Original Research Article

S-Prime Graph of S-meet Semilattice

Abstract

Let $\mathfrak{L}$ be a lattice and I be the proper ideal of a lattice $\mathfrak{L}$ and the set S be the multiplicative subset of $\mathfrak{L}$. In this paper, S-prime ideal in $\mathfrak{L}$ is introduced. Also it is shown that the prime ideal of $\mathfrak{L}$ is an S-prime ideal of $\mathfrak{L}$ studied with suitable examples. Further, multiplicative subset S and the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of an S-meet semilattice are introduced. Finally, a new graph called S-prime graph of S-prime ideal of S-meet semilattice is defined and their topological measures are generalized.

Keywords: Prime ideal; S-prime ideal; Partially ordered set; Lattice; Semilattice; S-meet semilattice; Ideal of a lattice

2010 Mathematics Subject Classification: 13A15; 03G10; 68R10

1 Introduction

In 1961, Gratzer and Schmidt [1] defined a standard ideal in $\mathfrak{L}$ and Noor and Latif [2] introduced and discussed about the standard n-ideal of $\mathfrak{L}$. In 1994, n-ideals in $\mathfrak{L}$ were introduced by Latif and Noor [3]. After that they studied finitely generated n-ideals of $\mathfrak{L}$ [4]. In 2000, the properties of standard n-ideal of $\mathfrak{L}$ were discussed by Noor and Latif [5].

In 2015, Meenakshi P and Karuna T [6], introduced the 2-absorbing and weakly 2-absorbing ideals of $\mathfrak{L}$ which was from [7 - 8]. A proper ideal I of $\mathfrak{L}$ is called a 2-absorbing ideal if $a \wedge b \wedge c$ is in I for a, b, c is in $\mathfrak{L}$ then either a meet b or a meet c or b meet c in I. Also, they define the triple zero in lattices and given some results related triple zero. In 2021, Ali Akbar and Toktam Haghdadi [9], introduced the n-absorbing ideals in L which is from [10]. Many authors introduced and studied different ideals in a lattice, they are: semiprime n-ideal of $\mathfrak{L}$ [11], modular n-ideals of $\mathfrak{L}$ [12] and so on.

In 2019, Ahmed Hamed and Achraf Malek [13], defined S-prime ideals of R. A proper ideal I of R is called an S-prime ideal I_s of R if x, y is in R and xy is in I_s then sx or sy is in I_s for some $s \in S$ where

---

S is the multiplicative subset of R. The multiplicative subset is the complement of the prime ideal of a ring R.

Recently, Kalamani and Mythily [14] introduced a graph, the vertices of the graph are from R and they are connected iff $sa \in I_s$ or $sb \in I_s$ for some $s \in S$ whenever $ab \in I$ where $a, b \in R$ and the set S is disjoint from I_s. Some of the properties of the I_s of R are discussed in [15] and they [16] studied the interplay of the semilattice theoretic properties of a poset with the ring theoretic properties.

In this article, the concept of I_s of R is defined in $\mathfrak{L}$, S-meet semilattice and some results are discussed. Also defined a new graph called S-prime graph and their topological measures are generalized. Refer [17 - 19] for background research related to the indices.

Throughout, this paper first (FZ) and second Zagreb (SZ) indices of $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ are denoted as $M_1(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ and $M_2(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$, first (FZ) and second Zagreb (SZ) coindices of $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ are denoted as $\overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ and $\overline{M_2}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$, Randi ´c index (RI) of $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ denoted as $R(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$.

This article is organized as follows. Section 2, recall some basic notions and definitions of lattice theory and topological indices of a graph. In section 3, the definitions of S-prime ideal of a lattice are given with the suitable examples. In section 4, the S-prime ideal and multiplicative subset of a lattice are introduced. In section 5, the S-prime ideal and multiplicative subset of S-meet semilattice are introduced. Also, a new graph called S-prime graph of S-prime ideal of S-meet semilattice is introduced with suitable examples. Some topological measures of the S-prime graph are discussed in sections 5, 6 and 7.

2 Preliminaries

In this section some primary definitions are recalled from [20], some topological indices of the graph definitions are given here.

Definition 2.1. *A relation $\mathcal{R}$ on a set A is said to be partial order relation if the relation R is reflexive, antisymmetric and transitive which may be described as follows: 1) Reflexivity: $a \sim a$ for all $a \in A$. 2) Antisymmetry: If $a \sim b$ and $b \sim a$ then $a = b$. 3) Transitivity: If $a \sim b$; $b \sim c$ then $a \sim c$. A set together with the partial order relation R is called poset.*

Definition 2.2. *A lattice is a poset $\mathfrak{L}$ in which every a, b has $\wedge$ and $\vee$.*

Definition 2.3. *Let $(\mathfrak{L}, \wedge, \vee)$ be a lattice and $M \subseteq \mathfrak{L}$. Then $(M, \wedge, \vee)$ is a sublattice of $(\mathfrak{L}, \wedge, \vee)$ iff M is closed under $\wedge$ and $\vee$.*

Definition 2.4. *The $(M, \wedge, \vee) = I$ of $\mathfrak{L}$ is an ideal iff $i \in I$ and for any a is in $\mathfrak{L}$ imply that $a \wedge i$ is in I.*

Definition 2.5. *The $(M, \wedge, \vee) = I$ of $\mathfrak{L}$ is prime iff a, b is in $\mathfrak{L}$ and $a \wedge b$ is in I imply that $a \in I$ or $b \in I$.*

Definition 2.6. *The topological measures are defined as,*
The FZI of $\mathfrak{G}$ is,

$$M_1(\mathfrak{G}) = \textstyle\sum_{\mathfrak{x} \in \mathcal{V}(\mathfrak{G})} \mathfrak{d}(\mathfrak{x})^2.$$

The SZI of $\mathfrak{G}$ is,

$$M_2(\mathfrak{G}) = \textstyle\sum_{\mathfrak{x}\mathfrak{y} \in \mathcal{E}(G)} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}).$$

The FZc of $\mathfrak{G}$ is,

$$\overline{M_1}(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G})}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})].$$

The SZc of $\mathfrak{G}$ is,

$$\overline{M_2}(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G})}\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}).$$

The RI of $\mathfrak{G}$ is,

$$R(\mathfrak{G}) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G})}\frac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}.$$

3 S-prime ideal of a lattice

In this section, the S-prime ideal of $\mathfrak{L}$ is defined with an example.

Definition 3.1. *Let $S \subseteq \mathfrak{L}$. Then the set S is called multiplicative subset of $L\mathfrak{L}$ if it contains 1) $1 \in S$ 2) $a \wedge b \in S \; \forall \; a, b \in S$*

Definition 3.2. *Let I be a proper ideal of a lattice $\mathfrak{L}$. The ideal I is said to be an S-prime ideal of $\mathfrak{L}$ if for any $x, y \in \mathfrak{L}$, $x \wedge y$ in I then $s \wedge x$ or $s \wedge y$ is in I for some $s \in S$, where S is the multiplicative subset of a lattice $\mathfrak{L}$ which is disjoint from I of $\mathfrak{L}$. The S-prime ideal of $\mathfrak{L}$ is denoted by I_s*

Example 3.3. *Consider $\mathfrak{L} = \{0, u, v, w, x, y, z, 1\}$ be a lattice whose Hasse diagram is given in the Figure 1. The I_S of $\mathfrak{L}+ = \{0, u, v, w, x, y, z, 1\}$ are, from Figure 1, $I_1 = \{0\}, I_2 = \{0, u\}, I_3 = \{0, u, v\}, I_4 = \{0, u, w\}, I_5 = \{0, u, x\}, I_6 = \{0, u, y\}$ and $I_7 = \{0, u, v, w, x, y, z\}$. The multiplicative subset of a lattice are $S_1 = \{1\}, S_2 = \{1, u\}, S_3 = \{1, v\}, S_4 = \{1, w\}, S_5 = \{1, x\}, S_6 = \{1, y\}, S_7 = \{1, z\}, S_8 = \{1, v, z\}, S_9 = \{1, v, z\}$ and $S_{10} = \{1, v, z\}$.*

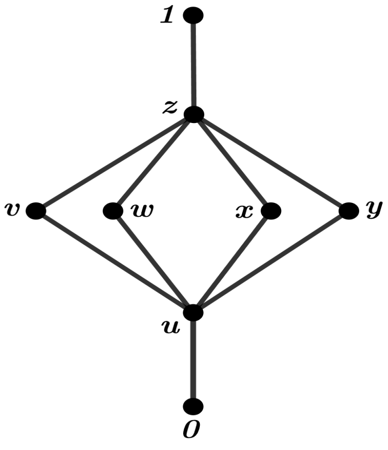

Figure 1: **Hasse diagram of L**

Theorem 3.4. *Every prime ideal P of $\mathfrak{L}$ is an S-prime ideal of $\mathfrak{L}$.*

Proof. Let P be the prime ideal of $\mathfrak{L}$ and $x, y \in \mathfrak{L} \ni x \wedge y \in P$ which implies that either x or $y \in P$. Let S be the multiplicative subset of $\mathfrak{L}$ which is disjoint from P of $\mathfrak{L}$. That is, $P \cap S = \emptyset$. Since the set S contains the multiplicative identity 1, $1 \wedge x \in P$ or $1 \wedge y \in P$. Therefore, $s \wedge x \in P$ or $s \wedge y \in P$ for some $s \in S$. Thus, P is I_s of $\mathfrak{L}$. $\square$

The converse of the Theorem 3.4 is not true. The following example provides confirmation of the converse.

Example 3.5. *Let us consider the example which is shown in Figure 1. Let* $I = \{0, u\}$ *be the* I_s *of* $\mathfrak{L} = \{0, u, v, w, x, y, z, 1\}$ *and the multiplicative subset of* $\mathfrak{L}$ *as* $S = \{b, f, 1\}$. *Now, let* $v, w \in \mathfrak{L}$ *if* $v \wedge w = u \in I_s$ *which implies that* $v \notin I_s$ *and* $w \notin I$. *Thus* I_s *is need not to be the prime ideal of* $\mathfrak{L}$.

Theorem 3.6. *Let* $I = P$ *of* $\mathfrak{L}$. *Then* $\mathfrak{L} - I$ *is the multiplicative subset of* $\mathfrak{L}$.

Proof. Let $\mathfrak{L}$ be the lattice, every pair of elements has $\vee$ and $\wedge$. Assume that $x \wedge y \in \mathfrak{L}$ and $I = P$ of $\mathfrak{L}$. Need to prove that, the set $\mathfrak{L} - I$ is a multiplicative subset of $\mathfrak{L}$. That is, 1 in $\mathfrak{L} - I$ and $x \wedge y$ in $\mathfrak{L} - I \; \forall \; x, \in \mathfrak{L} - I$.

If suppose $1 \notin \mathfrak{L} - I$ then 1 is in I and the ideal I becomes the improper ideal of a lattice $\mathfrak{L}$. i.e., $I = \mathfrak{L}$. It contradicts to I is P of $\mathfrak{L}$.

Let $x \wedge y$ in $\mathfrak{L} - I$ where $x, y \in \mathfrak{L}$ and $x \wedge y \in \mathfrak{L} \; \forall \; x \wedge y \in \mathfrak{L}$. Also, $x \wedge y \notin \mathfrak{L} - I$ which implies that $x \notin I$ and $y \notin I$. Therefore, $x, y \in \mathfrak{L} - I$. Thus, the set $\mathfrak{L} - I$ is the multiplicative subset of $\mathfrak{L}$. $\square$

Corollary 3.7. *Let* I, J *be the ideals of* $\mathfrak{L}$ *and* P *be the prime ideal of* $\mathfrak{L}$. *If* $I \wedge J \subseteq P$ *then* $s \wedge I$ *or* $s \wedge Y \subseteq P$ *for some* s *in* S, *where* S *is the multiplicative subset of* $\mathfrak{L}$ *disjoint from* P.

4 S-prime Graph of a S-meet Semilattice

In this section, multiplicative subset S and the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of a S-meet semilattice are defined. Also, a new graph called S-prime graph of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s is defined where the vertices are the elements of the S-meet semilattice $(L_s, \wedge, \subseteq)$.

Definition 4.1. *Let* $S \subseteq L_s$. *Then the set* S *is called multiplicative subset of a* S*-meet semilattice* L_s *if* $u \wedge v \in S \; \forall \; u, v \in S$.

Definition 4.2. *Let* $I \subseteq L_s$. *The ideal* I *is said to be an* S*-prime ideal* $\mathfrak{I}_\mathfrak{s}$ *of* L_s *if for any* $\mathfrak{u}, \mathfrak{v} \in L_s, \mathfrak{u} \wedge \mathfrak{v} \in \mathfrak{I}_\mathfrak{s}$ *then* $\exists \;\; s \in S$ *such that* $s \wedge \mathfrak{u}$ *or* $s \wedge \mathfrak{v}$ *in* $\mathfrak{I}_\mathfrak{s}$ *for some* $s \in S$, *where* S *is the multiplicative subset of* L_s *and* $S \cap \mathfrak{I}_\mathfrak{s} = \emptyset$.

Definition 4.3. *Let* $(L_s, \wedge, \subseteq)$ *be the* S*-meet semilattice where* L_s *is the collection of all* S*-prime ideals of* $\mathfrak{R}$. *The set of all elements of* L_s *are considered to be the vertices of the graph, the vertices* $\mathfrak{x}$ *and* $\mathfrak{y}$ *are adjacent if* $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$, *where* $\mathfrak{I}_\mathfrak{s}$ *is the* S*-prime ideal of* L_s. *It is an undirected graph called* S*-prime graph of the* S*-prime ideal* $\mathfrak{I}_\mathfrak{s}$, *denoted by* $\mathfrak{G}_{L_s}(\mathfrak{I}_\mathfrak{s})$, *simply* $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$.

Let $\mathfrak{R}$ be a ring of order $p^t q$. The S-prime graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ of $\mathfrak{I}_\mathfrak{s}$ is (i) a complete graph if the S-prime ideals $\mathfrak{I}_\mathfrak{s}$ of L_s are $\downarrow p, \downarrow q$ and $\downarrow pq$, (ii) a star graph if the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s is $\downarrow p^t q$ and (iii) a connected graph if the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s is $\downarrow p^k q, k < t$.

Example 4.4. *Let* $\mathfrak{R}$ *be a ring of order* 48 *and the* S*-prime graphs* $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ *are shown in Figure 3. The elements of* L_s *are* $< 2 >, < 3 >, < 6 >, < 12 >, < 24 >$ *and* $< 48 >$.

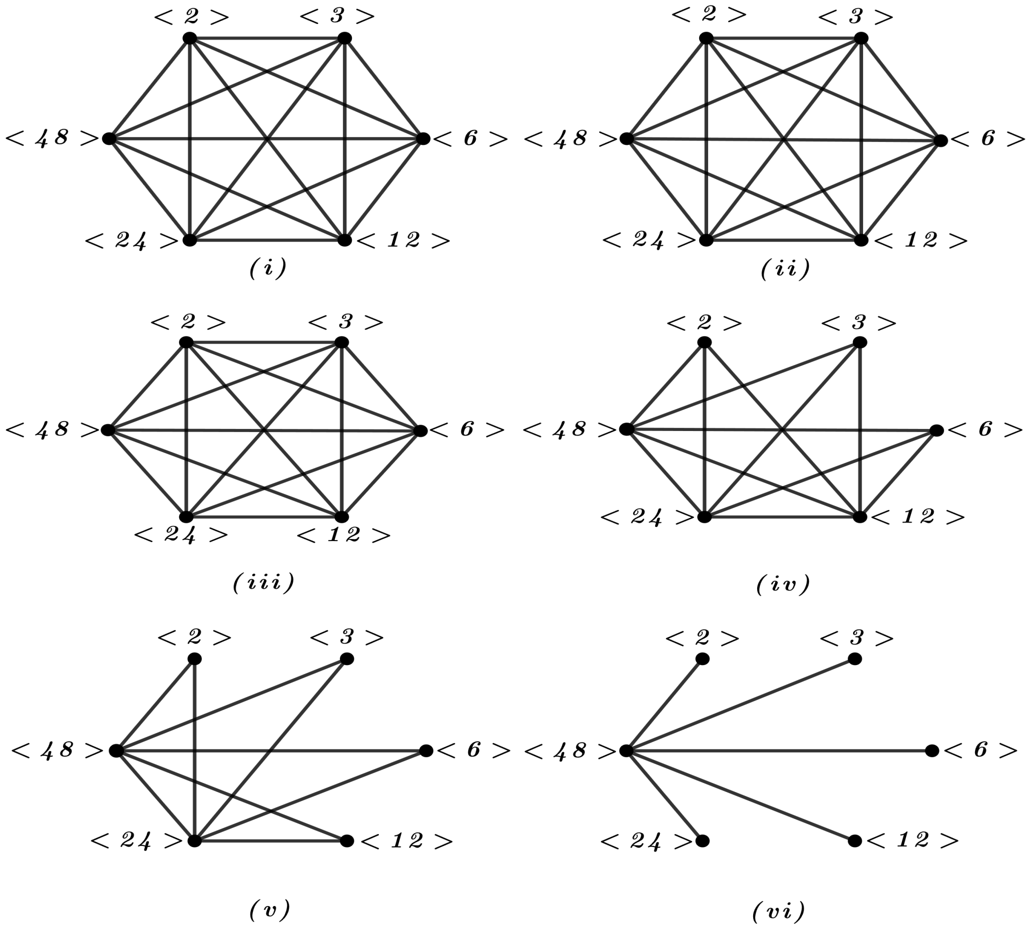

Figure 2: **S-prime graph of the S-prime ideal** $(i) \downarrow 2 \quad (ii) \downarrow 3 \quad (iii) \downarrow 6 \quad (iv) \downarrow 12 \quad (v) \downarrow 24 \quad (vi) \downarrow 48$

Let R be a ring of order pqr. Then the S-prime graph $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ is a complete graph if the S-prime ideals of L_s are $\downarrow p \cup \downarrow q, \downarrow p \cup \downarrow r, \downarrow q \cup \downarrow r, \downarrow pq \cup \downarrow pr \cup \downarrow qr, \downarrow p \cup \downarrow qr, \downarrow q \cup \downarrow pr$ and $\downarrow r \cup \downarrow pq$.

There are 3 distinct connected S-prime graphs $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}), \mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ where $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})$ is the S-prime graph for the S-prime ideals $\downarrow p,\ \downarrow q,\ \downarrow r,\ \downarrow pq \cup \downarrow pr,\ \downarrow pq \cup \downarrow qr,\ \downarrow pr \cup \downarrow qr$, $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ is the S-prime graph for the ideals $\downarrow pq,\ \downarrow pr,\ \downarrow qr$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ for the S-prime ideal $\downarrow pqr$.

Example 4.5. *Let $\mathfrak{R}$ be a ring of order 30 then the S-prime graph of S-meet semilattice is shown in Figure 4 whose vertex set is*

$\mathcal{V}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \{< 2 >, < 3 >, < 5 >, < 6 >, < 10 >, < 15 >, < 30 >\}.$

In the following sections, the topological measures $M_1(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$, $M_2(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$, $\overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$, $\overline{M_2}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$,$\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ and $R(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ of the connected S-prime graph of the S-prime ideals $\mathfrak{I}_{\mathfrak{s}}$ are studied.

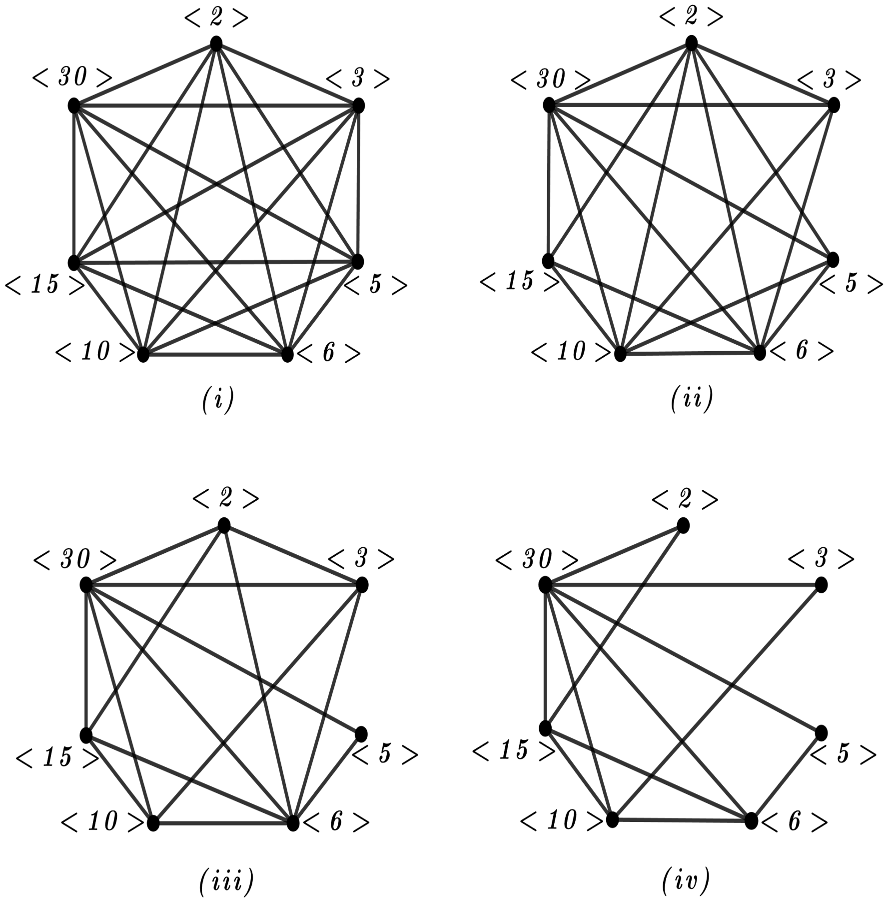

Figure 3: **S-prime graph of the S-prime ideal** $(i) \downarrow 2 \cup \downarrow 3 \;\; (ii) \downarrow 2 \;\; (iii) \downarrow 6 \;\; (iv) \downarrow 30$

5 First and Second Zagreb Indices of S-prime Graph

Let $\mathfrak{R}$ be a ring of order $p^t q$. The S-prime graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ is connected if the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ is the down-set of $p^k q$ where $k < t$. Let ϖ and ϑ be the order of the graph $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ and the ideal $\mathfrak{I}_\mathfrak{s}$ respectively.

Theorem 5.1. *Let $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s then*

$$M_1(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta\Big[(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta\Big].$$

Proof. Let $\mathfrak{x}$ be a vertex of $\mathfrak{G}(\mathfrak{I})$.

If $\mathfrak{x}$ is an element of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ then $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s} \; \forall \; \mathfrak{y} \in L_s$.

If $\mathfrak{x}$ is not an elements of $\mathfrak{I}_\mathfrak{s}$ then $\mathfrak{x} \wedge \mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$ only if $\mathfrak{y} \in \mathfrak{I}_\mathfrak{s}$.

Therefore, $\mathfrak{d}(\mathfrak{x})$ is given as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi - 1 \ \ if \ \ \mathfrak{x} \in \mathfrak{I}_{\mathfrak{s}} \\ \\ \vartheta \ \ if \ \ \mathfrak{x} \notin \mathfrak{I}_{\mathfrak{s}}. \end{cases}$$

Then, $M_1(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x} \in \mathcal{V}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x} \in \mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x} \notin \mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta^2$

$= \vartheta\Big[(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta\Big].$ ☐

Theorem 5.2. *Let $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph of the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of L_s then*

$$M_2(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta(\varpi - 1)\Big[\frac{(\vartheta - 1)}{2}(\varpi - 1) + \vartheta(\varpi - \vartheta)\Big].$$

Proof. Let $\mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ of the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of L_s. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}[\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})]$. This implies that either $\mathfrak{x}$ or $\mathfrak{y}$ is in $\mathfrak{I}_{\mathfrak{s}}$ and the $\mathfrak{d}(\mathfrak{x})$ and $\mathfrak{d}(\mathfrak{y})$ are defined in Theorem 5.1. Then,

$M_2(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \sum_{\mathfrak{x} \in \mathfrak{I}_{\mathfrak{s}}, \mathfrak{y} \in \mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x} \in \mathfrak{I}_{\mathfrak{s}}, \mathfrak{y} \notin \mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \vartheta\dfrac{(\vartheta - 1)}{2}(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)(\varpi - 1)\vartheta$

$= \vartheta(\varpi - 1)\Big[\dfrac{(\vartheta - 1)}{2}(\varpi - 1) + \vartheta(\varpi - \vartheta)\Big].$ ☐

Let $\mathfrak{R}$ be a ring of order pqr. There are 3 distinct S-prime connected graphs of $\mathfrak{I}_{\mathfrak{s}}$ namely $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})$, $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ and $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ defined earlier.

Theorem 5.3. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $M_1(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta\Big[(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)\Big].$

(ii) $M_1(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta\Big[(\varpi - 1)^2 + \vartheta)\Big] + (\vartheta + 2)^2.$

(iii) $M_1(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta(\varpi - 1)^2 + (\varpi - 4)\Big[(\vartheta + 1)^2 + (\vartheta + 3)^2\Big].$

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph of $\mathfrak{I}_{\mathfrak{s}}$ of L_s. Then,

$M_1(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x} \in \mathcal{V}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x} \in \mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x} \notin \mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi - 1)^2 + (\varpi - \vartheta)\vartheta^2$

$= \vartheta\Big[(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)\Big].$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph of $\mathfrak{I}_{\mathfrak{s}}$ of L_s.

In this case, the non-ideal elements are adjacent to all the ideal elements and some non-ideal elements. Here, the S-prime ideals are $\downarrow pq,$ $\downarrow pr$ and $\downarrow qr$.

Consider the S-prime ideal $\mathfrak{I}_\mathfrak{s} =\downarrow pq$ and $d(x)$ in $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ is as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi - 1 \ \ if \ \ x \in \mathfrak{I}_\mathfrak{s} \\ \vartheta \ \ if \ \ x = r \\ \vartheta + 2 \ \ if \ \ x \neq r. \end{cases}$$

Then,

$M_1(\mathfrak{G}_s^{(2)}(I)) = \sum_{\mathfrak{x}\in\mathcal{V}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}=r} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}\neq r} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi - 1)^2 + \vartheta^2 + (\vartheta + 2)^2$

$= \vartheta\Big[(\varpi - 1)^2 + \vartheta)\Big] + (\vartheta + 2)^2.$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph of $\mathfrak{I}_\mathfrak{s}$ of L_s.

In this case, the S-prime ideal is $\downarrow pqr$. The maximal elements of L_s are p, q, r and they are denoted as $\mathfrak{M}_k, k = 1, 2, 3$ and $\mathfrak{d}(\mathfrak{x})$ in $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ is as follows:

$$\mathfrak{d}(\mathfrak{x}) = \begin{cases} \varpi - 1 \ \ if \ \ \mathfrak{x} \in \mathfrak{I}_\mathfrak{s} \\ \vartheta + 1 \ \ if \ \ \mathfrak{x} = \mathfrak{M}_k \\ \vartheta + 3 \ \ if \ \ \mathfrak{x} \neq \mathfrak{M}_k. \end{cases}$$

Then,

$M_1(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\in\mathcal{V}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})^2$

$= \sum_{\mathfrak{x}\in\mathfrak{I}_\mathfrak{s}} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}=\mathfrak{M}_k} \mathfrak{d}(\mathfrak{x})^2 + \sum_{\mathfrak{x}\neq\mathfrak{M}_k} \mathfrak{d}(\mathfrak{x})^2$

$= \vartheta(\varpi - 1)^2 + (\varpi - 4)(\vartheta + 1)^2 + (\varpi - 4)(\vartheta + 3)^2$

$= \vartheta(\varpi - 1)^2 + (\varpi - 4)\Big[(\vartheta + 1)^2 + (\vartheta + 3)^2\Big].$ □

Theorem 5.4. *Let* $\mathfrak{R}$ *be a ring of order* pqr*. Then,*

(i) $M_2(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 1)\Big[(\varpi - 1)^2 + \vartheta^2(\varpi - \vartheta)\Big].$

(ii) $M_2(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 1)\Big[(\varpi - 1) + \vartheta^2\Big] + (\vartheta + 2)\Big[8(\vartheta - 1) + (\vartheta + 2)^2\Big].$

(iii) $M_2(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = 2(\varpi - 4)\Big[(\varpi - 1)(\eta - 4) + (\vartheta + 3)(\varpi + 2)\Big].$

Proof. (i) Let $\mathcal{E}[\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})]$ be the edge set of $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ of the S-prime ideal $\mathfrak{I}_\mathfrak{s}$ of L_s. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}[\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})]$.

This implies that either $\mathfrak{x}$ or $\mathfrak{y}$ is in $\mathfrak{I}_\mathfrak{s}$ and the degree of the vertices $\mathfrak{x}$ and $\mathfrak{y}$ are defined in (i) of

Theorem 5. 3. Then,

$M_2(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= (\varpi - 1)(\varpi - 1)^2 + \vartheta(\varpi - \vartheta)(\varpi - 1)\vartheta$

$= (\varpi - 1)\Big[(\varpi - 1)^2 + \vartheta^2(\varpi - \vartheta)\Big].$

(ii) Let $E[\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})]$ be the edge set of $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ of the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of L_s.

Consider the S-prime ideal $\mathfrak{I}_{\mathfrak{s}} =\downarrow pq$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ of $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ are defined in (ii) of Theorem 5.3. Then,

$M_2(\mathfrak{G}_s(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}[\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})]} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}) + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= (\varpi - 1)^2 + \Big[(\varpi - 1)\vartheta^2 + 8(\varpi - 1)(\vartheta + 2)\Big] + (\vartheta + 2)^3$

$= (\varpi - 1)\Big[(\varpi - 1) + \vartheta^2\Big] + (\vartheta + 2)\Big[8(\varpi - 1) + (\vartheta + 2)^2\Big].$

(iii) Let $\mathcal{E}[\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})]$ be the edge set of $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ of the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of L_s.

Consider the S-prime ideal is $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ of $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ are defined in (iii) of Theorem 5.3. Then,

$M_2(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}[\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})]} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})} + \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})$

$= \Big[(\varpi - 1)(\varpi - 5)(\varpi - 4) + (\varpi - 1)(\varpi - 3)(\varpi - 4)\Big] + \Big[(\vartheta + 1)(\vartheta + 3)(\varpi - 4) + (\vartheta + 3)^2(\varpi - 4)\Big]$

$= (\varpi - 1)(\varpi - 4)(2\varpi - 8) + (\varpi - 4)(\vartheta + 3)(2\varpi + 4)$

$= 2(\varpi - 1)(\varpi - 4)^2 + 2(\varpi - 4)(\vartheta + 3)(\varpi + 2)$

$= 2(\varpi - 4)\Big[(\varpi - 1)(\varpi - 4) + (\vartheta + 3)(\varpi + 2)\Big].$ ☐

6 First and Second Zagreb Coindex of S-prime Graph

The $M_1(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ and $M_2(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ of the S-prime graph are generalized in this section.

Theorem 6.1. *Let $\mathfrak{R}$ be a ring of order $p^t q$. Then,*

$$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta(\varpi - \vartheta)(\varpi - \vartheta - 1).$$

Proof. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ of the S-prime ideal $\mathfrak{I}_{\mathfrak{s}}$ of L_s. If $\mathfrak{x}\mathfrak{y}$ is an edge of

$\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$, then at least one of the end points of $\mathfrak{x}\mathfrak{y}$ must be in the ideal $\mathfrak{I}_\mathfrak{s}$. Then,

$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))} [\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

In this, there is no edge between the non-ideal elements $\mathfrak{x}, \mathfrak{y}$. Thus,

$\overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} (\varpi - \vartheta)(\varpi - \vartheta - 1)\vartheta$

$\therefore\ \overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \vartheta(\varpi - \vartheta)(\varpi - \vartheta - 1).$ □

Theorem 6.2. *Let $\mathfrak{R}$ be a ring of order $p^t q$. Then,*

$$\overline{M_2}(\mathfrak{G}_s(I)) = \overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})).\frac{\vartheta}{2}.$$

Proof. Let $\mathfrak{G}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Then,

$\overline{M_2}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s}))} \mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y}).$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} (\varpi - \vartheta)\dfrac{(\varpi - \vartheta - 1)}{2}\vartheta^2$

$= \dfrac{(\varpi - \vartheta)(\varpi - \vartheta - 1)}{2}.\vartheta^2$

$= \Big[\vartheta(\varpi - \vartheta)(\varpi - \vartheta - 1)\Big]\dfrac{\vartheta}{2}$

Hence, $\overline{M_2}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})) = \overline{M_1}(\mathfrak{G}(\mathfrak{I}_\mathfrak{s})).\dfrac{\vartheta}{2}.$ □

Theorem 6.3. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $\overline{M_1}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = 2(\varpi - 3)(\varpi - 4).$

(ii) $\overline{M_1}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 3)\Big[3(\vartheta - 1) + \varpi\Big].$

(iii) $\overline{M_1}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = 2\Big[(\vartheta + 1)(\varpi - 4) + (\varpi - 1)(\vartheta + 2)\Big].$

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Then,

$\overline{M_1}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))} [\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}} [\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= \Big[(\varpi - 3) + (\varpi - 3)\Big](\varpi - 4)$

$= 2(\varpi - 3)(\varpi - 4).$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_\mathfrak{s} =\downarrow pq$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are is defined in (ii) of Theorem 5.3. Then,

$\overline{M_1}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x}=r,\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x},\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= (\varpi - 3)\Big[\vartheta + (\varpi - 3)\Big] + \vartheta\Big[(\varpi - 3) + (\varpi - 3)\Big]$

$= (\varpi - 3)\Big[3(\vartheta - 1) + \varpi\Big].$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Consider the S-prime ideal is $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (iii) of Theorem 5.3. Then,

$\overline{M_1}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{x})]$

$= \sum_{\mathfrak{x},\mathfrak{y}=\mathfrak{M}_\mathfrak{k}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x}=\mathfrak{M}_\mathfrak{k},\mathfrak{y}\neq\mathfrak{M}_\mathfrak{k}}[\mathfrak{d}(\mathfrak{x}) + \mathfrak{d}(\mathfrak{y})]$

$= 2(\vartheta + 1)(\varpi - 4) + (\varpi - 1)\Big[\vartheta + 1 + (\vartheta + 3)\Big]$

$= 2(\vartheta + 1)(\varpi - 4) + (\varpi - 1)(2\vartheta + 4)$

$= 2(\vartheta + 1)(\varpi - 4) + 2(\varpi - 1)(\vartheta + 2)$

$= 2\Big[(\vartheta + 1)(\varpi - 4) + (\varpi - 1)(\vartheta + 2)\Big].$ □

Theorem 6.4. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $\overline{M_2}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = (\varpi - 4)\vartheta^2.$

(ii) $\overline{M_2}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = 2\vartheta(\vartheta + 2)^2.$

(iii) $\overline{M_2}(\mathfrak{G}^{(3)}(\mathfrak{I}_\mathfrak{s})) = (\vartheta + 1)\Big[(\vartheta + 1)(\varpi - 4) + (\vartheta + 3)(\varpi - 1)\Big].$

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Then,

$\overline{M_2}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_\mathfrak{s}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= (\varpi - 4)\vartheta\vartheta$

$= (\varpi - 4)\vartheta^2.$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_\mathfrak{s} = \downarrow pq$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (ii) of Theorem 5.3. Then,

$\overline{M_2}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_\mathfrak{s}))}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x}=r,\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x},\mathfrak{y}\neq r}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \vartheta(\vartheta+2)(\vartheta+2) + (\vartheta+2)(\vartheta+2)\vartheta$

$= 2\vartheta(\vartheta+2)^2.$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Consider the S-prime ideal is $\downarrow pqr$ and $\mathfrak{d}(\mathfrak{x}), \mathfrak{d}(\mathfrak{y})$ are defined in (iii) of Theorem 5.3. Then,

$\overline{M_2}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\notin\mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}}))}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= \sum_{\mathfrak{x},\mathfrak{y}=\mathfrak{M}_{\mathfrak{k}}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})] + \sum_{\mathfrak{x}=\mathfrak{M}_{\mathfrak{k}},\mathfrak{y}\neq\mathfrak{M}_{\mathfrak{k}}}[\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})]$

$= (\vartheta+1)(\vartheta+1)(\varpi-4) + (\vartheta+1)(\vartheta+3)(\varpi-1)$

$= (\vartheta+1)\Big[(\vartheta+1)(\varpi-4) + (\vartheta+3)(\varpi-1)\Big].$ ☐

7 Randi´c index of S-prime Graph

In this section, $R\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ of $\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})$ of the S-prime ideals $\mathfrak{I}_{\mathfrak{s}}$ are generalized.

Theorem 7.1. *Let $\mathfrak{R}$ be a ring of order $p^t q$. Then,*

$$R(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))) = \vartheta\Big[\frac{(\vartheta-1)}{2(\varpi-1)} + \frac{(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}\Big].$$

Proof. Let $\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of the S-prime graph and their degrees are defined earlier in Theorem 5.1. Then,

$R(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}}))}\dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$

$= \sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_{\mathfrak{s}}}\dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}}\dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$

$= \dfrac{\vartheta(\vartheta-1)}{2\sqrt{(\varpi-1)^2}} + \dfrac{\vartheta(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}$

Thus, $R(\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = \vartheta\Big[\dfrac{(\vartheta-1)}{2(\varpi-1)} + \dfrac{(\varpi-\vartheta)}{\sqrt{\vartheta(\varpi-1)}}\Big].$ ☐

Theorem 7.2. *Let $\mathfrak{R}$ be a ring of order pqr. Then,*

(i) $R(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = 1 + \dfrac{(\varpi-3)(\varpi-4)}{\sqrt{\vartheta(\varpi-1)}}$

(ii) $R\mathfrak{G}(\mathfrak{I}_{\mathfrak{s}})) = 1 + \dfrac{1}{(\varpi-1)} + \dfrac{\vartheta}{\sqrt{\vartheta(\varpi-1)}} + \dfrac{8}{\sqrt{(\varpi-1)(\vartheta+2)}}.$

(iii) $R(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = 3\Big[\dfrac{1}{\sqrt{(\varpi-1)(\vartheta+1)}} + \dfrac{1}{\sqrt{(\varpi-1)(\vartheta+3)}} + \dfrac{1}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \dfrac{1}{(\vartheta+3)}\Big].$

Proof. (i) Let $\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Then,

$$R(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}})) = \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(1)}(\mathfrak{I}_{\mathfrak{s}}))} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \textstyle\sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \frac{(\varpi-1)}{\sqrt{(\varpi-1)(\varpi-1)}} + \frac{(\varpi-3)(\varpi-4)}{\sqrt{(\varpi-1)\vartheta}}$$

$$= 1 + \frac{(\varpi-3)(\varpi-4)}{\sqrt{\vartheta(\varpi-1)}}.$$

(ii) Let $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_{\mathfrak{s}} =\downarrow pq$. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of $\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})$ and their degrees are defined earlier in (ii) of Theoreem 5.3. Then,

$$R(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}})) = \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(2)}(\mathfrak{I}_{\mathfrak{s}}))} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \textstyle\sum_{\mathfrak{x},\mathfrak{y}\in\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \textstyle\sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= 1 + \frac{1}{(\varpi-1)} + \frac{\vartheta}{\sqrt{\vartheta(\varpi-1)}} + \frac{8}{\sqrt{(\varpi-1)(\vartheta+2)}}.$$

(iii) Let $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ be the S-prime graph. Consider the S-prime ideal $\mathfrak{I}_{\mathfrak{s}} =\downarrow pqr$. Let $\mathfrak{x}\mathfrak{y} \in \mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}}))$ be the edge set of $\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})$ and their degrees are defined earlier in (iii) of Theoreem 5.3. Then,

$$R(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}})) = \textstyle\sum_{\mathfrak{x}\mathfrak{y}\in\mathcal{E}(\mathfrak{G}^{(3)}(\mathfrak{I}_{\mathfrak{s}}))} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \textstyle\sum_{\mathfrak{x}\in\mathfrak{I}_{\mathfrak{s}},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}} + \textstyle\sum_{\mathfrak{x},\mathfrak{y}\notin\mathfrak{I}_{\mathfrak{s}}} \dfrac{1}{\sqrt{\mathfrak{d}(\mathfrak{x})\mathfrak{d}(\mathfrak{y})}}$$

$$= \Big[\frac{3}{\sqrt{(\varpi-1)(\vartheta+1)}} + \frac{3}{\sqrt{(\varpi-1)(\vartheta+3)}}\Big] + \Big[\frac{3}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \frac{3}{\sqrt{(\vartheta+3)^2}}\Big]$$

$$= 3\Big[\frac{1}{\sqrt{(\varpi-1)(\vartheta+1)}} + \frac{1}{\sqrt{(\varpi-1)(\vartheta+3)}} + \frac{1}{\sqrt{(\vartheta+1)(\vartheta+3)}} + \frac{1}{(\vartheta+3)}\Big]. \qquad \square$$

8 Conclusion

In this paper, a new ideal called S-prime ideal in a lattice and S-meet semilattice are defined and it is shown that the prime ideal of a lattice is also an S-prime ideal of a lattice L. Also, multiplicative subset and S-prime ideal of S-meet semilattice are defined. Finally, a new graph from the S-meet semilattice $(L_s, \wedge, \subseteq)$ is introduced with examples and their topological measures are generalized.

References

[1] Gratzer. G, Schmidt E. T. Standard ideal in lattices. Acta math. Acad. Sci. Hung. 1961; 12: 17 - 86.

[2] Noor A. S. A, Latif M. A.Standard n-ideals of a lattice. Southeast Asian Bull. Math. 1997; 4: 185 - 192.

[3] Latif M. A, Noor A. S. A. n-Ideals of a Lattice. The Rajshahi University Studies, Part-B,1994; 22: 173-180.

[4] Noor A. S. A, Latif M. A. Finitely generated n-ideals of a lattice. Southeast Asian Bull. Math. 1998; 4: 73-79.

[5] Noor A. S. A, Latif M. A. Properties of Standard n-ideals of a lattice. Southeast Asian Bull. Math. 2000; 24: 403 - 409.

[6] Meenakshi P. Wasadikar, Karuna T. Gaikwad. Some properties of 2-absorbing primary ideals in lattices. AKCE International Journal of Graphs and Combinatorics.2019; 16: 18-26.

[7] Badawi A. On 2-absorbing ideals of commutative rings. Bull. Austral. Math. Soc. 2007; 75: 417-429.

[8] Badawi A, A. Y. Darani. On weakly 2-absorbing ideals of commutative rings. Houston J. Math. 2013; 39(2): 441-452.

[9] Ali Akbar Estaji, Toktam Haghdadi. On n-absorbing Ideals in a Lattice. Kragujevac Journal of Mathematics. 2021; 45(4): 597 - 605.

[10] Anderson D. F, Ayman Badawi. On n-absorbing ideals of commutative rings. 2011; 39: 1646 - 1672.

[11] Ali. A, Hafizur Rahman R. M, Noor A. S. A, Mizanur Rahman M. On semiprime n-ideals of lattices. Annals of Pure and Applied Mathematics. 2012; 21(1): 10-17.

[12] Ayub Ali M, Noor A. S. A. Some Properties of Modular n-Ideals of a Lattice. Journal of Physical Sciences. 2010;14: 173-180.

[13] Ahmed Hamed, Achraf Malek.S-prime ideals of a commutative ring. Beitrage zur Algebra and Geometrie. 2020; 61: 533 - 542.

[14] Kalamani D, Mythily C. V. S-prime ideal graph of a finite commutative ring. Advances and Applications in Mathematical Sciences. 2023; 22: 861-872.

[15] Mythily C. V, Kalamani D. Study on S-prime Ideal as Nilpotent Ideal. Journal of Applied Mathematics and Informatics. 2024; (In press).

[16] Kalamani D, Mythily C. V. Some New Results on S-prime ideals of a Finite Commutative ring as S-meet Semilattice. Applied and Computational Mathematics. 2024; 13 (4): 105 - 110.

[17] Ashrafi A. R., Doslic T, Hamzeh A. The Zagreb coindices of graph operations. Discrete Applied Mathematics. 2010; 158; 1571 - 1578.

[18] Khalifeh M. H, yousefi-Azari H, Ashrafi A. R. The first and second Zagreb indices of some graph operations. Discrete Applied Mathematics.2009; 157: 804 - 811.

[19] Kiruthika G, Kalamani D. Degree based partition of the power graph of the finite Abelian group. Malaya Journal of Matematik. 2020; 1: 66 -71.

[20] Gratzer. G. General Lattice Theory. Academic Press. New York San Francisco. 1978.