UNDER PEER REVIEW

Original Research Article

# $S$-Prime Graph of $S$-meet Semilattice

## Abstract

Let $\mathfrak{L}$ be a lattice and $I$ be the proper ideal of a lattice $\mathfrak{L}$ and the set $S$ be the multiplicative subset of $\mathfrak{L}$. In this paper, $S$-prime ideal in $\mathfrak{L}$ is introduced. Also it is shown that the prime ideal of $\mathfrak{L}$ is an $S$-prime ideal of $\mathfrak{L}$ studied with suitable examples. Further, multiplicative subset $S$ and the $S$-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of an $S$-meet semilattice are introduced. Finally, a new graph called $S$-prime graph of $S$-prime ideal of $S$-meet semilattice is defined and their topological measures are generalized.

*Keywords: Prime ideal; $S$-prime ideal; Partially ordered set; Lattice; Semilattice; $S$-meet semilattice; Ideal of a lattice*

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## 1 Introduction

In 1961, Gratzer and Schmidt [1] defined a standard ideal in $\mathfrak{L}$ and Noor and Latif [2] introduced and discussed about the standard $n$-ideal of $\mathfrak{L}$. In 1994, $n$-ideals in $\mathfrak{L}$ were introduced by Latif and Noor [3]. After that they studied finitely generated $n$-ideals of $\mathfrak{L}$ [4]. In 2000, the properties of standard $n$-ideal of $\mathfrak{L}$ were discussed by Noor and Latif [5].
In 2015, Meenakshi P and Karuna T [6], introduced the $2$-absorbing and weakly $2$-absorbing ideals of $\mathfrak{L}$ which was from [7 - 8]. A proper ideal $I$ of $\mathfrak{L}$ is called a $2$-absorbing ideal if $a \wedge b \wedge c$ is in $I$ for $a, b, c$ is in $\mathfrak{L}$ then either $a$ meet $b$ or $a$ meet $c$ or $b$ meet $c$ in $I$. Also, they define the triple zero in lattices and given some results related triple zero. In 2021, Ali Akbar and Toktam Haghdadi [9], introduced the $n$-absorbing ideals in $L$ which is from [10]. Many authors introduced and studied different ideals in a lattice, they are: semiprime $n$-ideal of $\mathfrak{L}$ [11], modular $n$-ideals of $\mathfrak{L}$ [12] and so on.
In 2019, Ahmed Hamed and Achraf Malek [13], defined $S$-prime ideals of $R$. A proper ideal $I$ of $R$ is called an $S$-prime ideal $I_s$ of $R$ if $x, y$ is in $R$ and $xy$ is in $I_s$ then $sx$ or $sy$ is in $I_s$ for some $s \in S$ where

$S$ is the multiplicative subset of $R$. The multiplicative subset is the complement of the prime ideal of a ring $R$.
Recently, Kalamani and Mythily [14] introduced a graph, the vertices of the graph are from $R$ and they are connected iff $sa \in I_s$ or $sb \in I_s$ for some $s \in S$ whenever $ab \in I$ where $a, b \in R$ and the set $S$ is disjoint from $I_s$. Some of the properties of the $I_s$ of $R$ are discussed in [15] and they [16] studied the interplay of the semilattice theoretic properties of a poset with the ring theoretic properties.
In this article, the concept of $I_s$ of $R$ is defined in $\mathfrak{L}$, $S$-meet semilattice and some results are discussed. Also defined a new graph called $S$-prime graph and their topological measures are generalized. Refer [17 - 19] for background research related to the indices.
Throughout, this paper first (FZ) and second Zagreb (SZ) indices of $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ are denoted as $M_1(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ and $M_2(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$, first (FZ) and second Zagreb (SZ) coindices of $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ are denoted as $\overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ and $\overline{M_2}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$, Randi´c index (RI) of $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ denoted as $R(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$.
This article is organized as follows. Section 2, recall some basic notions and definitions of lattice theory and topological indices of a graph. In section 3, the definitions of $S$-prime ideal of a lattice are given with the suitable examples. In section 4, the $S$-prime ideal and multiplicative subset of a lattice are introduced. In section 5, the $S$-prime ideal and multiplicative subset of $S$-meet semilattice are introduced. Also, a new graph called $S$-prime graph of $S$-prime ideal of $S$-meet semilattice is introduced with suitable examples. Some topological measures of the $S$-prime graph are discussed in sections 5, 6 and 7.

## 2 Preliminaries

In this section some primary definitions are recalled from [20], some topological indices of the graph definitions are given here.

**Definition 2.1.** *A relation $\mathcal{R}$ on a set $A$ is said to be partial order relation if the relation $R$ is reflexive, antisymmetric and transitive which may be described as follows: 1) Reflexivity: $a \sim a$ for all $a \in A$. 2) Antisymmetry: If $a \sim b$ and $b \sim a$ then $a = b$. 3) Transitivity: If $a \sim b$; $b \sim c$ then $a \sim c$. A set together with the partial order relation $R$ is called poset.*

**Definition 2.2.** *A lattice is a poset $\mathfrak{L}$ in which every $a, b$ has $\wedge$ and $\vee$.*

**Definition 2.3.** *Let $(\mathfrak{L}, \wedge, \vee)$ be a lattice and $M \subseteq \mathfrak{L}$. Then $(M, \wedge, \vee)$ is a sublattice of $(\mathfrak{L}, \wedge, \vee)$ iff $M$ is closed under $\wedge$ and $\vee$.*

**Definition 2.4.** *The $(M, \wedge, \vee) = I$ of $\mathfrak{L}$ is an ideal iff $i \in I$ and for any $a$ is in $\mathfrak{L}$ imply that $a \wedge i$ is in $I$.*

**Definition 2.5.** *The $(M, \wedge, \vee) = I$ of $\mathfrak{L}$ is prime iff $a, b$ is in $\mathfrak{L}$ and $a \wedge b$ is in $I$ imply that $a \in I$ or $b \in I$.*

**Definition 2.6.** *The topological measures are defined as,*
*The FZI of $\mathfrak{G}$ is,*

$$M_1(\mathfrak{G}) = \sum_{\mathfrak{x} \in \mathcal{V}(\mathfrak{G})} \mathfrak{d}(\mathfrak{x})^2.$$

*The SZI of $\mathfrak{G}$ is,*

$$M_2(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y} \in \mathcal{E}(G)} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}).$$

*The FZc of $\mathfrak{G}$ is,*

$$\overline{M_1}(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G})}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})].$$

*The SZc of $\mathfrak{G}$ is,*

$$\overline{M_2}(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G})} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}).$$

*The RI of $\mathfrak{G}$ is,*

$$R(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G})} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}.$$

# 3 $S$-prime ideal of a lattice

In this section, the $S$-prime ideal of $\mathfrak{L}$ is defined with an example.

**Definition 3.1.** *Let $S \subseteq \mathfrak{L}$. Then the set $S$ is called multiplicative subset of $L\mathfrak{L}$ if it contains 1) $1 \in S$ 2) $a \wedge b \in S\ \forall\ a, b \in S$*

**Definition 3.2.** *Let $I$ be a proper ideal of a lattice $\mathfrak{L}$. The ideal $I$ is said to be an $S$-prime ideal of $\mathfrak{L}$ if for any $x, y \in \mathfrak{L}$, $x \wedge y$ in $I$ then $s \wedge x$ or $s \wedge y$ is in $I$ for some $s \in S$, where $S$ is the multiplicative subset of a lattice $\mathfrak{L}$ which is disjoint from $I$ of $\mathfrak{L}$. The $S$-prime ideal of $\mathfrak{L}$ is denoted by $I_s$*

**Example 3.3.** *Consider $\mathfrak{L} = \{0, u, v, w, x, y, z, 1\}$ be a lattice whose Hasse diagram is given in the Figure 1. The $I_S$ of $\mathfrak{L}+ = \{0, u, v, w, x, y, z, 1\}$ are, from Figure 1, $I_1 = \{0\}, I_2 = \{0, u\}, I_3 = \{0, u, v\}, I_4 = \{0, u, w\}, I_5 = \{0, u, x\}, I_6 = \{0, u, y\}$ and $I_7 = \{0, u, v, w, x, y, z\}$. The multiplicative subset of a lattice are $S_1 = \{1\}, S_2 = \{1, u\}, S_3 = \{1, v\}, S_4 = \{1, w\}, S_5 = \{1, x\}, S_6 = \{1, y\}, S_7 = \{1, z\}, S_8 = \{1, v, z\}, S_9 = \{1, v, z\}$ and $S_{10} = \{1, v, z\}$.*

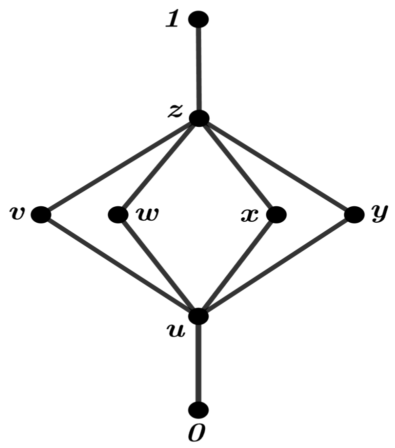


Figure 1: **Hasse diagram of** $L$

**Theorem 3.4.** *Every prime ideal $P$ of $\mathfrak{L}$ is an $S$-prime ideal of $\mathfrak{L}$.*

*Proof.* Let $P$ be the prime ideal of $\mathfrak{L}$ and $x, y \in \mathfrak{L} \ni x \wedge y \in P$ which implies that either $x$ or $y \in P$. Let $S$ be the multiplicative subset of $\mathfrak{L}$ which is disjoint from $P$ of $\mathfrak{L}$. That is, $P \cap S = \emptyset$. Since the set $S$ contains the multiplicative identity $1$, $1 \wedge x \in P$ or $1 \wedge y \in P$. Therefore, $s \wedge x \in P$ or $s \wedge y \in P$ for some $s \in S$. Thus, $P$ is $I_s$ of $\mathfrak{L}$. □

The converse of the Theorem 3.4 is not true. The following example provides confirmation of the converse.

**Example 3.5.** *Let us consider the example which is shown in Figure 1. Let $I = \{0, u\}$ be the $I_s$ of $\mathfrak{L} = \{0, u, v, w, x, y, z, 1\}$ and the multiplicative subset of $\mathfrak{L}$ as $S = \{b, f, 1\}$. Now, let $v, w \in \mathfrak{L}$ if $v \wedge w = u \in I_s$ which implies that $v \notin I_s$ and $w \notin I$. Thus $I_s$ is need not to be the prime ideal of $\mathfrak{L}$.*

**Theorem 3.6.** *Let $I = P$ of $\mathfrak{L}$. Then $\mathfrak{L} - I$ is the multiplicative subset of $\mathfrak{L}$.*

*Proof.* Let $\mathfrak{L}$ be the lattice, every pair of elements has $\vee$ and $\wedge$. Assume that $x \wedge y \in \mathfrak{L}$ and $I = P$ of $\mathfrak{L}$. Need to prove that, the set $\mathfrak{L} - I$ is a multiplicative subset of $\mathfrak{L}$. That is, $1$ in $\mathfrak{L} - I$ and $x \wedge y$ in $\mathfrak{L} - I \; \forall \; x, \in \mathfrak{L} - I$.

If suppose $1 \notin \mathfrak{L} - I$ then $1$ is in $I$ and the ideal $I$ becomes the improper ideal of a lattice $\mathfrak{L}$. i.e., $I = \mathfrak{L}$. It contradicts to $I$ is $P$ of $\mathfrak{L}$.

Let $x \wedge y$ in $\mathfrak{L} - I$ where $x, y \in \mathfrak{L}$ and $x \wedge y \in \mathfrak{L} \; \forall \; x \wedge y \in \mathfrak{L}$. Also, $x \wedge y \notin \mathfrak{L} - I$ which implies that $x \notin I$ and $y \notin I$. Therefore, $x, y \in \mathfrak{L} - I$. Thus, the set $\mathfrak{L} - I$ is the multiplicative subset of $\mathfrak{L}$. □

**Corollary 3.7.** *Let $I, J$ be the ideals of $\mathfrak{L}$ and $P$ be the prime ideal of $\mathfrak{L}$. If $I \wedge J \subseteq P$ then $s \wedge I$ or $s \wedge Y \subseteq P$ for some $s$ in $S$, where $S$ is the multiplicative subset of $\mathfrak{L}$ disjoint from $P$.*

# 4 $S$-prime Graph of a $S$-meet Semilattice

In this section, multiplicative subset $S$ and the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of a $S$-meet semilattice are defined. Also, a new graph called $S$-prime graph of the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of $L_s$ is defined where the vertices are the elements of the $S$-meet semilattice $(L_s, \wedge, \subseteq)$.

**Definition 4.1.** *Let $S \subseteq L_s$ . Then the set $S$ is called multiplicative subset of a $S$-meet semilattice $L_s$ if $u \wedge v \in S \; \forall \; u, v \in S$.*

**Definition 4.2.** *Let $I \subseteq L_s$. The ideal $I$ is said to be an $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of $L_s$ if for any $\mathfrak{u}, \mathfrak{v} \in L_s, \mathfrak{u} \wedge \mathfrak{v} \in \mathfrak{I}_\mathfrak{s}$ then $\exists \;\; s \in S$ such that $s \wedge \mathfrak{u}$ or $s \wedge \mathfrak{v}$ in $\mathfrak{I}_\mathfrak{s}$ for some $s \in S$, where $S$ is the multiplicative subset of $L_s$ and $S \cap \mathfrak{I}_\mathfrak{s} = \emptyset$.*

**Definition 4.3.** *Let $(L_s, \wedge, \subseteq)$ be the $S$-meet semilattice where $L_s$ is the collection of all $S$-prime ideals of $\mathfrak{R}$. The set of all elements of $L_s$ are considered to be the vertices of the graph, the vertices $\mathfrak{x}$ and $\mathfrak{y}$ are adjacent if $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$, where $\mathfrak{I}_\mathfrak{s}$ is the $S$-prime ideal of $L_s$. It is an undirected graph called $S$-prime graph of the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$, denoted by $\mathfrak{G}_{L_s}(\mathfrak{I}_\mathfrak{s})$, simply $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$.*

Let $\mathfrak{R}$ be a ring of order $p^t q$. The $S$-prime graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ of $\mathfrak{I}_\mathfrak{s}$ is $(i)$ a complete graph if the $S$-prime ideals $\mathfrak{I}_\mathfrak{s}$ of $L_s$ are $\downarrow p, \downarrow q$ and $\downarrow pq$, $(ii)$ a star graph if the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of $L_s$ is $\downarrow p^t q$ and $(iii)$ a connected graph if the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of $L_s$ is $\downarrow p^k q, k < t$.

**Example 4.4.** *Let $\mathfrak{R}$ be a ring of order $48$ and the $S$-prime graphs $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ are shown in Figure 3. The elements of $L_s$ are $< 2 >, < 3 >, < 6 >, < 12 >, < 24 >$ and $< 48 >$.*

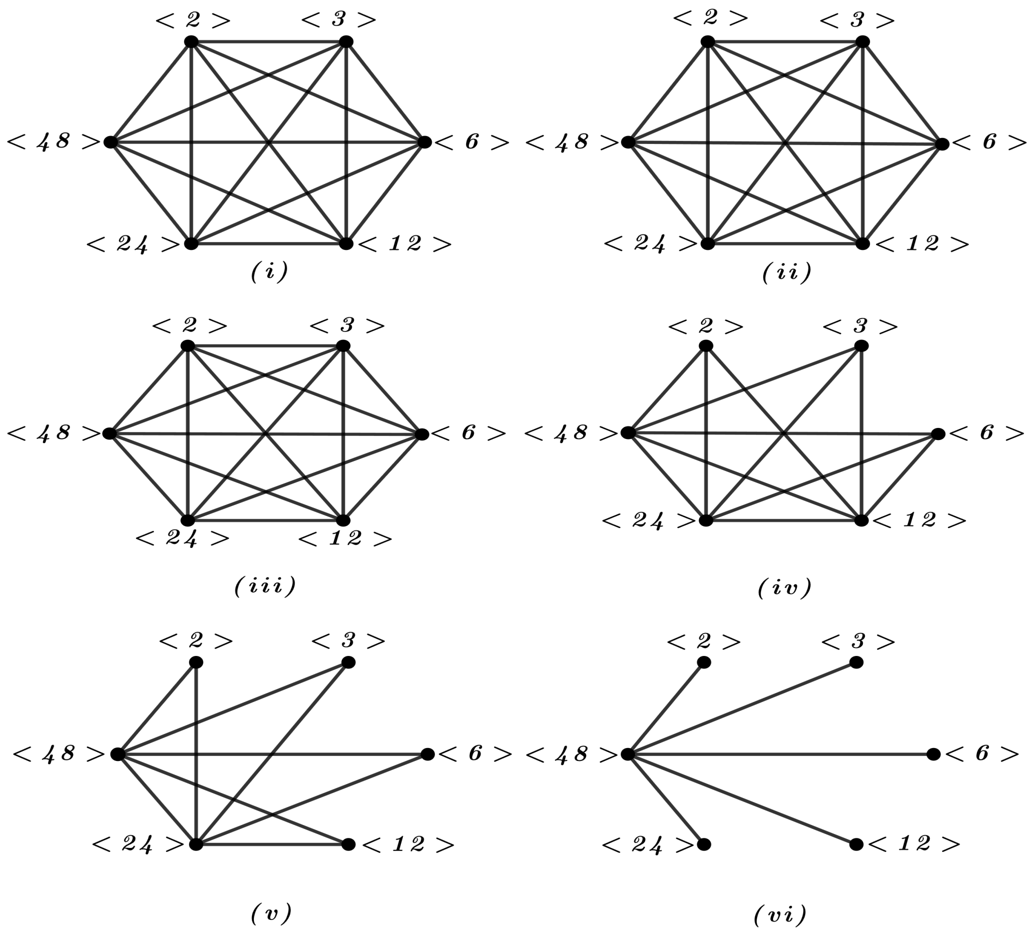


Figure 2: **$S$-prime graph of the $S$-prime ideal** $(i) \downarrow 2 \quad (ii) \downarrow 3 \quad (iii) \downarrow 6 \quad (iv) \downarrow 12 \quad (v) \downarrow 24 \quad (vi) \downarrow 48$

Let $R$ be a ring of order $pqr$. Then the $S$-prime graph $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ is a complete graph if the $S$-prime ideals of $L_s$ are $\downarrow p \cup \downarrow q, \downarrow p \cup \downarrow r, \downarrow q \cup \downarrow r, \downarrow pq \cup \downarrow pr \cup \downarrow qr, \downarrow p \cup \downarrow qr, \downarrow q \cup \downarrow pr$ and $\downarrow r \cup \downarrow pq$.

There are $3$ distinct connected $S$-prime graphs $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}), \mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ where $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})$ is the $S$-prime graph for the $S$-prime ideals $\downarrow p,\ \downarrow q,\ \downarrow r,\ \downarrow pq \cup \downarrow pr,\ \downarrow pq \cup \downarrow qr,\ \downarrow pr \cup \downarrow qr$, $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ is the $S$-prime graph for the ideals $\downarrow pq,\ \downarrow pr,\ \downarrow qr$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ for the $S$-prime ideal $\downarrow pqr$.

**Example 4.5.** *Let $\mathfrak{R}$ be a ring of order $30$ then the $S$-prime graph of $S$-meet semilattice is shown in Figure 4 whose vertex set is*

$\mathcal{V}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \{<2>, <3>, <5>, <6>, <10>, <15>, <30>\}$.

In the following sections, the topological measures $M_1(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$, $M_2(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$, $\overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$, $\overline{M_2}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$,$\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ and $R(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ of the connected $S$-prime graph of the $S$-prime ideals $\mathfrak{I}_{\mathfrak{s}}$ are studied.

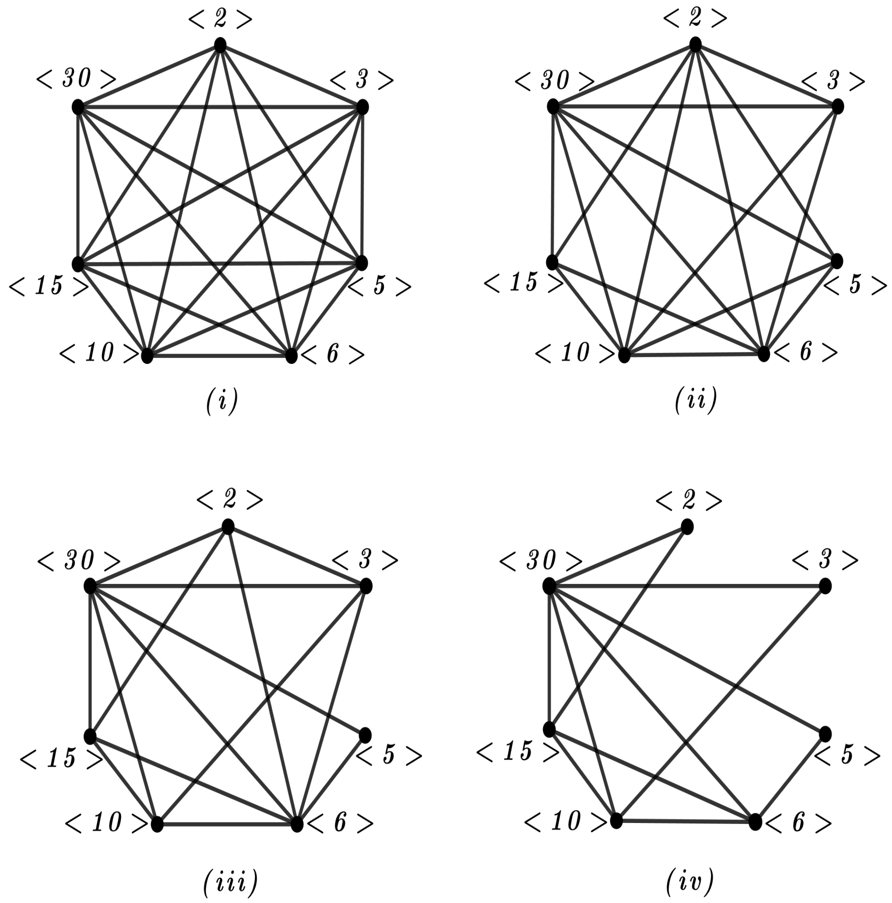


Figure 3: $S$**-prime graph of the** $S$**-prime ideal** $(i) \downarrow 2 \cup \downarrow 3 \;\; (ii) \downarrow 2 \;\; (iii) \downarrow 6 \;\; (iv) \downarrow 30$

## 5 First and Second Zagreb Indices of $S$-prime Graph

Let $\mathfrak{R}$ be a ring of order $p^t q$. The $S$-prime graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ is connected if the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ is the down-set of $p^k q$ where $k < t$. Let $\varpi$ and $\vartheta$ be the order of the graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ and the ideal $\mathfrak{I}_\mathfrak{s}$ respectively.

**Theorem 5.1.** *Let $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ be the $S$-prime graph of the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of $L_s$ then*

$$M_1(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta\Big[(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta\Big].$$

*Proof.* Let $\mathfrak{x}$ be a vertex of $\mathfrak{G}(\mathfrak{I})$.

If $\mathfrak{x}$ is an element of the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ then $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}\ \forall\ \mathfrak{y} \in L_s$.

If $\mathfrak{x}$ is not an elements of $\mathfrak{I}_\mathfrak{s}$ then $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$ only if $\mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$.

Therefore, $\mathfrak{d}(\mathfrak{x})$ is given as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi - 1 \ \ if \ \ \mathfrak{x} \in \mathfrak{I}_\mathfrak{s} \\ \vartheta \ \ if \ \ \mathfrak{x} \notin \mathfrak{I}_\mathfrak{s}. \end{cases}$$

Then, $M_1(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x} \in \mathcal{V}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x} \in \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x} \notin \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta^2$

$= \vartheta\Big[(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta\Big].$ □

**Theorem 5.2.** *Let $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ be the $S$-prime graph of the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of $L_s$ then*

$$M_2(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta(\varpi - 1)\Big[\frac{(\vartheta - 1)}{2}(\varpi - 1) + \vartheta(\varpi - \vartheta)\Big].$$

*Proof.* Let $\mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ be the edge set of $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ of the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of $L_s$. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}[\mathfrak{G}(\mathfrak{I}_\mathfrak{s})]$. This implies that either $\mathfrak{x}$ or $\mathfrak{y}$ is in $\mathfrak{I}_\mathfrak{s}$ and the $\mathfrak{d}(\mathfrak{x})$ and $\mathfrak{d}(\mathfrak{y})$ are defined in Theorem 5.1. Then,

$M_2(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \sum_{\mathfrak{x} \in \mathfrak{I}_\mathfrak{s}, \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x} \in \mathfrak{I}_\mathfrak{s}, \mathfrak{y} \notin \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \vartheta\frac{(\vartheta - 1)}{2}(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)(\varpi - 1)\vartheta$

$= \vartheta(\varpi - 1)\Big[\frac{(\vartheta - 1)}{2}(\varpi - 1) + \vartheta(\varpi - \vartheta)\Big].$ □

Let $\mathfrak{R}$ be a ring of order $pqr$. There are $3$ distinct $S$-prime connected graphs of $\mathfrak{I}_\mathfrak{s}$ namely $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}), \mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ defined earlier.

**Theorem 5.3.** *Let $\mathfrak{R}$ be a ring of order $pqr$. Then,*

*(i)* $M_1(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = \vartheta\Big[(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)\Big].$

*(ii)* $M_1(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = \vartheta\Big[(\varpi - 1)^2 + \vartheta)\Big] + (\vartheta + 2)^2.$

*(iii)* $M_1(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = \vartheta(\varpi - 1)^2 + (\varpi - 4)\Big[(\vartheta + 1)^2 + (\vartheta + 3)^2\Big].$

*Proof.* (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ be the $S$-prime graph of $\mathfrak{I}_\mathfrak{s}$ of $L_s$. Then,

$M_1(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x} \in \mathcal{V}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x} \in \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x} \notin \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta^2$

$= \vartheta\Big[(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)\Big].$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ be the $S$-prime graph of $\mathfrak{I}_\mathfrak{s}$ of $L_s$.

In this case, the non-ideal elements are adjacent to all the ideal elements and some non-ideal elements. Here, the $S$-prime ideals are $\downarrow pq,\ \downarrow pr$ and $\downarrow qr$.

Consider the $S$-prime ideal $\mathfrak{I}_\mathfrak{s} = \downarrow pq$ and $d(x)$ in $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ is as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi - 1 \ \ if \ \ x \in \mathfrak{I}_\mathfrak{s} \\ \vartheta \ \ if \ \ x = r \\ \vartheta + 2 \ \ if \ \ x \neq r. \end{cases}$$

Then,

$M_1(\mathfrak{G}_s^{(2)}(I)) = \sum_{\mathfrak{x} \in \mathcal{V}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x} \in \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x} = r} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x} \neq r} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi - 1)^2 + \vartheta^2 + (\vartheta + 2)^2$

$= \vartheta\Big[(\varpi - 1)^2 + \vartheta)\Big] + (\vartheta + 2)^2.$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ be the $S$-prime graph of $\mathfrak{I}_\mathfrak{s}$ of $L_s$.

In this case, the $S$-prime ideal is $\downarrow pqr$. The maximal elements of $L_s$ are $p, q, r$ and they are denoted as $\mathfrak{M}_k, k = 1, 2, 3$ and $\mathfrak{d}(\mathfrak{x})$ in $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ is as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi - 1 \ \ if \ \ \mathfrak{x} \in \mathfrak{I}_\mathfrak{s} \\ \vartheta + 1 \ \ if \ \ \mathfrak{x} = \mathfrak{M}_k \\ \vartheta + 3 \ \ if \ \ \mathfrak{x} \neq \mathfrak{M}_k. \end{cases}$$

Then,

$M_1(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x} \in \mathcal{V}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x} \in \mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x} = \mathfrak{M}_k} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x} \neq \mathfrak{M}_k} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi - 1)^2 + (\varpi - 4)(\vartheta + 1)^2 + (\varpi - 4)(\vartheta + 3)^2$

$= \vartheta(\varpi - 1)^2 + (\varpi - 4)\Big[(\vartheta + 1)^2 + (\vartheta + 3)^2\Big].$ □

**Theorem 5.4.** *Let $\mathfrak{R}$ be a ring of order $pqr$. Then,*

*(i)* $M_2(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 1)\Big[(\varpi - 1)^2 + \vartheta^2(\varpi - \vartheta)\Big].$

*(ii)* $M_2(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 1)\Big[(\varpi - 1) + \vartheta^2\Big] + (\vartheta + 2)\Big[8(\vartheta - 1) + (\vartheta + 2)^2\Big].$

*(iii)* $M_2(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = 2(\varpi - 4)\Big[(\varpi - 1)(\eta - 4) + (\vartheta + 3)(\varpi + 2)\Big].$

*Proof.* (i) Let $\mathcal{E}[\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})]$ be the edge set of $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ of the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of $L_s$. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}[\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})]$.

This implies that either $\mathfrak{x}$ or $\mathfrak{y}$ is in $\mathfrak{I}_\mathfrak{s}$ and the degree of the vertices $\mathfrak{x}$ and $\mathfrak{y}$ are defined in (i) of

Theorem 5. 3. Then,

$M_2(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= (\varpi-1)(\varpi-1)^2 + \vartheta(\varpi-\vartheta)(\varpi-1)\vartheta$

$= (\varpi-1)\Big[(\varpi-1)^2 + \vartheta^2(\varpi-\vartheta)\Big].$

(ii) Let $E[\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})]$ be the edge set of $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ of the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of $L_s$.

Consider the $S$-prime ideal $\mathfrak{I}_\mathfrak{s} =\downarrow pq$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ of $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ are defined in (ii) of Theorem 5.3. Then,

$M_2(\mathfrak{G}_s(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}[\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})]} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= (\varpi-1)^2 + \Big[(\varpi-1)\vartheta^2 + 8(\varpi-1)(\vartheta+2)\Big] + (\vartheta+2)^3$

$= (\varpi-1)\Big[(\varpi-1) + \vartheta^2\Big] + (\vartheta+2)\Big[8(\varpi-1) + (\vartheta+2)^2\Big].$

(iii) Let $\mathcal{E}[\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})]$ be the edge set of $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ of the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of $L_s$.

Consider the $S$-prime ideal is $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ of $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ are defined in (iii) of Theorem 5.3. Then,

$M_2(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}[\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})]} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})} + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \Big[(\varpi-1)(\varpi-5)(\varpi-4) + (\varpi-1)(\varpi-3)(\varpi-4)\Big] + \Big[(\vartheta+1)(\vartheta+3)(\varpi-4) + (\vartheta+3)^2(\varpi-4)\Big]$

$= (\varpi-1)(\varpi-4)(2\varpi-8) + (\varpi-4)(\vartheta+3)(2\varpi+4)$

$= 2(\varpi-1)(\varpi-4)^2 + 2(\varpi-4)(\vartheta+3)(\varpi+2)$

$= 2(\varpi-4)\Big[(\varpi-1)(\varpi-4) + (\vartheta+3)(\varpi+2)\Big].$ □

## 6 First and Second Zagreb Coindex of $S$-prime Graph

The $M_1(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ and $M_2(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ of the $S$-prime graph are generalized in this section.

**Theorem 6.1.** *Let $\mathfrak{R}$ be a ring of order $p^t q$. Then,*

$$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta(\varpi-\vartheta)(\varpi-\vartheta-1).$$

*Proof.* Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))$ be the edge set of $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ of the $S$-prime ideal $\mathfrak{I}_\mathfrak{s}$ of $L_s$. If $\mathfrak{x}\mathfrak{y}$ is an edge of

$\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$, then at least one of the end points of $\mathfrak{x}\mathfrak{y}$ must be in the ideal $\mathfrak{I}_{\mathfrak{s}}$. Then,

$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))} [\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

In this, there is no edge between the non-ideal elements $\mathfrak{x}, \mathfrak{y}$. Thus,

$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} (\varpi - \vartheta)(\varpi - \vartheta - 1)\vartheta$

$\therefore\ \overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta(\varpi - \vartheta)(\varpi - \vartheta - 1).$ □

**Theorem 6.2.** *Let $\mathfrak{R}$ be a ring of order $p^t q$. Then,*

$$\overline{M_2}(\mathfrak{G}_s(I)) = \overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})).\frac{\vartheta}{2}.$$

*Proof.* Let $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ be the $S$-prime graph. Then,

$\overline{M_2}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}).$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} (\varpi - \vartheta)\dfrac{(\varpi - \vartheta - 1)}{2}\vartheta^2$

$= \dfrac{(\varpi - \vartheta)(\varpi - \vartheta - 1)}{2}.\vartheta^2$

$= \Big[\vartheta(\varpi - \vartheta)(\varpi - \vartheta - 1)\Big]\dfrac{\vartheta}{2}$

Hence, $\overline{M_2}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})).\dfrac{\vartheta}{2}.$ □

**Theorem 6.3.** *Let $\mathfrak{R}$ be a ring of order $pqr$. Then,*

*(i)* $\overline{M_1}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = 2(\varpi - 3)(\varpi - 4).$

*(ii)* $\overline{M_1}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})) = (\varpi - 3)\Big[3(\vartheta - 1) + \varpi\Big].$

*(iii)* $\overline{M_1}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = 2\Big[(\vartheta + 1)(\varpi - 4) + (\varpi - 1)(\vartheta + 2)\Big].$

*Proof.* (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})$ be the $S$-prime graph. Then,

$\overline{M_1}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}))}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= \Big[(\varpi - 3) + (\varpi - 3)\Big](\varpi - 4)$

$= 2(\varpi - 3)(\varpi - 4).$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ be the $S$-prime graph. Consider the $S$-prime ideal $\mathfrak{I}_{\mathfrak{s}} = \downarrow pq$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are is defined in (ii) of Theorem 5.3. Then,

$\overline{M_1}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x}=r,\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x},\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= (\varpi - 3)\Big[\vartheta + (\varpi - 3)\Big] + \vartheta\Big[(\varpi - 3) + (\varpi - 3)\Big]$

$= (\varpi - 3)\Big[3(\vartheta - 1) + \varpi\Big].$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ be the $S$-prime graph. Consider the $S$-prime ideal is $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (iii) of Theorem 5.3. Then,

$\overline{M_1}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{x})]$

$= \sum_{\mathfrak{x},\mathfrak{y}=\mathfrak{M}_\mathfrak{k}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x}=\mathfrak{M}_\mathfrak{k},\mathfrak{y}\neq\mathfrak{M}_\mathfrak{k}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= 2(\vartheta + 1)(\varpi - 4) + (\varpi - 1)\Big[\vartheta + 1 + (\vartheta + 3)\Big]$

$= 2(\vartheta + 1)(\varpi - 4) + (\varpi - 1)(2\vartheta + 4)$

$= 2(\vartheta + 1)(\varpi - 4) + 2(\varpi - 1)(\vartheta + 2)$

$= 2\Big[(\vartheta + 1)(\varpi - 4) + (\varpi - 1)(\vartheta + 2)\Big].$ □

**Theorem 6.4.** *Let $\mathfrak{R}$ be a ring of order $pqr$. Then,*

*(i)* $\overline{M_2}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 4)\vartheta^2.$

*(ii)* $\overline{M_2}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = 2\vartheta(\vartheta + 2)^2.$

*(iii)* $\overline{M_2}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = (\vartheta + 1)\Big[(\vartheta + 1)(\varpi - 4) + (\vartheta + 3)(\varpi - 1)\Big].$

*Proof.* (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ be the $S$-prime graph. Then,

$\overline{M_2}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= (\varpi - 4)\vartheta\vartheta$

$= (\varpi - 4)\vartheta^2.$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ be the $S$-prime graph. Consider the $S$-prime ideal $\mathfrak{I}_\mathfrak{s} = \downarrow pq$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (ii) of Theorem 5.3. Then,

$\overline{M_2}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y} \notin \mathfrak{I}_{\mathfrak{s}}} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x}=r,\mathfrak{y}\neq r} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x},\mathfrak{y}\neq r} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \vartheta(\vartheta+2)(\vartheta+2) + (\vartheta+2)(\vartheta+2)\vartheta$

$= 2\vartheta(\vartheta+2)^2.$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ be the $S$-prime graph. Consider the $S$-prime ideal is $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (iii) of Theorem 5.3. Then,

$\overline{M_2}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y} \notin \mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}}))} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y} \notin \mathfrak{I}_{\mathfrak{s}}} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}=\mathfrak{M}_{\mathfrak{k}}} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x}=\mathfrak{M}_{\mathfrak{k}},\mathfrak{y}\neq\mathfrak{M}_{\mathfrak{k}}} [\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= (\vartheta+1)(\vartheta+1)(\varpi-4) + (\vartheta+1)(\vartheta+3)(\varpi-1)$

$= (\vartheta+1)\Big[(\vartheta+1)(\varpi-4) + (\vartheta+3)(\varpi-1)\Big].$ □

# 7 Randi´c index of $S$-prime Graph

In this section, $R\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ of $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ of the $S$-prime ideals $\mathfrak{I}_{\mathfrak{s}}$ are generalized.

**Theorem 7.1.** *Let $\mathfrak{R}$ be a ring of order $p^t q$. Then,*

$$R(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))) = \vartheta\Big[\frac{(\vartheta-1)}{2(\varpi-1)} + \frac{(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}\Big].$$

*Proof.* Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of the $S$-prime graph and their degrees are defined earlier in Theorem 5.1. Then,

$R(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$

$= \sum_{\mathfrak{x},\mathfrak{y} \in \mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x} \in \mathfrak{I}_{\mathfrak{s}},\mathfrak{y} \notin \mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$

$= \frac{\vartheta(\vartheta-1)}{2\sqrt{(\varpi-1)^2}} + \frac{\vartheta(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}$

Thus, $R(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta\Big[\frac{(\vartheta-1)}{2(\varpi-1)} + \frac{(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}\Big].$ □

**Theorem 7.2.** *Let $\mathfrak{R}$ be a ring of order $pqr$. Then,*

*(i)* $R(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = 1 + \frac{(\varpi-3)(\varpi-4)}{\sqrt{\vartheta(\varpi-1)}}$

*(ii)* $R\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = 1 + \frac{1}{(\varpi-1)} + \frac{\vartheta}{\sqrt{\vartheta(\varpi-1)}} + \frac{8}{\sqrt{(\varpi-1)(\vartheta+2)}}.$

*(iii)* $R(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = 3\Big[\frac{1}{\sqrt{(\varpi-1)(\vartheta+1)}} + \frac{1}{\sqrt{(\varpi-1)(\vartheta+3)}} + \frac{1}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \frac{1}{(\vartheta+3)}\Big].$

*Proof.* (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})$ be the $S$-prime graph. Then,

$$R(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}))} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \frac{(\varpi-1)}{\sqrt{(\varpi-1)(\varpi-1)}} + \frac{(\varpi-3)(\varpi-4)}{\sqrt{(\varpi-1)\vartheta}}$$

$$= 1 + \frac{(\varpi-3)(\varpi-4)}{\sqrt{\vartheta(\varpi-1)}}.$$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ be the $S$-prime graph. Consider the $S$-prime ideal $\mathfrak{I}_{\mathfrak{s}} =\downarrow pq$. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ and their degrees are defined earlier in (ii) of Theoreem 5.3. Then,

$$R(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}}))} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= 1 + \frac{1}{(\varpi-1)} + \frac{\vartheta}{\sqrt{\vartheta(\varpi-1)}} + \frac{8}{\sqrt{(\varpi-1)(\vartheta+2)}}.$$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ be the $S$-prime graph. Consider the $S$-prime ideal $\mathfrak{I}_{\mathfrak{s}} =\downarrow pqr$. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ and their degrees are defined earlier in (iii) of Theoreem 5.3. Then,

$$R(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}}))} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \Big[\frac{3}{\sqrt{(\varpi-1)(\vartheta+1)}} + \frac{3}{\sqrt{(\varpi-1)(\vartheta+3)}}\Big] + \Big[\frac{3}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \frac{3}{\sqrt{(\vartheta+3)^2}}\Big]$$

$$= 3\Big[\frac{1}{\sqrt{(\varpi-1)(\vartheta+1)}} + \frac{1}{\sqrt{(\varpi-1)(\vartheta+3)}} + \frac{1}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \frac{1}{(\vartheta+3)}\Big].$$

□

## 8 Conclusion

In this paper, a new ideal called $S$-prime ideal in a lattice and $S$-meet semilattice are defined and it is shown that the prime ideal of a lattice is also an $S$-prime ideal of a lattice $L$. Also, multiplicative subset and $S$-prime ideal of $S$-meet semilattice are defined. Finally, a new graph from the $S$-meet semilattice $(L_s, \wedge, \subseteq)$ is introduced with examples and their topological measures are generalized.

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