On some stochastic parabolic systems driven by new fractional Brownian motions

Abstract
Vector-valued functions of new fractional Brownian motions are considered. The concept of stochastic integrals are generalized. Formulas of Ito are also generalized. Some stochastic parabolic systems driven by new fractional Brownian motions are studied. Uniqueness and existence theorems are proved.
Keywords: Fractional normal distribution- Riemann stochastic integrals- Vectors of fractional Brownian motion- Fractional stochastic parabolic systems.
Mathematics Subject Classifications: 35A05, 47D60, 47D62, 77D09, 60H05, 60H10, 60G18

1-Introduction

Let $(\Omega, \mathfrak{A}, P)$ **be a probability space,**
(Ω *is a set*, $\mathfrak{A}$ *is a* $\sigma -$ *algebra of subsets of* $\Omega, P: \mathfrak{A} \to$ $[0.1]$ *is a probability measure*).
According to our previous results, [1], [2], we say that a random variable $X: \boldsymbol{\Omega} \to R$ **has a fractional Gaussian (or fractional normal) distribution, if** X **has a probability density function** f **defined by:**

$$\boldsymbol{f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi t^\alpha \theta}} \zeta_\alpha(\theta)\ exp\,(-\frac{-(x-m)^2}{2t^\alpha\theta}) d\theta} .$$

Where R **is the set of all real numbers,** $0 < \alpha \le 1$و $\boldsymbol{\zeta_\alpha(\theta)}$ **is the stable probability density function. It is clear that** X **has mean** $\boldsymbol{m}$ **and variance** $\frac{t^\alpha}{\Gamma(\alpha+1)}$ **, where** $\Gamma(.)$ ***is the gamma function.*** **In this case we write** X **is** $\boldsymbol{N_\alpha(m, \frac{t^\alpha}{\Gamma(\alpha+1)})}$ **(see [2-5]).**
Again according to our previous results, [1], we call a real valued stochastic process $W_\alpha(.)$ **a fractional Brownian motion if the following conditions are satisfied:**
i- $W_\propto(0) = 0$
ii- $\boldsymbol{W_\alpha(t) - w_\alpha(s)}$ ***is*** $\boldsymbol{N_\alpha(0, \frac{t^\alpha - s^\alpha}{\Gamma(\alpha+1)})}$**, for all** $\boldsymbol{0} < s < t$,
iii-forall times $\boldsymbol{0 < t_1 < \cdots < t_n}$ the random variables $\boldsymbol{W_\alpha(t_1)}$, $\boldsymbol{W_\alpha(t_2) - W_\alpha(t_1)}$, … , $\boldsymbol{W_\alpha(t_n) - W_\alpha(t_{n-1})}$ are independent, (with independent increments).
It is easy to see that:

$$\boldsymbol{E(W_\alpha(t)) = 0, E(W_\alpha^2(t)) = \frac{t^\alpha}{\Gamma(\alpha+1)}, E(W_\alpha(t) W_\alpha(s)) = \frac{s^\alpha}{\Gamma(\alpha+1)}},\ \boldsymbol{s \le t.}$$

Where $\boldsymbol{E(X)}$ **is the expectation of** $\boldsymbol{X}$**.**
Let $\boldsymbol{\mathcal{L}^2(0,T)}$ **be the space of all real-valued, progressively measurable stochastic processes G(.) such that** $\boldsymbol{E(\int_0^T G^2 dt) < \infty}$ **.**
The fractional stochastic integral $\boldsymbol{\int_0^T G dW_\alpha}$ **is defined in [1].**
It is proved that:

$$\boldsymbol{\int_0^T W_\alpha\, dW_\alpha = \frac{W_\alpha^2(T)}{2} - \frac{T^\alpha}{2\Gamma(\alpha+1)}},$$

$$\boldsymbol{d(tW_\alpha) = t dW_\alpha + W_\alpha dt},$$

$$\boldsymbol{\int_0^T (aG + bH) dW_\alpha = a\int_0^T G dW_\alpha + b\int_0^T H dW_\alpha},$$

$$\boldsymbol{E(\int_0^T G dW_\alpha) = 0.}$$

$$\boldsymbol{E(\int_0^T G dW_\alpha \int_0^T H dW_\alpha) = \frac{1}{\Gamma(\alpha)} E(\int_0^T t^{\alpha-1} G H dt)},$$

for all $H, G \epsilon \mathcal{L}^2(0,T)$ and all real numbers a, b.

2-Vectors of fractional Brownian motions

Let $\mathcal{L}^1(0,T)$ be the space of all real-valued, progressively measurable stochastic processes F such that $E[\int_0^T |F|dt] < \infty$.
Suppose that $W_\alpha(.) = (W_\alpha^1(.), \ldots, W_\alpha^m(.))$ is an m-dimensional fractional Brownian motion. We say that Z^{nxm}-valued stochastic process $G = (G^{ij})$ belongs to $\mathcal{L}_{nxm}^2(0,T)$ if $G^{ij} \epsilon \mathcal{L}^2(0,T)$. An R^n-valued stochastic process $F(.) = \left(F^1(.), \ldots, F^n(.)\right)$ belongs to $\mathcal{L}_n^1(0,T)$ if $F^i(.) \epsilon \mathcal{L}^1(0,T), i = 1, \ldots, n, j = 1, \ldots, m$.
If G belongs to $\mathcal{L}_{nxm}^2(0,T)$, then $\int_0^T G dW_\alpha$ is an R^n-valued random variable, whose $i-th$ component is $\sum_{j=1}^m \int_0^T G^{ij} dW_\propto^j$, $i = 1, \ldots n$.
If $G \in \mathcal{L}_{nxm}^2(0,T)$, then:

$$E[\int_0^T GdW$$

If $X(.) = (X^1(.), \ldots, X^n(.))$ is an R^n −valued stochastic process such that:

For some F$\in \mathcal{L}_n^1(0,T)$, $G \in \mathcal{L}_{nxm}^2(0,T)$, *we say that* $X(.)$ has the fractional stochastic differential:

This means that:

Let $u: R^n X[0,T] \to R$ be continuous, with continuous partial derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, i,j = 1, \ldots, n$, then a direct generalization of our formula in [1] is given by

$$du(X(t),t) = \frac{\partial u}{\partial t} dt + \sum_{i=1}^n \frac{\partial u}{\partial x_i} dX^i + \frac{t^{\alpha-1}}{2\Gamma(\alpha)} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \sum_{\ell=1}^m G^{i\ell} G^{j\ell},$$

Where the argument of the partial derivatives of u is $(X(t), t)$. (R is the set of all real numbers, R^n is the n −dimensional Euclidean space), see [3-8].

3- New fractional stochastic parabolic systems

Let $W_\propto(.)$ be an k-dimensional fractional Brownian. We will henceforth take

The $\sigma - algebra$ generated by the history of the fractional Wiener process up to (and including) time t.
Consider the following fractional stochastic differential system

$$du(x,t) = [\sum_{|q| \le 2m} A_q(x,t) D^q u(x,t) + f(x,t,u]dt + g(x,t,u) dW_\propto(t), t > 0, \quad (3.1)$$

With the initial condition

$u(x,0) = \varphi(x),$ (3.2)

(where *$x \in R^r$, $\{A_q , |q| \le 2m\}$ is a family of square matrices of order k ,whose* ***elements are sufficiently smooth functions on*** *$R^r x[0,T], q = (q_1, \dots q_r)$* ***is a multi-index,*** *$|q| = q_1 + \cdots + q_r, D^q = D_1^{q_1}, \dots, D_r^{q_r}$, $D_j = \frac{\partial}{\partial x_j}$, $j = 1, \dots, r$, f , $\varphi \in R^k$, g is a square matrix of order k.*

The elements of φ are continuous bounded deterministic functions on R^r.

Let us suppose that the real parts of the$\lambda - roots$ of the equation

Satisfy the inequality

$Re\lambda(x,t,\sigma) \le -\delta|\sigma|^{2m},$

σ $\in R^r$, $|\sigma|^2 = \sigma_1^2 + \cdots + \sigma_r^2, \sigma^q = \sigma_1^{q_1} \dots \sigma_r^{q_r}$, δ *is a positive constant* **and I is the unit matrix of order n.**

Following Edelman [ ] we suppose that the elements of thecoefficientsA_q, $|q| = 2m$, are continuous in $t \in [0,T]$, moreover , the continuity in t is uniform with respect to $x \in R^r$.It is supposed also that these elements are deterministic bounded on $R^r x[0,T]$ and satisfy the Holder condition with respect to x.

Under these conditions there exists for the system:

a fundamental solution matrix $Q(x,y,t,s)$ satisfies the following conditions:

$\frac{\partial Q}{\partial t}, D^q Q \in C(R^{2r} x[0,T]x[0,T])$, *where $C(S)$ is the set of all matrices with continuous bounded element on a region S, also G satisfies the following inequality:*

$|D^q Q(x,t,y,s)| \le \frac{A_1}{(t-s)^\beta} exp\Lambda, t > s, \Lambda = \frac{-A_2|x-y|^{2m}}{t-s}, |x|^2 = x_1^2 + \cdots + x_r^2$, A_1, A_2 *are positive* **constants ,** $\beta = \frac{1}{2}(r + |q|), |q| \le 2m$, $|Q|^2 = \sum_{i,j=1}^{k}(\eta_{i,j})^2$, $G = (\eta_{i,j})$,[ ].

Now the problem (3.1), (3.2) can be written in the form:

$$u(x,t) = \int_{R^r} Q(x,y,t,0)\varphi(+ \int_0^t \int_{R^r} Q(x,y,t,s)j$$

We say that an R^k –valued stochastic process $u(.,.)$ is a solution of the fractional stochastic integral system (3.3) for$x \in R^r 0 \le t \le T$, provided

(i) $u(.,.)$ **is progressively measurable with respect to $\mathcal{F}(.)$ for every fixed** $x \epsilon R^r$

(ii) $(x,t,u) \in \mathcal{L}_k^1(0,T)$***, for every fixedx*** $\in R^r$

(iii) $g(x,t,u) \in \mathcal{L}_{kxk}^2(0,T)$ ***, for every fixed*** $x \epsilon R^r$

and $u(x,t)$ ***satisfies equation (3.3).***

Theorem. Suppose that $f: R^r x[0,T]xR^k \to R^k$ and $g: R^r x[0,T]xR^k \to Z^{kxk}$ are continuous and satisfy **the following conditions:**

(i) $|f(x,t,u) - f(x,t,v)| \leq L|u - v|$
$|g(x,t,u) - g(x,t,v)| \leq L|u - v|$,
For all $0 \leq t \leq T, x \epsilon R^r, u, v \in R^k$

(ii) $|f(x,t,u)| \leq L(1 + |u|)$

for all $0 \leq t \leq T, x \in R^r, u \in R^k$
for some positive constant L
Then there exists a unique stochastic solution $u \in \mathcal{L}_k^2(0,T)$, for every $x \epsilon R^r$, of the fractional stochastic integral equation (3.3), (by unique solution , we mean that if $u, u^* \in \mathcal{L}_k^2(0,T)$, *for every* $x \in R^r$, *with continuous sample paths almost surely , and both solves equation (3.3), then* $P(u(x,t) = u^*(x,t), for\ all\ x \in R^r, 0 \leq t \leq T) = 1$.
Notice that hypothesis (i) says that $f\ and\ g$ **are uniformly Lipschitz continuous in the variable** u**. Notice also that hypothesis (ii) actually follows from (i).**
Proof. Let nus prove now the uniqueness. Suppose that $u\ and\ v$ **are solutions of (3.3). Then for all** $x \in R^r$, $0 \leq t \leq T$,

$$u(x,t) - v(x,$$

It is easy to get the following estimation

$$E\{|u(x,t) - v(x,t)|^2\} \leq$$
$$2E\left\{\left|\int_0^t \int_{R^r} Q(x,y,t,s)[f(y,s,u(y,s)) - f(y,s,v((y,s)))]\, dyds\right|^2\right\} +$$
$$2E\left\{\ \left|\int_0^t \int_{R^r} Q(x,y,t,s)\big[g(y,s,u(y,s)) - g(y,s,v(y,s))\big]dydW_\propto(s)\right|^2\right\}.$$

According to our results in [1] and the properties of the fundamental matrix solution Q **we get by using Schwartz inequality and the Lipchitz condition (i) the following estimation:**

$$Sup_{x \epsilon R^r} E\{|u - v|$$

for some constant $K > 0$ *and all* $0 \leq t \leq T$.
Consequently

$$Su$$

For some constant $K^* > 0$ and all $0 \leq t \leq T$.
Thus according to fractional $Gronwall'lemma$**, we get** $u(x,t) = v(x,t)$ **almost surely for all** $x \in R^r, 0 \leq t \leq T$. As $u\ and\ v$ have continuous sample paths almost surely,

$$P(|$$

To prove the existence, we define a sequence $\{u^n(x,t)\}$ **by:**

$$u^{n+1}(x,t) = \int_{R^r} Q(x,y,t,0)\varphi(y)dy + \int_0^t \int_{R^r} Q(x,y,t,s)f(y,s,u^n(y,s)dyds +$$
$$\int_0^t \int_{R^r} Q(x,y,t,s)g(y,s,u^n(y,s)dydW_\alpha(s),$$

for $n = 0,1, \ldots and\ x \in R^r, 0 \leq t \leq T, u^0(x,t) = \varphi(x)$.Define also

For $n = 0$, we have

$d^0(x,t) \leq$

Thus we can find a constant $K > 0$ *such that:*

$d^0(x,t) \leq K \frac{t^\alpha}{\Gamma(\alpha+1)}$.

It is easy to get by induction

for some constant $K > 0$.
According to the Martingale inequality

we get
$E[Max_{0 \leq t \leq T}|u^{n+1}(x,t) - u^n(x.t)|^2] \leq \boldsymbol{K^{n+1}} \frac{\boldsymbol{T^{(n+1)\alpha}}}{\boldsymbol{n!(\Gamma(\alpha+1))^{n+1}}}$.

Applying Borel-Cantelli lemma, we deduce that for every $\omega \in \Omega$

Converges uniformly on $R^r x[0,T]$ to a stochastic process $u(x,t)$.
We pass to limits in the definition of $u^n(x,t)$, to prove

$u(x,$

For all $x \epsilon R^r, 0 \leq t \leq T$.

Let us prove now that the considered stochastic solution u is an element of $\mathcal{L}_k^2(0,T)$.

We can find a constant $K > 0$ such that:

$E[|u^{n+1}($

Using condition (ii) and the properties of $Q(x, y, t, s)$, we get by induction:

$E\{|u^{n+}$

For all $x \in R^r, 0 \leq t \leq T. Taking\ the\ limit\ as\ n\ tends\ to\ infinity\ we\ get$ **the required** *result.*

4-Conclussion

New fractional Brownian motion is studied. Some results of Ito about vectors of fractional Brownian motion are generalized. Parabolic systems driven by new vectors of fractionalBrownian motion are studied, see [8-16].

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