On the number of combinations generating a sum

Abstract

We present some elementary arguments that are applicable to problems in additive combinatorics. In particular, we study the problem of multiple pairs generating the same sum, establishing a general result in the context of a group.

Keywords: Additive combinatorics; number theory; groups

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1 Introduction

One may perceive Szemerédi's theorem as a crown jewel in the field of additive combinatorics (Szemerédi [1969], Szemerédi [1975], Szemerédi [1990]). A corollary is that a sum may be generated in an arbitrarily large number of ways, if one assumes positive upper density. However, different assumptions and elementary methods may be supplied to achieve the desired conclusion. We present some elementary combinatorial arguments to prove a general result in the context of a group. We also demonstrate how the argument allows one to prove existence of arithmetic progression of length three in the difference set i.e. the set minus itself. Prior research in this line has been conducted and may be found in Hardy and Ramanujan [1918], Hardy and Wright [1979], Erdös and Turán [1936], Erdös [1946], Erdös et al. [1968], Roth [1953], Van der Waerden [1927], Furstenberg [1977], Gowers [2001] and Tao and Vu [2006].

2 Generating a sum in different ways

Let $\mathbb{N}$ denote the set of natural numbers, including zero. For $X, Y \subseteq \mathbb{N}$ and $a \in \mathbb{N}$, we define $aX := \{ax : x \in X\}$ and $X + Y := \{x + y : x \in X, y \in Y\}$. A subset $X \subseteq N$ is a *basis* if $X + X = \mathbb{N}$. Clearly, if X is a basis and $X \subseteq Y$, then Y is also a basis. A basis X is said to be *minimal*, if for each strict subset $Y \subset X$, Y is not a basis. It then follows from the definition, that X is a minimal basis if and only if, for each $x \in X$, $X \backslash \{x\}$ is not a basis. As an example, for a fixed $n \geq 1$, consider the set

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$X_n := \{0, 1, ..., n\} \cup \{kn : k \in \mathbb{N}\}$. Then, we may show that X_n is a minimal basis, for each $n > 1$. For every basis X, there exists $Y \subseteq X$, such that Y is a minimal basis (see Erdös and Nathanson [1975]).

For any $X \subseteq \mathbb{N}$, define

$$Q_n := X \cap \{0, 1, ..., n\}. \tag{2.1}$$

We provide an elementary proof for the following theorem Suppose that $X \subseteq \mathbb{N}$. Further, suppose that we have

$$\liminf_{n \to +\infty} \frac{|Q_n|}{n} > 0. \tag{2.2}$$

Then, for each $k \geq 1$, there exists $m \in \mathbb{N}$ such that the sum $x + y = m$ may be generated in at least k possible combinations, for $x, y \in X$.

Proof. Let $k \geq 1$ and $\delta > 0$ be the lim inf in (2.13). Then, let $n \in \mathbb{N}$ such that

$$q_n \geq \delta n \tag{2.3}$$

and

$$\frac{q_n(q_n + 1)}{2} \geq k(2n + 1). \tag{2.4}$$

where $q_n = |Q_n|$. We define $\bar{Q} = \{(x, y) \in Q_n \times Q_n : x \leq y\}$. Hence, $|\bar{Q}| = \frac{q_n(q_n+1)}{2}$. For the $2n + 1$ parallels lines $L_m = \{(x, y) : x + y = m\}$, where $m \in \{0, ..., 2n\}$, we have that

$$\bar{Q} \subseteq \bigcup_{m=0}^{2n} \bar{Q} \cap L_m, \tag{2.5}$$

by the definition of Q_n. By the pigeon-hole principle, there exists m such that $|\bar{Q} \cap L_m| \geq k$, which proves the theorem. ☐

Note that X_n as defined above satisfies 2.13. The conclusion of the theorem for X_n follows also from the fact that each X_n contains an infinite arithmetic progression. If X is a basis and satisfies 2.13, then the above theorem implies we may find natural numbers that can be generated in an arbitrarily large number of possibilities. If X satisfies 2.13, then we say that X has *positive upper density*.

We also prove the following theorem, for sets with a finite doubling constant (see Freiman [2006], Ruzsa [1999]). Suppose that $X \subseteq \mathbb{N}$ is an infinite set and $c \geq 1$. Further, suppose that

$$|Q_n + Q_n| \leq c|Q_n|, \tag{2.6}$$

for infinitely many n. Then, for each $k \geq 1$, there exists $m \in \mathbb{N}$ such that the sum $x + y = m$ may be generated in at least k possible combinations, for $x, y \in X$.

Proof. Let $k \geq 1$. Let n be such that

$$ckq_n \leq \frac{q_n(q_n + 1)}{2}. \tag{2.7}$$

Note that $\bar{Q}$ is covered by at most cq_n many parallel lines L_m. Hence, by the pigeon-hole principle, there exists L_m such that $|\bar{Q} \cap L_m| \geq k$. ☐

Note again that the basis X_n satisfies the condition in the above theorem since X_n may be represented as the union of an arithmetric progression and a finite subset.

From Schinzel's theorem (see Sierpinski [2014]), one may show that the above conclusion holds for the set of squares of natural numbers. The set would however have limit frequency equal to zero. Note that from the Green-Tao theorem (see Tao and Vu [2006]), we may show that the set of all primes P has the property that for each $k \geq 1$, there exists m and l such that the sum $x_1 + ... + x_l = m$ can be generated in at least k possible combinations, where $x_u \in P$, for each $1 \leq u \leq l$. Yet still, for the primes $X = P$, the limit frequency $\frac{|Q_n|}{n}$ is zero.

One may also consider countable ordered groups $(G, *, \succeq)$ such that $\succeq$ is a total order on G, which satisfies the following conditions : i) for any $x, y, z \in G$, if $z \succeq x$ and $z \succeq y$, then $z * z \succeq x * y$ and ii) for any $x \in G$, the set $\{x' \in G : x \succeq x'\}$ is finite.

We define the following sets in G.

$$N_x = \{x' \in G : x \succeq x'\}; Q_x = X \cap N_x. \tag{2.8}$$

We also prove the following theorem concerning ordered groups. Suppose that $(G, *)$ is a countable group and $X \subseteq G$. Suppose that $\succeq$ is a total order on G such that $(G, *, \succeq)$ satisfies conditions i) and ii).

If there exists a real number $\delta > 0$ and $c \geq 1$ such that for each $x_0 \in G$, there exists $x \succeq x_0$ such that

$$\frac{|Q_x|}{|N_x|} \geq \delta \tag{2.9}$$

and for each $x \in G$, we have that

$$|N_{x*x}| \leq c|N_x|, \tag{2.10}$$

then for each $k \geq 1$, there exists $m \in G$ such that $x * y = m$ may be generated in at least k possible combinations, for $x, y \in X$.

Proof. Let $k \geq 1$. Let $x \in G$ such that

$$q_x \geq \delta n_x \tag{2.11}$$

and

$$\frac{q_x(q_x + 1)}{2} \geq ckn_x, \tag{2.12}$$

where $q_x = |\{x' \in X : x \succeq x'\}|$ and $n_x = |\{x' \in G : x \succeq x'\}|$. Since condition i) is satisfied, we have that $Q_x + Q_x \subseteq N_{x*x}$. Hence, we obtain $|Q_x + Q_x| \leq cn_x$. Since, for $\bar{Q} = \{(x, y) \in Q_x \times Q_x : y \succeq x\}$, we have that $|\bar{Q}| = \frac{q_x(q_x+1)}{2}$, it follows that $\bar{Q}$ is covered by at most cn_x many parallel lines $L_m = \{(x, y) : x * y = m\}$. Hence, by the pigeon-hole principle, there exists L_m such that $|\bar{Q} \cap L_m| \geq k$. □

The next theorem concerns difference sets i.e the set $X - X$ (see Roth [1953], Bourgain [1990]). Suppose that $X \subseteq \mathbb{N}$. Further, suppose that we have

$$\liminf_{n \to +\infty} \frac{|Q_n|}{n} > 0. \tag{2.13}$$

Then, $X - X$ contains an arithmetic progression of length three.

Proof. Let n be such that

$$\sum_{j=1}^{q_n-2} \binom{q_n-j}{2} > 8n. \tag{2.14}$$

In the above strict inequality, the LHS represents the cardinality of the set $\bar{Q}_3 := \{(x,y,z) \in Q_n^3 : x < y < z\}$. Now, consider the set of numbers

$$Q_3^+ = \{z - 2y + x : (x,y,z) \in \bar{Q}_3\}.$$

Note that the absolute value of each number in Q_3^+ is at most $4n$. If $0 \in Q_3^+$, then X would contain an arithmetic progression of length three and hence, so would $X - X$. If not, from the inequality, it follows that there exist $(x,y,z), (x',y',z') \in Q_n^3$ such that $z - 2y + x$ and $z' - 2y' + x'$ have the same sign and leave the same remainder when divided by either $4n$ or $-4n$. However, since both are in $[-4n, 4n]$, it follows that

$$z - 2y + x = z' - 2y' + x'. \tag{2.15}$$

Hence, $x' - x, y' - y$ and $z' - z$ are in arithmetic progression. ☐

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