}$. Then $(M, \wedge, \vee)$ is a **sublattice** of $(\mathfrak{L}, \wedge, \vee)$ iff M is closed under $\wedge$ and $\vee$.*

Definition 2.4. *The sublattice I of $\mathfrak{L}$ is an **ideal** of $\mathfrak{L}$ if $a \wedge i \in I$ for every $i \in I$ and $a \in \mathfrak{L}$.*

Definition 2.5. *The sublattice I of $\mathfrak{L}$ is **prime ideal** of $\mathfrak{L}$ if $a \wedge b \in I$ implies $a \in I$ or $b \in I$ for every $a, b \in \mathfrak{L}$.*

Definition 2.6. *The topological measures of the graph $\mathfrak{G}$ are defined as follows:*

*The **first Zagreb index of a graph** $\mathfrak{G}$ is*

$$M_1(\mathfrak{G}) = \sum_{\mathfrak{x}\in\mathcal{V}(\mathfrak{G})} \mathfrak{d}(\mathfrak{x})^2.$$

*The **second Zagreb index of a graph** $\mathfrak{G}$ is*

$$M_2(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(G)} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}).$$

*The **first Zagreb coindex of a graph** $\mathfrak{G}$ is*

$$\overline{M_1}(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G})} [\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})].$$

The ***second Zagreb coindex of a graph*** $\mathfrak{G}$ *is*

$$\overline{M_2}(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\notin \mathcal{E}(\mathfrak{G})} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}).$$

The ***Randi´c index of a graph*** $\mathfrak{G}$ *is*

$$R(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\in \mathcal{E}(\mathfrak{G})} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}.$$

3 S-prime ideal of a lattice

In this section, the S-prime ideal of $\mathfrak{L}$ is defined with an example and some of its results are discussed.

Definition 3.1. *Let $S \subseteq \mathfrak{L}$. Then the set S is called meet subset of $\mathfrak{L}$ if $a \wedge b \in S$ for all $a, b \in S$.*

Definition 3.2. *Let I be a proper ideal of a lattice $\mathfrak{L}$. The ideal I is said to be an S-prime ideal of $\mathfrak{L}$ if $x \wedge y$ in I then $s \wedge x$ or $s \wedge y$ is in I for any $x, y \in \mathfrak{L}$ and for some $s \in S$, where S is the meet subset of a lattice $\mathfrak{L}$ which is disjoint from I of $\mathfrak{L}$. The S-prime ideal of $\mathfrak{L}$ is denoted by I_s*

Example 3.3. *Consider $\mathfrak{L} = \{0, u, v, w, x, y, z, 1\}$ be a lattice whose Hasse diagram is given in Fig.1. The S-prime ideal I_s of $\mathfrak{L} = \{0, u, v, w, x, y, z, 1\}$ are, from Fig.1, $I_1 = \{0\}, I_2 = \{0, u\}, I_3 = \{0, u, v\}, I_4 = \{0, u, w\}, I_5 = \{0, u, x\}, I_6 = \{0, u, y\}$ and $I_7 = \{0, u, v, w, x, y, z\}$. The meet subset of a lattice are $S_1 = \{1\}, S_2 = \{1, u\}, S_3 = \{1, v\}, S_4 = \{1, w\}, S_5 = \{1, x\}, S_6 = \{1, y\}, S_7 = \{1, z\}, S_8 = \{1, v, z\}, S_9 = \{1, v, z\}$, $S_{10} = \{1, v, z\}$ and so on.*

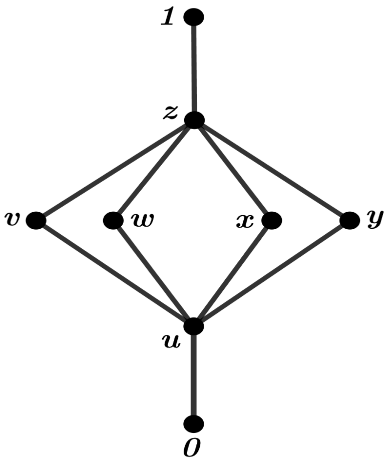

Fig. 1: **Hasse diagram of $\mathfrak{L}$**

Theorem 3.4. *Every prime ideal P of $\mathfrak{L}$ is an S-prime ideal of $\mathfrak{L}$.*

Proof. Let P be the prime ideal of $\mathfrak{L}$.
Let S be the meet subset of $\mathfrak{L}$ which is disjoint from P of $\mathfrak{L}$. That is, $P \cap S = \emptyset$.
Choose $x, y \in \mathfrak{L}$ such that $x \wedge y \in P$.
Since P is prime, x or $y \in P$.
If $s \in S$ then $s \wedge x$ or $s \wedge y \in P$.
Thus, the prime ideal P is the S-prime ideal I_s of $\mathfrak{L}$. $\square$

The converse of Theorem 3.4 is not true and is explained in the following example.

Example 3.5. *Let us consider the example which is shown in Fig.1. Let $I_s = \{0, u\}$ be the S-prime ideal of $\mathfrak{L} = \{0, u, v, w, x, y, z, 1\}$ and the meet subset of $\mathfrak{L}$ as $S = \{1, z, y\}$. Now, let $v, w \in \mathfrak{L}$ if $v \wedge w = u \in I_s$ which implies that $v \notin I_s$ and $w \notin I_s$. Thus I_s is not the prime ideal of $\mathfrak{L}$.*

Theorem 3.6. *Let P be the prime ideal of $\mathfrak{L}$. Then $\mathfrak{L} - P$ is the meet subset of $\mathfrak{L}$.*

Proof. Let $\mathfrak{L}$ be the lattice and P be the prime ideal of $\mathfrak{L}$. It is needed to prove that the set $\mathfrak{L} - P$ is a meet subset of $\mathfrak{L}$. Let $x, y \in \mathfrak{L} - P$.
This implies that $x, y \in \mathfrak{L}$ and $x, y \notin P$.
Suppose $x \wedge y \in P$. As P is prime, either x or y is in P.
This contradicts to $x, y \notin P$. Therefore, $x \wedge y \notin P$.
The elements $x, y \in \mathfrak{L}$ implies that $x \wedge y \in \mathfrak{L}$. Therefore, $x \wedge y \in \mathfrak{L} - P$.
Thus, the set $\mathfrak{L} - P$ is the meet subset of a lattice $\mathfrak{L}$. $\square$

4 S-prime Graph of an S-meet Semilattice

In this section, the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of an S-meet semilattice L_s is defined. Also, a new graph called the S-prime graph of L_s is explained with necessary examples. The concept of the meet subset and the S-prime ideal of $\mathfrak{L}$ are applicable to the S-meet semilattice L_s of a ring $\mathfrak{R}$.

Definition 4.1. *Let I be the proper ideal of L_s. The ideal I is said to be an S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s if for any $\mathfrak{u}, \mathfrak{v} \in L_s, \mathfrak{u} \wedge \mathfrak{v} \in \mathfrak{I}_\mathfrak{s}$ then $\exists\ s \in S$ such that $s \wedge \mathfrak{u}$ or $s \wedge \mathfrak{v}$ in $\mathfrak{I}_\mathfrak{s}$ for some $s \in S$, where S is the meet subset of L_s and $S \cap \mathfrak{I}_\mathfrak{s} = \emptyset$.*

Definition 4.2. *Let $(L_s, \wedge, \subseteq)$ be the S-meet semilattice where L_s is the collection of all S-prime ideals of $\mathfrak{R}$. The set of all elements of L_s are considered to be the vertices of the graph, the vertices $\mathfrak{x}$ and $\mathfrak{y}$ are adjacent if $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$, where $\mathfrak{I}_\mathfrak{s}$ is the S-prime ideal of L_s. It is an undirected graph called S-prime graph of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$, denoted by $\mathfrak{G}_{L_s}(\mathfrak{I}_\mathfrak{s})$, simply $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$.*

Let $\mathfrak{R}$ be a ring of order $p^t q$. The S-prime graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ of $\mathfrak{I}_\mathfrak{s}$ is (i) a complete graph if the S-prime ideals $\mathfrak{I}_\mathfrak{s}$ of L_s are $\downarrow p, \downarrow q$ and $\downarrow pq$, (ii) a star graph if the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s is $\downarrow p^t q$ and (iii) a connected graph if the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s is $\downarrow p^k q, k < t$.

Example 4.3. *Let* $\mathfrak{R} = \mathbb{Z}_{48}$ *and the* S*-prime graphs* $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ *are shown in Fig.2. The elements of* L_s *are* $<2>, <3>, <6>, <12>, <24>$ *and* $<48>$*.*

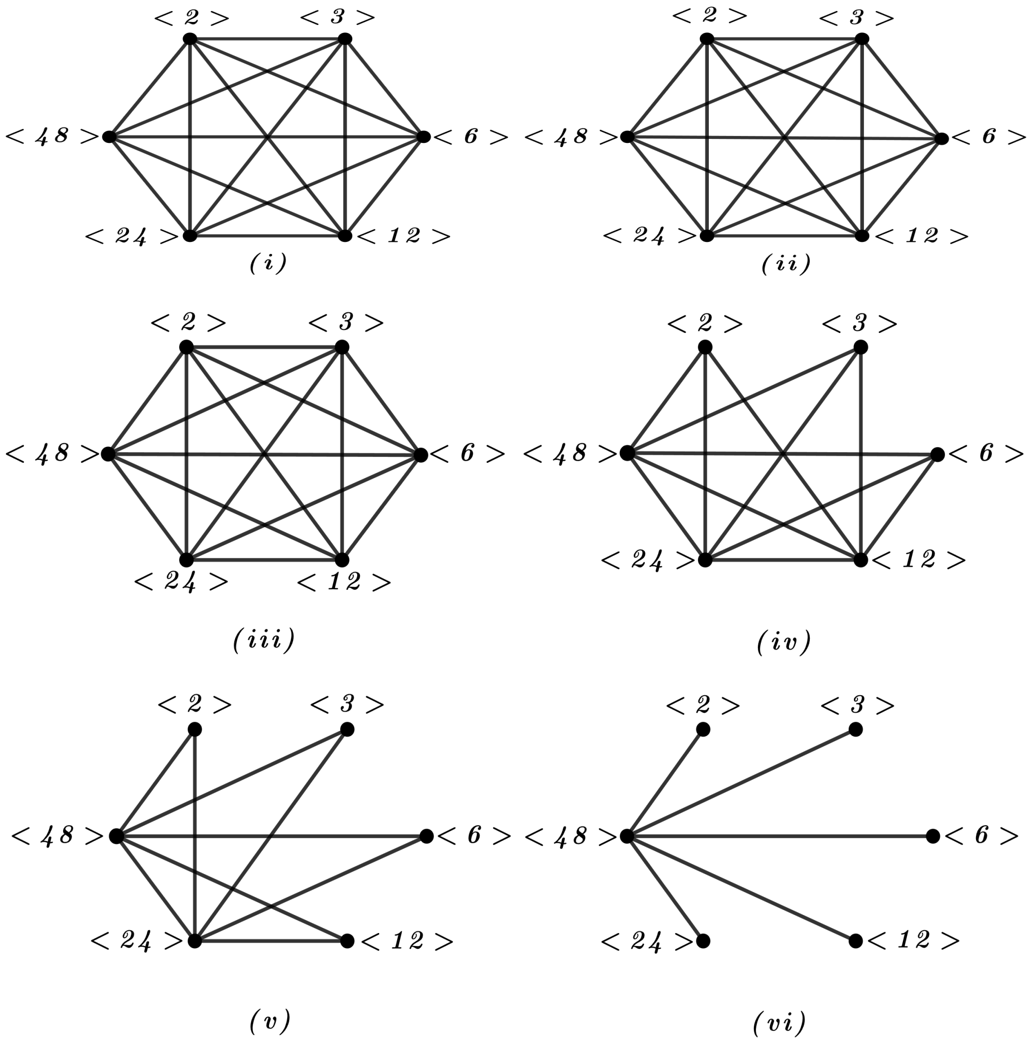

Fig. 2: **S-prime graph of the S-prime ideal** $(i) \downarrow 2 \;\; (ii) \downarrow 3 \;\; (iii) \downarrow 6 \;\; (iv) \downarrow 12 \;\; (v) \downarrow 24 \;\; (vi) \downarrow 48$

Let $\mathfrak{R}$ be a ring of order pqr. Then the S-prime graph $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ is a complete graph if the S-prime ideals of L_s are $\downarrow p \cup \downarrow q, \downarrow p \cup \downarrow r, \downarrow q \cup \downarrow r, \downarrow pq \cup \downarrow pr \cup \downarrow qr, \downarrow p \cup \downarrow qr, \downarrow q \cup \downarrow pr$ and $\downarrow r \cup \downarrow pq$.

There are 3 distinct connected S-prime graphs $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}), \mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ where $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})$ is the S-prime graph for the S-prime ideals $\downarrow p, \; \downarrow q, \; \downarrow r, \; \downarrow pq \cup \downarrow pr, \; \downarrow pq \cup \downarrow qr, \; \downarrow pr \cup \downarrow qr$, $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ is the S-prime graph for the ideals $\downarrow pq, \; \downarrow pr, \; \downarrow qr$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ for the S-prime ideal $\downarrow pqr$.

Example 4.4. *Let $\mathfrak{R} = \mathbb{Z}_{30}$ and the S-prime graph of S-meet semilattice is shown in Fig.3 whose vertex set is*

$\mathcal{V}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \{< 2 >, < 3 >, < 5 >, < 6 >, < 10 >, < 15 >, < 30 >\}.$

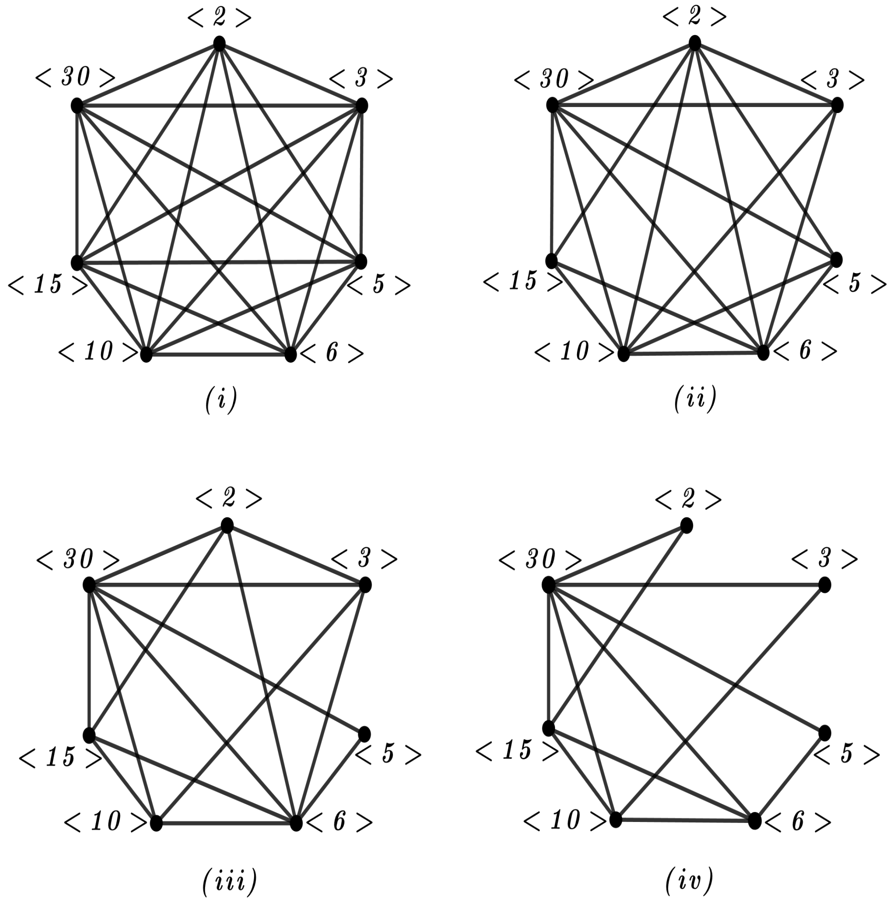

Fig. 3: **S-prime graph of the S-prime ideal** $(i) \downarrow 2 \cup \downarrow 3 \;\; (ii) \downarrow 2 \;\; (iii) \downarrow 6 \;\; (iv) \downarrow 30$

In the following sections, the topological measures $M_1(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$, $M_2(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$, $\overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$, $\overline{M_2}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$,$\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ and $R(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ of the connected S-prime graph of the S-prime ideals $\mathfrak{I}_\mathfrak{s}$ are studied.

5 First and Second Zagreb Indices of S-prime Graph

Let $\mathfrak{R}$ be a ring of order $p^t q$. The S-prime graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ is connected if the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ is the down-set of $p^k q$ where $k < t$. Let ϖ and ϑ be the order of the graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ and the ideal $\mathfrak{I}_\mathfrak{s}$ respectively.

Theorem 5.1. *Let $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s then*

$$M_1(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta\Big[(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta\Big].$$

Proof. Let $\mathfrak{x}$ be a vertex of $\mathfrak{G}(\mathfrak{I})$.

If $\mathfrak{x}$ is an element of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$, then $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s} \ \forall \ \mathfrak{y} \in L_s$.

If $\mathfrak{x}$ is not an element of $\mathfrak{I}_\mathfrak{s}$, then $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$ only if $\mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$.

Therefore, $\mathfrak{d}(\mathfrak{x})$ is given as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi - 1 \ \ if \ \ \mathfrak{x} \in \mathfrak{I}_\mathfrak{s} \\ \\ \vartheta \ \ otherwise. \end{cases} \tag{5.1}$$

Then, $M_1(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x} \in \mathcal{V}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2$

$$= \sum_{\mathfrak{x} \in \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x} \notin \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2$$
$$= \vartheta(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta^2$$

$$= \vartheta\Big[(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta\Big].$$

☐

Theorem 5.2. *Let $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s then*

$$M_2(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta(\varpi - 1)\Big[\frac{(\vartheta - 1)}{2}(\varpi - 1) + \vartheta(\varpi - \vartheta)\Big].$$

Proof. Let $\mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ be the edge set of $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}[\mathfrak{G}(\mathfrak{I}_\mathfrak{s})]$.

This implies that either $\mathfrak{x}$ or $\mathfrak{y}$ is in $\mathfrak{I}_\mathfrak{s}$ and $\mathfrak{d}(\mathfrak{x})$ and $\mathfrak{d}(\mathfrak{y})$ are defined in (5.1). Then,

$$M_2(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$$

$$= \sum_{\mathfrak{x} \in \mathfrak{I}_\mathfrak{s}, \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x} \in \mathfrak{I}_\mathfrak{s}, \mathfrak{y} \notin \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$$

$$= \vartheta\frac{(\vartheta - 1)}{2}(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)(\varpi - 1)\vartheta$$

$$= \vartheta(\varpi - 1)\Big[\frac{(\vartheta - 1)}{2}(\varpi - 1) + \vartheta(\varpi - \vartheta)\Big].$$

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Let $\mathfrak{R}$ be a ring of order pqr. There are 3 distinct S-prime connected graphs of $\mathfrak{I}_\mathfrak{s}$ namely $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}), \mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ which are explained earlier.

Theorem 5.3. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $M_1(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = \vartheta\Big[(\varpi-1)^2 + \vartheta(\varpi-\vartheta)\Big]$.

(ii) $M_1(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = \vartheta\Big[(\varpi-1)^2 + \vartheta\Big] + (\vartheta+2)^2$.

(iii) $M_1(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = \vartheta(\varpi-1)^2 + (\varpi-4)\Big[(\vartheta+1)^2 + (\vartheta+3)^2\Big]$.

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph of $\mathfrak{I}_\mathfrak{s}$ of L_s. Then,

$$\begin{aligned}
M_1(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) &= \textstyle\sum_{\mathfrak{x}\in\mathcal{V}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2 \\
&= \textstyle\sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 \\
&= \vartheta(\varpi-1)^2 + (\varpi-\vartheta)\vartheta^2 \\
&= \vartheta\Big[(\varpi-1)^2 + \vartheta(\varpi-\vartheta)\Big].
\end{aligned}$$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph of $\mathfrak{I}_\mathfrak{s}$ of L_s.

In this case, the non-ideal elements are adjacent to all the ideal elements and some non-ideal elements. Here, the S-prime ideals are $\downarrow pq$, $\downarrow pr$ and $\downarrow qr$.

Consider the S-prime ideal $\mathfrak{I}_\mathfrak{s} = \downarrow pq$ and $d(x)$ in $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ is as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi - 1 \;\; if \;\; \mathfrak{x}\in\mathfrak{I}_\mathfrak{s} \\ \vartheta \;\; if \;\; \mathfrak{x} = r \\ \vartheta + 2 \;\; otherwise. \end{cases} \tag{5.2}$$

Then,

$$\begin{aligned}
M_1(\mathfrak{G}_s^{(2)}(I)) &= \textstyle\sum_{\mathfrak{x}\in\mathcal{V}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2 \\
&= \textstyle\sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}=r} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}\neq r} \mathfrak{d}(\mathfrak{x})^2 \\
&= \vartheta(\varpi-1)^2 + \vartheta^2 + (\vartheta+2)^2 \\
&= \vartheta\Big[(\varpi-1)^2 + \vartheta\Big] + (\vartheta+2)^2.
\end{aligned}$$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph of $\mathfrak{I}_\mathfrak{s}$ of L_s.

In this case, the S-prime ideal is $\downarrow pqr$. The maximal elements of L_s are p, q, r and they are denoted as $\mathfrak{M}_k, k = 1, 2, 3$ and $\mathfrak{d}(\mathfrak{x})$ in $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ is as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi - 1 \;\; if \;\; \mathfrak{x} \in \mathfrak{I}_\mathfrak{s} \\ \vartheta + 1 \;\; if \;\; \mathfrak{x} = \mathfrak{M}_k \\ \vartheta + 3 \;\; otherwise. \end{cases} \tag{5.3}$$

Then,

$$M_1(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = \textstyle\sum_{\mathfrak{x}\in\mathcal{V}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2$$

$$= \textstyle\sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}=\mathfrak{M}_k} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}\neq\mathfrak{M}_k} \mathfrak{d}(\mathfrak{x})^2$$

$$= \vartheta(\varpi - 1)^2 + (\varpi - 4)(\vartheta + 1)^2 + (\varpi - 4)(\vartheta + 3)^2$$

$$= \vartheta(\varpi - 1)^2 + (\varpi - 4)\Big[(\vartheta + 1)^2 + (\vartheta + 3)^2\Big]. \qquad \square$$

Theorem 5.4. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $M_2(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 1)\Big[(\varpi - 1)^2 + \vartheta^2(\varpi - \vartheta)\Big]$.

(ii) $M_2(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 1)\Big[(\varpi - 1) + \vartheta^2\Big] + (\vartheta + 2)\Big[8(\vartheta - 1) + (\vartheta + 2)^2\Big]$.

(iii) $M_2(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = 2(\varpi - 4)\Big[(\varpi - 1)(\eta - 4) + (\vartheta + 3)(\varpi + 2)\Big]$.

Proof. (i) Let $\mathcal{E}[\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})]$ be the edge set of $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}[\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})]$.

This implies that either $\mathfrak{x}$ or $\mathfrak{y}$ is in $\mathfrak{I}_\mathfrak{s}$ and the degrees of the vertices $\mathfrak{x}$ and $\mathfrak{y}$ are defined in (5.1). Then,

$$M_2(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$$

$$= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$$

$$= (\varpi - 1)(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)(\varpi - 1)\vartheta$$

$$= (\varpi - 1)\Big[(\varpi - 1)^2 + \vartheta^2(\varpi - \vartheta)\Big].$$

(ii) Let $E[\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})]$ be the edge set of $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_S.

Consider the S-prime ideal $\mathfrak{I}_\mathfrak{s} =\downarrow pq$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ of $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ are defined in (5.2). Then,

$$M_2(\mathfrak{G}_s(\mathfrak{I}_\mathfrak{s})) = \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}[\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})]} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$$

$$= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$$

$$= (\varpi - 1)^2 + \Big[(\varpi - 1)\vartheta^2 + 8(\varpi - 1)(\vartheta + 2)\Big] + (\vartheta + 2)^3$$

$$= (\varpi - 1)\Big[(\varpi - 1) + \vartheta^2\Big] + (\vartheta + 2)\Big[8(\varpi - 1) + (\vartheta + 2)^2\Big].$$

(iii) Let $\mathcal{E}[\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})]$ be the edge set of $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ of the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of L_s.

Consider the S-prime ideal is $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ of $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ are defined in (5.3). Then,

$$\begin{aligned}
M_2(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}[\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})]} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})\\
&= \textstyle\sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})} + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})\\
&= \Big[(\varpi-1)(\varpi-5)(\varpi-4)+(\varpi-1)(\varpi-3)(\varpi-4)\Big]+\Big[(\vartheta+1)(\vartheta+3)(\varpi-4)+(\vartheta+3)^2(\varpi-4)\Big]\\
&= (\varpi-1)(\varpi-4)(2\varpi-8)+(\varpi-4)(\vartheta+3)(2\varpi+4)\\
&= 2(\varpi-1)(\varpi-4)^2+2(\varpi-4)(\vartheta+3)(\varpi+2)\\
&= 2(\varpi-4)\Big[(\varpi-1)(\varpi-4)+(\vartheta+3)(\varpi+2)\Big].
\end{aligned}$$

□

6 First and Second Zagreb Coindex of S-prime Graph

The $M_1(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ and $M_2(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ of the S-prime graph are generalized in this section.

Theorem 6.1. *Let $\mathfrak{R}$ be a ring of order $p^t q$. Then,*

$$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta(\varpi-\vartheta)(\varpi-\vartheta-1).$$

Proof. Let $\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ of the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of L_s. If $\mathfrak{x}\mathfrak{y}$ is an edge of $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$, then at least one of the end points of $\mathfrak{x}\mathfrak{y}$ must be in the ideal $\mathfrak{I}_{\mathfrak{s}}$. Then,

$$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))} [\mathfrak{d}(\mathfrak{x})+\mathfrak{d}(\mathfrak{y})]$$

In this, there is no edge between the non-ideal elements $\mathfrak{x},\mathfrak{y}$. Thus,

$$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} (\varpi-\vartheta)(\varpi-\vartheta-1)\vartheta$$

$$\therefore\ \overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta(\varpi-\vartheta)(\varpi-\vartheta-1).$$

□

Theorem 6.2. *Let $\mathfrak{R}$ be a ring of order $p^t q$. Then,*

$$\overline{M_2}(\mathfrak{G}_s(I)) = \overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})).\frac{\vartheta}{2}.$$

Proof. Let $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Then,

$$\begin{aligned}
\overline{M_2}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}).\\
&= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} (\varpi-\vartheta)\dfrac{(\varpi-\vartheta-1)}{2}\vartheta^2\\
&= \frac{(\varpi-\vartheta)(\varpi-\vartheta-1)}{2}.\vartheta^2\\
&= \Big[\vartheta(\varpi-\vartheta)(\varpi-\vartheta-1)\Big]\frac{\vartheta}{2}
\end{aligned}$$

Hence, $\overline{M_2}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})).\dfrac{\vartheta}{2}$.

□

Theorem 6.3. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $\overline{M_1}(\mathfrak{G}^{(1)}(\mathfrak{J}_{\mathfrak{s}})) = 2(\varpi - 3)(\varpi - 4)$.

(ii) $\overline{M_1}(\mathfrak{G}^{(2)}(\mathfrak{J}_{\mathfrak{s}})) = (\varpi - 3)\Big[3(\vartheta - 1) + \varpi\Big]$.

(iii) $\overline{M_1}(\mathfrak{G}^{(3)}(\mathfrak{J}_{\mathfrak{s}})) = 2\Big[(\vartheta + 1)(\varpi - 4) + (\varpi - 1)(\vartheta + 2)\Big]$.

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{J}_{\mathfrak{s}})$ be the S-prime graph. Then,

$$\begin{aligned}\overline{M_1}(\mathfrak{G}^{(1)}(\mathfrak{J}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{J}_{\mathfrak{s}}))}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]\\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{J}_{\mathfrak{s}}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]\\ &= \Big[(\varpi - 3) + (\varpi - 3)\Big](\varpi - 4)\\ &= 2(\varpi - 3)(\varpi - 4).\end{aligned}$$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{J}_{\mathfrak{s}})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{J}_{\mathfrak{s}} = \downarrow pq$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (5.2). Then,

$$\begin{aligned}\overline{M_1}(\mathfrak{G}^{(2)}(\mathfrak{J}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{J}_{\mathfrak{s}}))}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]\\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{J}_{\mathfrak{s}}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]\\ &= \textstyle\sum_{\mathfrak{x}=r,\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x},\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]\\ &= (\varpi - 3)\Big[\vartheta + (\varpi - 3)\Big] + \vartheta\Big[(\varpi - 3) + (\varpi - 3)\Big]\\ &= (\varpi - 3)\Big[3(\vartheta - 1) + \varpi\Big].\end{aligned}$$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{J}_{\mathfrak{s}})$ be the S-prime graph. Consider the S-prime ideal is $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (5.3). Then,

$$\begin{aligned}\overline{M_1}(\mathfrak{G}^{(3)}(\mathfrak{J}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{J}_{\mathfrak{s}}))}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]\\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{J}_{\mathfrak{s}}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{x})]\\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}=\mathfrak{M}_{\mathfrak{k}}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x}=\mathfrak{M}_{\mathfrak{k}},\mathfrak{y}\neq\mathfrak{M}_{\mathfrak{k}}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]\\ &= 2(\vartheta + 1)(\varpi - 4) + (\varpi - 1)\Big[\vartheta + 1 + (\vartheta + 3)\Big]\\ &= 2(\vartheta + 1)(\varpi - 4) + (\varpi - 1)(2\vartheta + 4)\\ &= 2(\vartheta + 1)(\varpi - 4) + 2(\varpi - 1)(\vartheta + 2)\\ &= 2\Big[(\vartheta + 1)(\varpi - 4) + (\varpi - 1)(\vartheta + 2)\Big].\end{aligned}$$

□

Theorem 6.4. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $\overline{M_2}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 4)\vartheta^2.$

(ii) $\overline{M_2}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = 2\vartheta(\vartheta + 2)^2.$

(iii) $\overline{M_2}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = (\vartheta + 1)\Big[(\vartheta + 1)(\varpi - 4) + (\vartheta + 3)(\varpi - 1)\Big].$

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Then,

$$\begin{aligned}\overline{M_2}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] \\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] \\ &= (\varpi - 4)\vartheta\vartheta \\ &= (\varpi - 4)\vartheta^2.\end{aligned}$$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_\mathfrak{s} = \downarrow pq$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (5.2). Then,

$$\begin{aligned}\overline{M_2}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] \\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] \\ &= \textstyle\sum_{\mathfrak{x}=r,\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x},\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] \\ &= \vartheta(\vartheta + 2)(\vartheta + 2) + (\vartheta + 2)(\vartheta + 2)\vartheta \\ &= 2\vartheta(\vartheta + 2)^2.\end{aligned}$$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Consider the S-prime ideal is $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (5.3). Then,

$$\begin{aligned}\overline{M_2}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] \\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] \\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}=\mathfrak{M}_\mathfrak{k}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x}=\mathfrak{M}_\mathfrak{k},\mathfrak{y}\neq\mathfrak{M}_\mathfrak{k}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] \\ &= (\vartheta + 1)(\vartheta + 1)(\varpi - 4) + (\vartheta + 1)(\vartheta + 3)(\varpi - 1) \\ &= (\vartheta + 1)\Big[(\vartheta + 1)(\varpi - 4) + (\vartheta + 3)(\varpi - 1)\Big].\end{aligned}$$

☐

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7 Randi´c index of S-prime Graph

In this section, $R(\mathfrak{G}(\mathfrak{J}_\mathfrak{s}))$ of $\mathfrak{G}(\mathfrak{J}_\mathfrak{s})$ of the S-prime ideals $\mathfrak{J}_\mathfrak{s}$ are generalized.

Theorem 7.1. *Let* $\mathfrak{R}$ *be a ring of order* $p^t q$. *Then,*

$$R(\mathfrak{G}(\mathfrak{J}_\mathfrak{s}))) = \vartheta\Big[\frac{(\vartheta-1)}{2(\varpi-1)} + \frac{(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}\Big].$$

Proof. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}(\mathfrak{J}_\mathfrak{s}))$ be the edge set of the S-prime graph and the degrees of the vertices are defined in (5.1). Then,

$$R(\mathfrak{G}(\mathfrak{J}_\mathfrak{s})) = \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}(\mathfrak{J}_\mathfrak{s}))} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{J}_\mathfrak{s}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \textstyle\sum_{\mathfrak{x}\in\mathfrak{J}_\mathfrak{s},\mathfrak{y}\notin\mathfrak{J}_\mathfrak{s}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \frac{\vartheta(\vartheta-1)}{2\sqrt{(\varpi-1)^2}} + \frac{\vartheta(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}$$

Thus, $R(\mathfrak{G}(\mathfrak{J}_\mathfrak{s})) = \vartheta\Big[\dfrac{(\vartheta-1)}{2(\varpi-1)} + \dfrac{(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}\Big].$ □

Theorem 7.2. *Let* $\mathfrak{R}$ *be a ring of order* pqr. *Then,*

(i) $R(\mathfrak{G}^{(1)}(\mathfrak{J}_\mathfrak{s})) = 1 + \dfrac{(\varpi-3)(\varpi-4)}{\sqrt{\vartheta(\varpi-1)}}$

(ii) $R(\mathfrak{G}(\mathfrak{J}_\mathfrak{s})) = 1 + \dfrac{1}{(\varpi-1)} + \dfrac{\vartheta}{\sqrt{\vartheta(\varpi-1)}} + \dfrac{8}{\sqrt{(\varpi-1)(\vartheta+2)}}.$

(iii) $R(\mathfrak{G}^{(3)}(\mathfrak{J}_\mathfrak{s})) = 3\Big[\dfrac{1}{\sqrt{(\varpi-1)(\vartheta+1)}} + \dfrac{1}{\sqrt{(\varpi-1)(\vartheta+3)}} + \dfrac{1}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \dfrac{1}{(\vartheta+3)}\Big].$

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{J}_\mathfrak{s})$ be the S-prime graph. Then,

$$R(\mathfrak{G}^{(1)}(\mathfrak{J}_\mathfrak{s})) = \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{J}_\mathfrak{s}))} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{J}_\mathfrak{s}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \textstyle\sum_{\mathfrak{x}\in\mathfrak{J}_\mathfrak{s},\mathfrak{y}\notin\mathfrak{J}_\mathfrak{s}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \frac{(\varpi-1)}{\sqrt{(\varpi-1)(\varpi-1)}} + \frac{(\varpi-3)(\varpi-4)}{\sqrt{(\varpi-1)\vartheta}}$$

$$= 1 + \frac{(\varpi-3)(\varpi-4)}{\sqrt{\vartheta(\varpi-1)}}.$$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{J}_\mathfrak{s})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{J}_\mathfrak{s} =\downarrow pq$. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{J}_\mathfrak{s}))$ be the edge set of $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ and the degrees of the vertices are defined in (5.2). Then,

$$\begin{aligned} R(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}}))} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} \\ &= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} \\ &= 1 + \frac{1}{(\varpi-1)} + \frac{\vartheta}{\sqrt{\vartheta(\varpi-1)}} + \frac{8}{\sqrt{(\varpi-1)(\vartheta+2)}}. \end{aligned}$$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_{\mathfrak{s}} =\downarrow pqr$. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ and the degrees of the vertices are defined in (5.3). Then,

$$\begin{aligned} R(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) &= \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}}))} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} \\ &= \textstyle\sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} \\ &= \Big[\frac{3}{\sqrt{(\varpi-1)(\vartheta+1)}} + \frac{3}{\sqrt{(\varpi-1)(\vartheta+3)}}\Big] + \Big[\frac{3}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \frac{3}{\sqrt{(\vartheta+3)^2}}\Big] \\ &= 3\Big[\frac{1}{\sqrt{(\varpi-1)(\vartheta+1)}} + \frac{1}{\sqrt{(\varpi-1)(\vartheta+3)}} + \frac{1}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \frac{1}{(\vartheta+3)}\Big]. \qquad \square \end{aligned}$$

8 Conclusion

In this paper, the meet subset and a new ideal called S-prime ideal in a lattice and S-meet semilattice are defined and it is shown that the prime ideal of a lattice is also an S-prime ideal of a lattice L. Then, a new graph for the S-meet semilattice $(L_s, \wedge, \subseteq)$ is introduced and explained with necessary examples and their degree based topological measures are generalized for the S-prime graph of S-meet semilattice L_s.

References

[1] Gratzer. G, Schmidt E. T. Standard ideal in lattices. Acta math. Acad. Sci. Hung. 1961; 12: 17 - 86.

[2] Noor A. S. A, Latif M. A.Standard n-ideals of a lattice. Southeast Asian Bull. Math. 1997; 4: 185 - 192.

[3] Latif M. A, Noor A. S. A. n-Ideals of a Lattice. The Rajshahi University Studies, Part-B,1994; 22: 173-180.

[4] Noor A. S. A, Latif M. A. Finitely generated n-ideals of a lattice. Southeast Asian Bull. Math. 1998; 4: 73-79.

[5] Noor A. S. A, Latif M. A. Properties of Standard n-ideals of a lattice. Southeast Asian Bull. Math. 2000; 24: 403 - 409.

[6] Meenakshi P. Wasadikar, Karuna T. Gaikwad. On 2-absorbing and weakly 2-absorbing ideals of lattices. Mathematical Sciences International Research Journal. 2015;4(2): 82 - 85.

[7] Badawi A. On 2-absorbing ideals of commutative rings. Bull. Austral. Math. Soc. 2007; 75: 417-429.

[8] Badawi A, A. Y. Darani. On weakly 2-absorbing ideals of commutative rings. Houston J. Math. 2013; 39(2): 441-452.

[9] Ali Akbar Estaji, Toktam Haghdadi. On n-absorbing Ideals in a Lattice. Kragujevac Journal of Mathematics. 2021; 45(4): 597 - 605.

[10] Anderson D. F, Ayman Badawi. On n-absorbing ideals of commutative rings. 2011; 39: 1646 - 1672.

[11] Ali. A, Hafizur Rahman R. M, Noor A. S. A, Mizanur Rahman M. On semiprime n-ideals of lattices. Annals of Pure and Applied Mathematics. 2012; 21(1): 10-17.

[12] Ayub Ali M, Noor A. S. A. Some Properties of Modular n-Ideals of a Lattice. Journal of Physical Sciences. 2010;14: 173-180.

[13] Ahmed Hamed, Achraf Malek.S-prime ideals of a commutative ring. Beitrage zur Algebra and Geometrie. 2020; 61: 533 - 542.

[14] Kalamani D, Mythily C. V. S-prime ideal graph of a finite commutative ring. Advances and Applications in Mathematical Sciences. 2023; 22: 861-872.

[15] Mythily C. V, Kalamani D. Study on S-prime Ideal as Nilpotent Ideal. Journal of Applied Mathematics and Informatics. 2024; (In press).

[16] Kalamani D, Mythily C. V. Some New Results on S-prime ideals of a Finite Commutative ring as S-meet Semilattice. Applied and Computational Mathematics. 2024; 13 (4): 105 - 110.

[17] Ashrafi A. R., Doslic T, Hamzeh A. The Zagreb coindices of graph operations. Discrete Applied Mathematics. 2010; 158; 1571 - 1578.

[18] Khalifeh M. H, yousefi-Azari H, Ashrafi A. R. The first and second Zagreb indices of some graph operations. Discrete Applied Mathematics.2009; 157: 804 - 811.

[19] Kiruthika G, Kalamani D. Degree based partition of the power graph of the finite Abelian group. Malaya Journal of Matematik. 2020; 1: 66 -71.

[20] Gratzer. G. General Lattice Theory. Academic Press. New York San Francisco. 1978.