

ON FLC-FOCAL CURVES ACCORDING FLC-FRAME

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ABSTRACT. In this study, we first characterize focal curves by examining the FLC-frame in three-dimensional Euclidean space. We then derive the relationship between the curvatures of a curve and the focal curvatures. Finally, we present some new conditions for curves with constant curvatures in \mathbb{E}^3 .

Keywords: Flc-frame, focal curve, focal curvatures.

1. BACKGROUND ON FLC-FRAME

Consider the tridimensional Euclidean space \mathbb{E}^3 with inner product

$$\langle \cdot, \cdot \rangle = dx^2 + dy^2 + dz^2,$$

where $(x, y, z) \in \mathbb{E}^3$ is a rectangular coordinate system. Consider the curve $\alpha : I \rightarrow \mathbb{E}^3$, which is differentiable in the Euclidean 3-space and is defined on an open interval I . The Frenet frame is defined as follows [15]

$$t = \frac{\alpha'}{\|\alpha'\|}, \quad b = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \quad n = b \wedge t, \quad (1.1)$$

satisfying

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \|\alpha'\| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}. \quad (1.2)$$

Here, κ and τ are differentiable functions defined on I , referred to as the *curvature* and *torsion* of α , respectively. The vectors t , n and b represent the tangent, principal normal, and binormal vectors of α , respectively.

In addition to the Frenet frame, it is possible to establish other frame along a three-dimensional curve such as: Bishop frame, q-frame, alternative frame and others [4, 12]. Recently, Dede [5] introduced a newframe along a polynomial space curve, called as Flc-frame. The computation of Flc-frame is easier than the Frenet frame and has some advantages, such as (see [6]):

- i) There are no singular points of order 1, that is, there is no $t_0 \in I$ such that $\alpha''(t_0) = 0$, in the FLC frame, whereas the Frenet frame exhibits irregular behavior at a first-order singular point.
- ii) The chances of inflection points occurring, that is, points $t_0 \in I$ such that $\alpha(t_0) \wedge \alpha''(t_0) = 0$, are lower in the FLC frame than in the Frenet frame.

Discussion of the Flc-frame and its application to the tube surfaces can be found in [5], on geometry of focal surfaces in [16], in the analysis of Smarandache ruled surfaces in [17].

Let $\alpha(t)$ be a polynomial space curve of degree n . The Flc-frame is given by

$$t = \frac{\alpha'}{\|\alpha'\|}, \quad D_1 = \frac{\alpha' \wedge \alpha^{(n)}}{\|\alpha' \wedge \alpha^{(n)}\|}, \quad D_2 = D_1 \wedge t, \quad (1.3)$$

where the prime ' indicates the differentiation with respect to t and the notation $\alpha^{(n)}$ expresses the n th derivative of the curve α with respect to t [5]. The new vectors D_1 and D_2 are called as *binormal-like vector* and *normal-like vector*, respectively.

Calculations show that the derivatives of the Flc-frame satisfy [5]

$$\begin{bmatrix} t' \\ D_2' \\ D_1' \end{bmatrix} = \|\alpha'\| \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ D_2 \\ D_1 \end{bmatrix}, \quad (1.4)$$

where

$$d_1 = \frac{\langle \alpha' \wedge \alpha'', \alpha' \wedge \alpha^{(n)} \rangle}{\|\alpha'\|^3 \|\alpha' \wedge \alpha^{(n)}\|}, \quad d_2 = \frac{\det(\alpha'', \alpha', \alpha^{(n)})}{\|\alpha'\|^2 \|\alpha' \wedge \alpha^{(n)}\|}, \quad d_3 = \frac{\det(\alpha', \alpha'', \alpha^{(n)}) \langle \alpha', \alpha^{(n)} \rangle}{\|\alpha'\|^2 \|\alpha' \wedge \alpha^{(n)}\|} \quad (1.5)$$

are called the curvatures of the Flc-frame.

Corollary 1.1. *If the degree of polynomial space curve is two, then the Flc-frame coincides with the Frenet frame with curvatures $d_1 = \kappa, d_2 = 0$ and $d_3 = \tau = 0$ [6].*

A focal curve or generalized evolute is the geometric locus given by the centers of the osculating circles of a given curve. With applications ranging from Dynamical Systems Theory to Surface Engineering [14], the focal curve can be expressed in terms of the frame of the initial curve and in a parametrization the coefficients called focal curvatures are obtained. These curves are studied in different spaces and frames, for example in [1, 2, 7, 8, 9, 11]. Motivated by these, in this paper, we study the focal curves according Flc-frame in \mathbb{E}^3 .

2. FOCAL CURVES ACCORDING FLC-FRAME IN \mathbb{E}^3

Let $\alpha : I \rightarrow \mathbb{E}^3$ be a regular space curve in the three-dimensional Euclidean space \mathbb{E}^3 with nonzero curvature κ and torsion τ . The focal curve of α is the curve given by the equation

$$\beta(t) = \alpha(t) + \varphi_1(t)n(t) + \varphi_2(t)b(t), \quad (2.6)$$

where n is a principal unit normal vector field of α , b is a binormal unit vector field of α . The coefficients $\varphi_1(t)$ and $\varphi_2(t)$ are smooth functions called focal curvatures of α [10].

In terms of the Flc-frame, the focal curve of α is given by

$$\beta(t) = \alpha(t) + \varphi_1(t)D_2(t) + \varphi_2(t)D_1(t). \quad (2.7)$$

Theorem 2.1. *Consider a unit speed curve $\alpha : I \rightarrow \mathbb{E}^3$ and its corresponding focal curve β . Then,*

$$\begin{aligned} \beta(s) &= \alpha(s) + e^{-\int \frac{d_1 d_3}{d_2} ds} \left[\int e^{\int \frac{d_1 d_3}{d_2} ds} \frac{d_3}{d_2} ds + C \right] D_2 \\ &+ \left\{ \frac{1}{d_2} - \frac{d_1}{d_2} e^{-\int \frac{d_1 d_3}{d_2} ds} \left[\int e^{\int \frac{d_1 d_3}{d_2} ds} \frac{d_3}{d_2} ds + C \right] \right\} D_1, \end{aligned} \quad (2.8)$$

where C is a constant of integration.

Proof. Suppose α is a curve with unit speed and β represents its focal curve in \mathbb{E}^3 .

Differentiating the equation (2.7) and using (1.4), we obtain

$$\beta' = (1 - d_1\varphi_1 - d_2\varphi_2)t + (\varphi_1' - d_3\varphi_2)D_2 + (d_3\varphi_1 + \varphi_2')D_1. \quad (2.9)$$

From equation (2.9), the first two components vanish, we get

$$1 - d_1\varphi_1 - d_2\varphi_2 = 0, \quad (2.10)$$

$$\varphi_1' - d_3\varphi_2 = 0. \quad (2.11)$$

From equation (2.10),

$$\varphi_2 = \frac{1 - d_1\varphi_1}{d_2}.$$

In (2.11),

$$\varphi_1' + \frac{d_1 d_3}{d_2} \varphi_1 = \frac{d_3}{d_2}.$$

By integrating this equation, we find

$$\varphi_1 = e^{-\int \frac{d_1 d_3}{d_2} ds} \left[\int e^{\int \frac{d_1 d_3}{d_2} ds} \frac{d_3}{d_2} ds + C \right],$$

$$\varphi_2 = \frac{1}{d_2} - \frac{d_1}{d_2} e^{-\int \frac{d_1 d_3}{d_2} ds} \left[\int e^{\int \frac{d_1 d_3}{d_2} ds} \frac{d_3}{d_2} ds + C \right].$$

Substituting in the equation (2.7), the result follows. \square

As an immediate consequence of the above theorem, we have:

Corollary 2.2. Consider a unit speed curve $\alpha : I \rightarrow \mathbb{E}^3$ and its corresponding focal curve β in \mathbb{E}^3 . Then, the focal curvatures of β are

$$\varphi_1 = e^{-\int \frac{d_1 d_3}{d_2} ds} \left[\int e^{\int \frac{d_1 d_3}{d_2} ds} \frac{d_3}{d_2} ds + C \right],$$

$$\varphi_2 = \frac{1}{d_2} - \frac{d_1}{d_2} e^{-\int \frac{d_1 d_3}{d_2} ds} \left[\int e^{\int \frac{d_1 d_3}{d_2} ds} \frac{d_3}{d_2} ds + C \right].$$

Based on Theorem 2.1, we can state the following corollary:

Corollary 2.3. Consider a unit speed curve $\alpha : I \rightarrow \mathbb{E}^3$ and its focal curve β in \mathbb{E}^3 . If d_1, d_2 and d_3 are constants, then the focal curvatures of β are

$$\varphi_1 = \frac{1}{d_1} + C e^{-\frac{d_1 d_3}{d_2} s},$$

$$\varphi_2 = \frac{1}{d_2} - \frac{d_1}{d_2} \left(\frac{1}{d_1} + C e^{-\frac{d_1 d_3}{d_2} s} \right).$$

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