Original Research Article

ON FLC-FOCAL CURVES ACCORDING FLC-FRAME

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ABSTRACT. In this study, we firstly characterize focal curves by considering Flc-frame in the Euclidean 3-space. Then, we obtain the relation of each curvatures of curve in terms of focal curvatures. Finally, we give some new conditions with constant curvatures in $\mathbb{E}^3$.
Keywords: Flc-frame, focal curve, focal curvatures.

1. BACKROUND ON FLC-FRAME

Consider the tridimensional Euclidean space $\mathbb{E}^3$ with inner product

$$\langle \cdot, \cdot \rangle = dx^2 + dy^2 + dz^2$$

where $(x, y, z) \in \mathbb{E}^3$ is a rectangular coordinate system. Let $\alpha : I \to \mathbb{E}^3$ be a differentiable curve in the Euclidean 3-space defined on an open interval I. The Frenet frame is defined as follows [13]

$$t = \frac{\alpha'}{\| \alpha' \|}, \quad b = \frac{\alpha' \wedge \alpha''}{\| \alpha' \wedge \alpha'' \|}, \quad n = b \wedge t, \tag{1.1}$$

satisfying

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \| \alpha' \| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{1.2}$$

where κ e τ are differentiable functions on I called the *curvature* and the *torsion* of α, respectively, t is the tangent vector, n is the principal normal vector and b is the binormal vector of α.

Apart from Frenet frame we can define more frame along a space curve [4, 11]. Recently, Dede [5] introduced a newframe along a polynomial space curve, called as Flc-frame. The computation of Flc-frame is easier than the both Frenet and Bishop frames [4, 3]. Moreover they showed that to have a inflection point on Flc-frame is less possible than Frenet frame. Discussion of the Flc-frame and its application to the tube surfaces can be found in [5].

Let $\alpha(t)$ be a polynomial space curve of degree n. The Flc-frame is given by

$$t = \frac{\alpha'}{\| \alpha' \|}, \quad D_1 = \frac{\alpha' \wedge \alpha^{(n)}}{\| \alpha' \wedge \alpha^{(n)} \|}, \quad D_2 = D_1 \wedge t, \tag{1.3}$$

where the prime ' indicates the differentiation with respect to t [5]. If the order of derivative exceeds three, we replace prime by the superscript (n), such as $\alpha'''' = \alpha^{(4)}$. The new vectors D_1 and D_2 are called as *binormal-like vector* and *normal-like vector*, respectively.

The local rate of change of the Flc-frame called as the Frenet-like formulas can be expressed in the following form

$$\begin{bmatrix} t' \\ D_2' \\ D_1' \end{bmatrix} = \| \alpha' \| \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} t \\ D_2 \\ D_1 \end{bmatrix}. \tag{1.4}$$

The curvatures of the Flc-frame are given by

$$d_1 = \frac{\langle \alpha' \wedge \alpha'', \alpha' \wedge \alpha^{(n)} \rangle}{\| \alpha' \|^3 \| \alpha' \wedge \alpha^{(n)} \|}, \quad d_2 = \frac{\det(\alpha'', \alpha', \alpha^{(n)})}{\| \alpha' \|^2 \| \alpha' \wedge \alpha^{(n)} \|}, \quad d_3 = \frac{\det(\alpha', \alpha'', \alpha^{(n)}) \langle \alpha', \alpha^{(n)} \rangle}{\| \alpha' \|^2 \| \alpha' \wedge \alpha^{(n)} \|} \tag{1.5}$$

Corollary 1.1. *If the degree of polynomial space curve is two, then the Flc-frame coincides with the Frenet frame with curvatures $d_1 = \kappa, d_2 = 0$ and $d_3 = \tau = 0$ [6].*

More about Flc-frame can be found at [6, 14, 15].

2. Focal Curves According Flc-frame in $\mathbb{E}^3$

Let $\alpha : [a, b] \to \mathbb{E}^3$ be a regular space curve in the three-dimensional Euclidean space $\mathbb{E}^3$ with nonzero curvature κ and torsion τ. The focal curve of α is the curve given by the equation

$$\beta(t) = \alpha(t) + \varphi_1(t)n(t) + \varphi_2(t)b(t) \tag{2.6}$$

where n is a principal unit normal vector field of α, b is a binormal unit vector field of α. The coefficients $\varphi_1(t)$ and $\varphi_2(t)$ are smooth functions called focal curvatures of α [9].

In terms of the Flc-frame, the focal curve of α is given by

$$\beta(t) = \alpha(t) + \varphi_1(t)D_2(t) + \varphi_2(t)D_1(t) \tag{2.7}$$

Theorem 2.1. *Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve and β its focal curve on $\mathbb{E}^3$. Then,*

$$\begin{aligned} \beta &= \alpha + e^{-\int \frac{d_1 d_3}{d_2} ds} \left[\int e^{\int \frac{d_1 d_3}{d_2} ds} \frac{d_3}{d_2} ds + C \right] D_2 \\ &+ \left\{ \frac{1}{d_2} - \frac{d_1}{d_2} e^{-\int \frac{d_1 d_3}{d_2} ds} \left[\int e^{\int \frac{d_1 d_3}{d_2} ds} \frac{d_3}{d_2} ds + C \right] \right\} D_1, \end{aligned} \tag{2.8}$$

where C is a constant of integration.

Proof. Assume that α is a unit speed curve and β its focal curve in $\mathbb{E}^3$.
So, by differentiating of the formula (2.7) and using (1.4), we get

$$\beta' = (1 - d_1\varphi_1 - d_2\varphi_2)t + (\varphi_1' - d_3\varphi_2)D_2 + (d_3\varphi_1 + \varphi_2')D_1. \tag{2.9}$$

From equation (2.9), the first two components vanish, we get

$$1 - d_1\varphi_1 - d_2\varphi_2 = 0, \tag{2.10}$$

$$\varphi_1' - d_3\varphi_2 = 0. \tag{2.11}$$

From equation (2.10),

$$\varphi_2 = \frac{1 - d_1\varphi_1}{d_2}.$$

In (2.11),

$$\varphi_1' + \frac{d_1 d_3}{d_2}\varphi_1 = \frac{d_3}{d_2}.$$

By integrating this equation, we find

$$\varphi_1 = e^{-\int \frac{d_1 d_3}{d_2} ds}\left[\int e^{\int \frac{d_1 d_3}{d_2} ds}\frac{d_3}{d_2} ds + C\right],$$

$$\varphi_2 = \frac{1}{d_2} - \frac{d_1}{d_2} e^{-\int \frac{d_1 d_3}{d_2} ds}\left[\int e^{\int \frac{d_1 d_3}{d_2} ds}\frac{d_3}{d_2} ds + C\right].$$

This completes the proof of the theorem. □

As an immediate consequence of the above theorem, we have:

Corollary 2.2. *Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve and β its focal curve on $\mathbb{E}^3$. Then, the focal curvatures of β are*

$$\varphi_1 = e^{-\int \frac{d_1 d_3}{d_2} ds}\left[\int e^{\int \frac{d_1 d_3}{d_2} ds}\frac{d_3}{d_2} ds + C\right],$$

$$\varphi_2 = \frac{1}{d_2} - \frac{d_1}{d_2} e^{-\int \frac{d_1 d_3}{d_2} ds}\left[\int e^{\int \frac{d_1 d_3}{d_2} ds}\frac{d_3}{d_2} ds + C\right].$$

In the light of Theorem 2.1, we express the following Corollary:

Corollary 2.3. *Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve and β its focal curve on $\mathbb{E}^3$. If d_1, d_2, d_3 are constant then, the focal curvatures of β are*

$$\varphi_1 = \frac{1}{d_1} + Ce^{-\frac{d_1 d_3}{d_2} s},$$

$$\varphi_2 = \frac{1}{d_2} - \frac{d_1}{d_2}\left(\frac{1}{d_1} + Ce^{-\frac{d_1 d_3}{d_2} s}\right).$$

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