# Addressing Autocorrelation, Multicollinearity, And Heavy-Tail Errors in Linear Regression Model


## ABSTRACT

*The most popular estimator for estimating parameters of linear regression models is the Ordinary Least Squares(OLS) Estimator. The OLS is considered the Best Linear Unbiased Estimator when certain assumptions are not violated. However, when autocorrelation multicollinearity and heavy-tail error are jointly present in the dataset theOLS estimator is found to be inefficient and imprecise. In this paper, we developed an estimator of linear regression model parameters that handle multicollinearity, autocorrelation, and heavy tail errors jointly. The new estimator, LADHLKL, was derived by combining the Hildreth-Lu (HL), the Kibria Lukman (KL) and the Least Absolute Deviation (LAD) estimators. The LADHLKL poses both the characteristics of the LAD, HL, and KL estimators which makes it resistant to both problems. We examine the properties of the proposed estimator and compare its performance with other existing estimators in terms of mean square error. An application to real-life data and simulation study revealed that the proposed estimator dominates other estimators in all the considered conditions in terms of mean square error.*



**Keywords**: LAD estimator; Hildreth-Lu estimator; Kibria-Lukman estimator; MSE; Multicollinearity; Autocorrelation; Heavy-Tail residuals; Simulation Study.


## 1. INTRODUCTION

Multiple linear regression is a technique in statistics that describes the relationship between the response variable and at least two other variables called explanatory variables. The OLS is the most popular estimator used in estimating the linear regression model parameters. In the linear regression model, both the assumptions of uncorrelated regressors and uncorrelated residuals can jointly be violated [1]. If these two assumptions are jointly violated, the OLS as the usual estimator used in estimating the parameters of

the linear regression model will be inefficient. To jointly combat these problems, [2] Proposed some combined estimators that efficiently estimate the parameters of linear regression models in the presence of multicollinearity and autocorrelated errors. [3], [4] proposed Two Stages Ridge estimator when the assumptions of uncorrelated predictors and uncorrelated errors in the linear regression model are jointly violated. [5]combined the Praise-Winston estimator with the Kibria-Lukman estimator to handle the joint effects of multicollinearity and serially correlated residuals in the linear regression model.

Multicollinearity and outliers are among the problems that are jointly encountered in regression analysis. The usual OLS estimator gives unfavorable results when Multicollinearity and outliers jointly exist in data [6]. In the joint presence of outliers and multicollinearity, the ridge regression, Liu, principal components, stein estimators, etc, and other robust estimators such as M, MM, S, LTS, LMS, and LAD yield inefficient results[7]. [6] proposed a two-parameter estimator called Ridge-Type Modified M-Estimator to circumvent the joint effect of multicollinearity and outliers in the linear regression model. [7] also proposed some quantile-based ridge M-estimators to combat y-direction outliers and multicollinearity simultaneously under various distributions of error terms.

The threat posed by the autocorrelation becomes more complicated when this assumption violation comes together with outliers [8]. Researchers who erroneously assumed independent residuals and the complete absence of outliers obtained inefficient estimates as consequences [8]. In the joint presence of autocorrelation and outliers, the OLS and some of its modifications are inefficient [9], [10]. In regression models, it is commonly found that the response variable contains a significant number of outliers, and the residual term is serially correlated. These assumptions violations unfavorably affect model estimation [11]. To address these problems, [10] proposed a robust modified two-stage least squares to handle the simultaneous effects of autocorrelation and outliers in non-linear regression.

The aim of this study is to develop estimator of linear regression model parameters that handle multicollinearity, autocorrelation, and heavy-tail errors jointly with the following objectives:

i. To propose estimator that jointly handle multicollinearity, autocorrelation, and heavy-tail outliers in the residuals.

ii. To examine the properties of the proposed estimators.

iii. To identify the most efficient estimator among the proposed and considered estimators when autocorrelation, multicollinearity, and heavy-tail errors jointly exist.

iv. To apply the proposed estimators to real-life data.

## 2. MATERIAL AND METHODS

Consider the linear regression model given by:

$$y = X\beta + u \tag{1}$$

Where $y$ is a nx1 random vector of a response variable, X is a nxp matrix with full rank, $\beta$ is a px1 vector of estimable parameters, and $u$ is a nx1 random vector of residuals, distributed as $N(0, \sigma^2 I_n)$.

Let T be an orthogonal matrix, satisfying $T^T X^T XT = \Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_p)$ where $\Lambda$ is a diagonal matrix of order p x p with diagonal elements $\lambda_1, \lambda_2, ..., \lambda_p$ as the eigenvalues of $X^T X$. T and $\Lambda$ are the matrices of eigenvectors and eigenvalues, respectively. Hence, the canonical form of model (1) is:

$$y = Z\alpha + u \tag{2}$$

Where $Z = XT$ and $\alpha = T^T \beta$.

The OLS estimator of $\alpha$ is:

$$\hat{\alpha}_{OLS} = \Lambda^{-1} Z^T y \tag{3}$$

$$MSE(\hat{\alpha}_{OLS}) = \sigma^2 \Lambda^{-1} \tag{4}$$

If the residuals of model (2) above are generated by a first-order autoregressive process given by:

$$u_t = \rho u_{t-1} + \varepsilon_t \tag{5}$$

Where $|\rho| < 1$, $\varepsilon_t$ are not necessarily normal but are independent and identically distributed.

Here, $E(u_t) = 0,$ and $E(u_t u_t^T) = \delta_u^2 \Omega \neq \delta_u^2 I$ where $\Omega$ is a positive definite symmetricmatrix. Under this situation, $\hat{\alpha}_{OLS}$, although linear unbiased, is found to be inefficient.

The feasible Generalized Least Squares estimator suggested by [12] is among the appropriate estimators for estimating linear regression model's parameters when the residuals are correlated. The procedure is:

From $E(u_t u_t^{'}) = \delta_u^2 \Omega = \delta_u^2 \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-1} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-1} & \rho^{n-1} & \cdots & 1 \end{pmatrix}$ (6)

Let V be an n×n non-singular matrix such that $\Omega^{-1} = V^{'}V$ defined by:

$$V = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{pmatrix} \quad (7)$$

The variables in equation (2) above are transformed as:

$$y^* = Vy,\ Z^* = VZ,\ u^* = Vu \quad (8)$$

Hence,

$$y^* = Z^* \alpha + u^* \quad (9)$$

Using the OLS estimator, estimate the coefficients of the model in equation (9). The resulting estimator $\hat{\alpha}_{GLS}$ of $\alpha_{GLS}$ is called the HL estimator and is given by

$$\hat{\alpha}_{GLS} = (Z^T \Omega^{-1} Z)^{-1} Z^T \Omega^{-1} y$$

$$\hat{\alpha}_{GLS} = \Lambda^{*-1} Z^{*T} y^{*} \tag{10}$$

MSE of GLS:

$$MSE(\hat{\alpha}_{GLS}) = \frac{\delta^2}{\Lambda^*} \tag{11}$$

Often, the regressors involved in regression analysis are found to be linearly related. This condition renders the OLS estimator inefficient by inflating the coefficient estimates' variances, leading to erroneous conclusions. To address this problem, [13] introduced a biased estimator, called KL defined as:

$$\hat{\alpha}_{KL} = (\Lambda + kI_p)^{-1}(\Lambda - kI_p)\hat{\alpha}_{OLS}$$

Where k>0 is the biasing parameter.

$$\hat{\alpha}_{KL} = R(k)H(k)\hat{\alpha}_{OLS} \tag{12}$$

Where $R(k) = (\Lambda + kI_p)^{-1}$ and $H(k) = (\Lambda - kI_p)$

Bias, variance-covariance, and Mean Square Error matrix of the KL estimator are:

$$Bias(\hat{\alpha}_{KL}) = (R(k)H(k) - I_P)\alpha \tag{13}$$

$$\mathrm{cov}(\hat{\alpha}_{KL}) = \delta^2 R(k)H(k)\Lambda^{-1}(H(k)R(k))^T \tag{14}$$

$$MSEM(\hat{\alpha}_{KL}) = \delta^2 R(k)H(k)\Lambda^{-1}(H(k)R(k))^T + (R(k)H(k) - I_P)\alpha\alpha^T(R(k)H(k) - I_P)^T \tag{15}$$

When the residuals in equation (2) above have a heavy-tail distribution, the usual OLS estimator will be affected. [14] proposed the LAD estimator that minimizes the sum of the absolute deviations of the residuals. This estimator is less affected by the heavy-tailed errors than the OLS estimator in equation (3), the HL estimator in equation (10), and the KL estimator in equation (12). The LAD estimator is defined as:

$$\hat{\alpha}_{LAD} = \min \sum_{t=1}^{n} |u_t| \tag{16}$$

The MSEM of the LAD estimator is given by

$$MSE(\hat{\alpha}_{LAD}) = tr\left(\Omega\right) \text{ where } \Omega = \text{cov}(\hat{\alpha}_{LAD}) = \delta_{LAD}^{2}\Lambda^{-1} \tag{17}$$

But apart from the autocorrelated residuals, if at least two of the independent variables in equation (2) above are linearly related, then the OLS estimator in equation (3), the HL estimator in equation (10) and the KL estimator in equation (12) would be inefficient. To simultaneously combat these two problems, [5]suggested the generalized KL (GKL) estimator given by

$$\hat{\alpha}_{GLKL} = R^{*}(k)H^{*}(k)\hat{\alpha}_{GLS}, k > 0 \tag{18}$$

Where $R^{*}(k) = (\Lambda^{*} + kI_{p})^{-1}$ and $H^{*}(k) = (\Lambda^{*} - kI_{p})$

Their estimator is also biased. The Bias, Covariance and the MSE matrices of their estimator are:

$$Bias(\hat{\alpha}_{GLKL}) = (R^{*}(k)H^{*}(k) - I_{P})\alpha \tag{19}$$

$$\text{cov}(\hat{\alpha}_{GLKL}) = \delta^{2}R^{*}(k)H^{*}(k)\Lambda^{*-1}(R^{*}(k)H^{*}(k))^{T} \tag{20}$$

$$MSEM(\hat{\alpha}_{GLKL}) = \delta^{2}R^{*}(k)H^{*}(k)\Lambda^{*-1}(R^{*}(k)H^{*}(k))^{T} + (R^{*}(k)H^{*}(k) - I_{P})\alpha\alpha^{T}(R^{*}(k)H^{*}(k) - I_{P})^{T} \tag{21}$$

If, from the model (2), the independent variables involved are correlated and the residual term has heavy-tail outliers, the $\hat{\alpha}_{OLS}$, $\hat{\alpha}_{KL}$ and $\hat{\alpha}_{LAD}$ in equations (3), (12), and (16) respectively will be inefficient. [15]suggested a robust KL M estimator to jointly address outliers in the response variable and multicollinearity given by:

$$\hat{\alpha}_{MKL} = (\Lambda + kI)^{-1}(\Lambda - kI)\hat{\alpha}_{M} \tag{22}$$

Where $\hat{\alpha}_{M}$ is the robust-M estimator.

Bias, covariance, and the MSE matrices of their Robust KL M estimator are:

$$Bias(\hat{\alpha}_{MKL}) = (R(k)H(k) - I_{p})\alpha \tag{23}$$

$$\operatorname{cov}(\hat{\alpha}_{MKL}) = R(k)H(k)\Omega(R(k)H(k))^T \tag{24}$$

where $\Omega = \operatorname{var}(\hat{\alpha}_M)$

$$MSEM(\hat{\alpha}_{MKL}) = R(k)H(k)\Omega(R(k)H(k))^T + (R(k)H(k) - I_P)\alpha\alpha^T(R(k)H(k) - I_P)^T \tag{25}$$

Hence, the robust KL base on LAD estimator is used to combat multicollinearity and heavy-tail outliers in the residuals simultaneously, and is given by:

$$\hat{\alpha}_{LADKL} = (\Lambda + kI)^{-1}(\Lambda - kI)\hat{\alpha}_{LAD}$$

$$\hat{\alpha}_{LADKL} = R(k)H(k)\hat{\alpha}_{LAD} \tag{26}$$

The Bias, variance-covariance, and MSE matrices of the LAD KL estimator are:

$$Bias(\hat{\alpha}_{LADKL}) = (R(k)H(k) - I_P)\alpha \tag{27}$$

$$\operatorname{cov}(\hat{\alpha}_{LADKL}) = R(k)H(k)\Omega(R(k)H(k))^T \tag{28}$$

Where $\Omega = \operatorname{cov}(\hat{\alpha}_{LAD}) = \delta_{LAD}^2\Lambda^{-1}$

$$MSEM(\hat{\alpha}_{LADKL}) = R(k)H(k)\Omega(R(k)H(k))^T + (R(k)H(k) - I_P)\alpha\alpha^T(R(k)H(k) - I_P)^T \tag{29}$$

However, in many reported cases where the residuals, apart from following the autoregressive of first order, have heavy-tail outliers, the $\hat{\alpha}_{OLS}$, $\hat{\alpha}_{GLS}$ and $\hat{\alpha}_{LAD}$ were found to be inefficient and imprecise. To overcome these two challenges, Robust Least Absolute Deviation (LAD) is applied to the transformed data (autocorrelated free data). The robust LAD estimator was used to estimate the parameters of the model in equations (9). The resulting estimator is called the LAD Hildreth-Lu estimator and is given by:

$$\hat{\alpha}_{LADHL} = \min \sum_{t=1}^{n} \left| u_t^* \right| \tag{30}$$

Where $u_t^*$ is the autocorrelated residual in equation (9).

The MSE of the LADHL estimator is given by

$$MSE(\hat{\alpha}_{LADHL}) = tr\left(\Omega^*\right) \text{ where } \Omega^* = \operatorname{cov}(\hat{\alpha}_{LADHL}) = \frac{\delta^2_{LADHL}}{\Lambda^*} \tag{31}$$

## 2.1 The Proposed Estimator

$\hat{\alpha}_{LADKL}$ is resistant to both heavy-tail outliers and multicollinearity, though sensitive to autocorrelation. $\hat{\alpha}_{HLKL}$ is sensitive to heavy-tail outliers but resistant to autocorrelation and multicollinearity. $\hat{\alpha}_{LADHL}$ is resistant to heavy-tail outliers and autocorrelation but sensitive to multicollinearity. To jointly address the problems of heavy-tail errors, autocorrelation, and multicollinearity, we proposed the LAD HL KL estimator given by

$$\hat{\alpha}_{LADHLKL} = (\Lambda^* + kI_p)^{-1}(\Lambda^* - kI_p)\hat{\alpha}_{LADHL}, k > 0 \tag{32}$$

$$\hat{\alpha}_{LADHLKL} = R^*(k)H^*(k)\hat{\alpha}_{LADHL} \tag{33}$$

Where $\hat{\alpha}_{LADHL}$ is the LAD HL estimator in equation (30).

### 2.1.1 Properties of the Proposed Estimator

Bias, variance-covariance, and Mean Square Error matrix of the LAD HL KL estimator are:

**Bias:**

$$Bias(\hat{\alpha}_{LADHLKL}) = E[\hat{\alpha}_{LADHLKL}] - \alpha$$

$$= R^*(k)H^*(k)E(\hat{\alpha}_{LADHL}) - \alpha$$

$$= R^*(k)H^*(k)\alpha - \alpha$$

$$Bias(\hat{\alpha}_{LADHLKL}) = (R^*(k)H^*(k) - I_P)\alpha \tag{34}$$

**Variance:**

$$\operatorname{cov}(\hat{\alpha}_{LADHLKL}) = \operatorname{cov}(R^*(k)H^*(k)\hat{\alpha}_{LADHL})$$

$$\operatorname{cov}(\hat{\alpha}_{LADHLKL}) = (R^*(k)H^*(k)\operatorname{cov}(\hat{\alpha}_{LADHL})((R^*(k)H^*(k))^T$$

$$\operatorname{cov}(\hat{\alpha}_{LADHLKL}) = R^*(k)H^*(k)\Omega^*(R^*(k)H^*(k))^T \tag{35}$$

Where $\Omega^* = \operatorname{cov}(\hat{\alpha}_{LADHL}) = \dfrac{\delta^2_{LADHL}}{\Lambda^*}$

**Mean Square Error Matrix:**

The mean square error matrix is given by

$$MSEM(\hat{\alpha}_{LADHLKL}) = E((\hat{\alpha}_{LADHLKL} - \alpha)(\hat{\alpha}_{LADHLKL} - \alpha)^T)$$

$$MSEM(\hat{\alpha}_{LADHLKL}) = \operatorname{var}(\hat{\alpha}_{LADHLKL}) + Bias(\hat{\alpha}_{LADHLKL})Bias(\hat{\alpha}_{LADHLKL})^T)$$

$$MSEM(\hat{\alpha}_{LADHLKL}) = R^*(k)H^*(k)\Omega^*(R^*(k)H^*(k))^T + (R^*(k)H^*(k) - I_P)\alpha\alpha^T((R^*(k)H^*(k) - I_P))^T$$
(36)

Hence, the scalar MSE is given by

$$MMSE(\hat{\alpha}_{LADPWKL}) = tr\left[\operatorname{var}(\hat{\alpha}_{LADHLKL}) + Bias(\hat{\alpha}_{LADHLKL})(Bias(\hat{\alpha}_{LADHLKL}))^T\right]$$

$$SMSE(\hat{\alpha}_{LADHLKL}) = \sum_{j=1}^{p} \frac{(\lambda_j^* - k)^2}{(\lambda_j^* + k)^2}\Omega^*{}_{jj} + 4k^2 \sum_{j=1}^{p} \frac{\alpha_j^2}{(\lambda_j^* + k)^2} \tag{37}$$

## 2.2 Robust Choice of The Biasing Parameter

Using the procedure of optimization to derive the biasing parameter of an estimator became a tradition [16]. Rewriting equation (37) as a function of k.

$$f(k) = \sum_{j=1}^{p} \frac{(\lambda_j^* - k)^2}{(\lambda_j^* + k)^2}\Omega^*{}_{jj} + 4k^2 \sum_{j=1}^{p} \frac{\alpha_j^2}{(\lambda_j^* + k)^2} \tag{38}$$

The procedure is to minimize equation (38). This is achieved by setting $\frac{\partial f(k)}{\partial k} = 0$ and by proceeding will yield an estimation for the biasing parameter k which is complex. Hence,

we proposed to use a robust equivalence of the biasing parameter used for the KL estimator [13]. The estimate of shirinkage parameter used for the KL estimator is presented in equation (39).

$$\hat{k} = \min\left(\frac{\hat{\delta}^2}{2\hat{\alpha}_j^2 + (\hat{\delta}^2/\lambda_j)}\right) \tag{39}$$

The robust equivalence of the biasing parameter in equation (39) is given by

$$\hat{k}_{LADHL} = \min\left(\frac{\hat{\sigma}^2_{LADHL}}{2\hat{\alpha}^2_{j,LADHL} + \hat{\delta}^2_{LADHL}/\lambda_j^*}\right) \tag{40}$$

Where $\hat{\alpha}_{LADHL}$ is the estimator of $\alpha_{LADHL}$ as given in equation (30), p is the number of explanatory variables, $\lambda_j^*$ is the j[th] diagonal element of $\Lambda^*$, and $\hat{\sigma}^2_{LADHL} = \frac{\sum_{i=1}^{n} u^2_{LADHL}}{n-p}$, with the assumption that $\hat{\alpha}_{LADHL} \sim N\left(\alpha, \omega^2\left(\frac{1}{n}\Lambda^*\right)^{-1}\right)$, and it holds since $\sqrt{n}(\hat{\alpha}_{LADHL} - \alpha) \to N\left(0, \omega^2\left(\frac{1}{n}\Lambda^*\right)^{-1}\right)$ where $\omega = [2f(0)]^{-1}$ is a nuisance parameter, and $f(0)$ is the height of the density of the errors at zero, their median [17].

### 2.3 Monte Carlo Simulation

#### 2.3.1 Simulation Technique

A simulation study was performed using R software to evaluate the performances of OLS, HL, LAD, KL, LADHL, LADKL, HLKL, and LADHLKL. The regressors were generated using the procedure adopted by [13], [18]–[20]. The procedure is:

$$x_{tj} = \left(1-\gamma^2\right)^{\frac{1}{2}} c_{tj} + \gamma c_{t,p+1} \tag{41}$$

$$t = 1, 2, ..., n,\ j = 1, 2, ..., p,$$

Where $c_{tj}$ are the pseudo-random numbers from the standard normal distribution and $\gamma$ denote the correlation between any two explanatory variables which are set as $\gamma$ = 0.85, 0.95, and 0.99, p is the number of the regressors considered, which is 3. The variables are standardized such that $X^T X$ and $X^T y$ are in correlation forms. The n observations of the response variable are generated by

$$y_i = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} + ... + \beta_p x_{tp} + u_t \tag{42}$$

$$t = 1, 2, ..., n,$$

$$u_t = \rho u_{t-1} + \varepsilon_t \text{ such that } u_t \sim N(0, \delta^2 \Omega) \tag{43}$$

Where $\rho$ is the autocorrelation coefficient and was set to $\rho$ = 0.85, 0.95, and 0.99, and the error variance was set as $\delta_u^2$ =0.5, 1, and 4. For the model, we assume zero intercept and the values of the coefficients are selected such that $\beta^T \beta = 1$.

Outliers are being introduced by increasing the magnitude of the tail values of $\varepsilon_t$. The procedure is

$$\varepsilon_{(t:1 \le t < (n-n_{outlier}))} + \left(25 * \max\left(\varepsilon_t\right) + \varepsilon_{(t:(n-n_{outlier}) \le t \le n)}\right) \tag{44}$$

Where $n_{outlietr}$ is the number of outliers along the tail of $\varepsilon_t$.

Three (3) sample sizes were used: n = 10, 30, and 50, and three (3) levels of percentages of outliers were considered ($n_{outlier}$%= 10%, 20%, 30%).

By implication, 1000 replications of these experiments are carried out and the MSE is estimated as

$$MSE(\hat{\alpha}) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\alpha}_{tj} - \alpha_t)^T(\hat{\alpha}_{tj} - \alpha_t) \tag{45}$$

## 3. RESULTS AND DISCUSSION

### 3.1 Simulation Results and Discussion

The Monte Carlo simulation's results of the estimators are presented in Table 1-9.

From the simulation results, the following observations were made:

1. Our proposed LADHLKL estimator generally outperformed other estimators in all the combinations of situations considered.
2. The OLS estimator generally performed worst when autocorrelation, multicollinearity, and heavy-tail errors jointly exist.
3. The MSE values generally increases for all the estimators as the error variance increases.
4. The MSE values of the estimators decreases when the sample size increases, generally.
5. Increase in the degree of multicollinearity lead to the increase in the MSE values of the estimators.
6. The MSE values of the estimators increase when the degree of autocorrelation increases.
7. Increase in the percentage of outliers leads to the increase in MSE values.

### 3.2 Numerical Application

We used the data that represents the product in the manufacturing sector, the imported intermediate and capital commodities, and imported raw materials, in Iraq in the period from 1960 to 1975. Though we considered the 1960 to 1975 period for the data, [3], [21]–[24] used the same dataset for the period from 1960 to 1990. The regression model is given as follows:

$$y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \beta_3 X_{3t} + u_t \tag{46}$$

The method of Ordinary Least Squares was used to estimate the parameters of model (46). Based on the fitted model, the statistics required for the various diagnostics are presented in Table 10.

From the correlation matrix of the predictors, it can be seen that the estimated correlations between the independent variables $\gamma_j$ are in excess of 0.870 and the Variance Inflation Factor $VIF_j$ for j = 1, 2, 3, as can be seen from Table 10 are all greater than 30. This implies that the data suffer from multicollinearity.

Also, the Durbin-Watson statistic value DW (0.803) <dl (0.860), with a p-value of 0.001. This implies that the model suffers from autoregressive of order 1 (AR (1)).

Table 10 also shows that the value of the Kurtosis for the residuals, K is greater than three (K>3), which indicate that the residuals is heavily skewed to the right.

Therefore, apart from having a heavy-tail residual, the data suffers from both multicollinearity and autocorrelation.

We used the proposed LADHLKL and seven other considered estimators to estimate the coefficients of model (46). The estimates of the regression coefficients, their corresponding average MSE values, and ranks are presented in Table 11.

The results of the real-life application show that our proposed LADHLKL estimator had the smallest MSE value and therefore, performed better than other considered estimators. This result is consistent with the simulation results.

## 4. CONCLUSION

The assumptions of the absence of multicollinearity among the regressors, independent and normally distributed residuals are sometimes jointly violated in multiple linear regression models, yet have not been sufficiently studied.

In this research, we introduced a Robust Feasible Generalized Kibria-Lukman (LADHLKL) estimator to mitigate the joint effect of multicollinearity, autocorrelation, and heavy-tail outliers in residuals in the linear regression models. We derived the statistical properties of the proposed estimator. We compared the performance of the proposed LADHLKL estimator with other existing estimators based on the mean square error (MSE) criterion through simulation and real-life application. Both the results from the numerical example and the simulation study showed that our proposed estimator outperformed other estimators. The proposed estimator is, therefore, recommended for researchers in handling the joint

effect of multicollinearity, autocorrelation, and heavy-tail errors in a linear regression model. For future research, we plan to extend this study to generalized linear models.

## COMPETING INTERESTS

The authors declare that they have no conflicts of interest.

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**Table 1: Estimated MSE values when the error variance = 0.5, n = 10**

| % of Outlier | ρ | ▪ | OLS | HL | LAD | HLLAD | KL | HLKL | LADKL | HLLADKL |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.850 | 171.850 | 167.313 | 3.948 | 2.979 | 57.120 | 52.581 | 0.854 | 0.545 |
| | 0.85 | 0.950 | 521.054 | 515.607 | 11.095 | 8.751 | 170.142 | 159.604 | 2.027 | 1.376 |
| | | 0.990 | 2662.446 | 2640.594 | 56.205 | 46.216 | 854.159 | 805.721 | 9.329 | 7.371 |
| | | 0.850 | 172.782 | 167.744 | 10.752 | 7.225 | 56.735 | 51.954 | 3.358 | 2.351 |
| 10% | 0.95 | 0.950 | 524.316 | 514.052 | 26.886 | 19.562 | 170.030 | 155.752 | 7.022 | 5.963 |
| | | 0.990 | 2682.937 | 2628.849 | 133.217 | 95.701 | 859.987 | 787.690 | 29.946 | 28.565 |
| | | 0.850 | 173.945 | 172.518 | 53.383 | 29.718 | 53.625 | 51.701 | 21.682 | 14.212 |
| | 0.99 | 0.950 | 528.623 | 523.409 | 127.379 | 80.150 | 163.743 | 152.859 | 45.126 | 37.759 |
| | | 0.990 | 2720.707 | 2660.798 | 544.818 | 390.588 | 851.880 | 759.942 | 167.828 | 185.367 |
| | | 0.850 | 687.952 | 197.826 | 4.305 | 1.659 | 223.138 | 28.763 | 0.801 | 0.150 |
| | 0.85 | 0.950 | 2003.240 | 508.808 | 12.276 | 5.101 | 622.531 | 59.613 | 2.043 | 0.332 |
| | | 0.990 | 9945.791 | 2283.137 | 61.016 | 25.098 | 2986.876 | 223.022 | 9.461 | 1.263 |
| | | 0.850 | 758.495 | 233.403 | 11.567 | 1.601 | 247.299 | 35.392 | 2.916 | 0.144 |
| 20% | 0.95 | 0.950 | 2214.230 | 611.247 | 31.043 | 4.819 | 693.740 | 76.874 | 7.165 | 0.315 |
| | | 0.990 | 11014.820 | 2778.822 | 150.442 | 23.366 | 3343.373 | 298.453 | 32.651 | 1.279 |
| | | 0.850 | 787.724 | 248.173 | 56.008 | 2.021 | 253.732 | 38.246 | 18.573 | 0.212 |
| | 0.99 | 0.950 | 2304.366 | 654.002 | 145.678 | 6.105 | 719.248 | 84.430 | 45.017 | 0.392 |
| | | 0.990 | 11493.084 | 2986.618 | 702.271 | 29.184 | 3507.063 | 331.958 | 209.842 | 1.561 |
| | | 0.850 | 1421.815 | 391.260 | 4.753 | 2.545 | 416.174 | 86.622 | 1.015 | 0.238 |
| | 0.85 | 0.950 | 4235.762 | 1184.063 | 12.399 | 12.064 | 1217.035 | 261.653 | 2.246 | 0.806 |
| | | 0.990 | 21416.884 | 6008.384 | 61.372 | 92.610 | 6056.272 | 1317.118 | 10.300 | 5.616 |
| 30% | | 0.850 | 1724.258 | 498.788 | 12.788 | 2.344 | 514.410 | 114.816 | 3.382 | 0.234 |
| | 0.95 | 0.950 | 5134.572 | 1497.830 | 30.468 | 13.066 | 1502.639 | 341.403 | 7.018 | 1.126 |
| | | 0.990 | 25950.784 | 7554.185 | 143.468 | 98.629 | 7471.364 | 1696.965 | 31.518 | 8.427 |
| | 0.99 | 0.850 | 1858.946 | 548.156 | 60.894 | 2.657 | 558.709 | 127.776 | 19.151 | 0.306 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.950 | 5539.193 | 1641.920 | 141.751 | 24.052 | 1639.743 | 377.947 | 40.013 | 3.142 |
| | | 0.990 | 28020.038 | 8263.977 | 653.303 | 172.310 | 8194.601 | 1870.526 | 180.424 | 25.883 |

**Table 2: Estimated MSE values when the error variance = 0.5, n = 30**

| % of Outlier | ρ | ▪ | OLS | HL | LAD | HLLAD | KL | HLKL | LADKL | HLLADKL |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.850 | 106.719 | 8.108 | 1.265 | 0.155 | 35.820 | 1.175 | 0.112 | 0.033 |
| | 0.85 | 0.950 | 201.888 | 23.009 | 3.373 | 0.411 | 45.893 | 3.302 | 0.233 | 0.023 |
| | | 0.990 | 752.410 | 110.860 | 15.865 | 1.943 | 120.280 | 15.718 | 0.880 | 0.077 |
| | | 0.850 | 125.960 | 10.229 | 5.255 | 0.149 | 40.749 | 1.578 | 1.115 | 0.032 |
| 10% | 0.95 | 0.950 | 247.390 | 29.720 | 14.075 | 0.401 | 58.534 | 4.882 | 2.378 | 0.021 |
| | | 0.990 | 951.318 | 145.659 | 65.821 | 1.916 | 180.921 | 25.252 | 9.704 | 0.071 |
| | | 0.850 | 138.080 | 11.381 | 35.785 | 0.180 | 46.359 | 2.020 | 16.982 | 0.035 |
| | 0.99 | 0.950 | 280.813 | 33.300 | 90.733 | 0.493 | 75.824 | 6.327 | 36.016 | 0.026 |
| | | 0.990 | 1111.871 | 164.108 | 397.612 | 2.378 | 251.031 | 32.913 | 144.675 | 0.093 |
| | | 0.850 | 316.520 | 11.455 | 1.288 | 0.185 | 93.776 | 1.652 | 0.071 | 0.037 |
| | 0.85 | 0.950 | 475.720 | 33.685 | 3.402 | 0.485 | 106.222 | 4.438 | 0.119 | 0.022 |
| | | 0.990 | 1669.627 | 168.706 | 16.217 | 2.300 | 236.746 | 22.500 | 0.425 | 0.063 |
| | | 0.850 | 491.612 | 14.656 | 5.504 | 0.179 | 146.191 | 1.884 | 0.312 | 0.036 |
| 20% | 0.95 | 0.950 | 760.786 | 43.963 | 14.313 | 0.470 | 167.560 | 5.033 | 0.557 | 0.020 |
| | | 0.990 | 2665.315 | 222.332 | 68.524 | 2.229 | 395.432 | 25.812 | 2.105 | 0.058 |
| | | 0.850 | 591.930 | 16.742 | 33.414 | 0.229 | 179.059 | 2.137 | 4.513 | 0.040 |
| | 0.99 | 0.950 | 936.987 | 50.667 | 89.597 | 0.609 | 205.221 | 5.678 | 9.848 | 0.025 |
| | | 0.990 | 3315.553 | 257.444 | 394.421 | 2.888 | 496.005 | 29.182 | 32.877 | 0.076 |
| | | 0.850 | 997.066 | 2.946 | 2.147 | 0.203 | 456.666 | 0.198 | 0.139 | 0.028 |
| | 0.85 | 0.950 | 2507.966 | 7.638 | 6.358 | 0.537 | 671.345 | 0.480 | 0.224 | 0.018 |
| 30% | | 0.990 | 12984.931 | 36.108 | 31.029 | 2.537 | 3104.076 | 1.611 | 0.571 | 0.061 |
| | 0.95 | 0.850 | 1564.191 | 7.528 | 6.349 | 0.194 | 737.149 | 0.420 | 0.604 | 0.028 |
| | | 0.950 | 3604.272 | 20.622 | 18.923 | 0.512 | 1051.750 | 1.261 | 0.880 | 0.018 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.990 | 18017.443 | 98.269 | 90.898 | 2.424 | 3354.627 | 5.685 | 1.809 | 0.057 |
| | | 0.850 | 1926.157 | 12.207 | 28.125 | 0.232 | 909.472 | 0.678 | 4.596 | 0.029 |
| | 0.99 | 0.950 | 4270.783 | 34.247 | 102.255 | 0.621 | 1287.733 | 2.047 | 9.733 | 0.021 |
| | | 0.990 | 20926.636 | 164.958 | 500.156 | 2.947 | 4039.963 | 9.707 | 33.906 | 0.074 |

**Table 3: Estimated MSE values when the error variance = 0.5, n = 50**

| % of Outlier | ρ | ▪ | OLS | HL | LAD | HLLAD | KL | HLKL | LADKL | HLLADKL |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.850 | 38.623 | 1.488 | 0.413 | 0.060 | 3.906 | 0.265 | 0.063 | 0.036 |
| | 0.85 | 0.950 | 109.631 | 3.961 | 1.112 | 0.162 | 10.967 | 0.278 | 0.039 | 0.014 |
| | | 0.990 | 535.311 | 18.845 | 5.531 | 0.778 | 53.133 | 0.710 | 0.081 | 0.018 |
| | | 0.850 | 57.477 | 2.030 | 1.378 | 0.055 | 6.745 | 0.338 | 0.161 | 0.042 |
| 10% | 0.95 | 0.950 | 163.408 | 5.438 | 4.122 | 0.148 | 18.900 | 0.362 | 0.208 | 0.013 |
| | | 0.990 | 799.318 | 26.169 | 21.668 | 0.718 | 91.837 | 0.966 | 0.798 | 0.014 |
| | | 0.850 | 69.634 | 2.339 | 11.048 | 0.063 | 9.488 | 0.376 | 2.595 | 0.047 |
| | 0.99 | 0.950 | 197.702 | 6.297 | 37.641 | 0.168 | 26.334 | 0.409 | 8.371 | 0.014 |
| | | 0.990 | 968.013 | 30.487 | 201.833 | 0.794 | 128.207 | 1.126 | 45.023 | 0.017 |
| | | 0.850 | 44.017 | 0.439 | 0.450 | 0.069 | 12.604 | 0.027 | 0.057 | 0.051 |
| | 0.85 | 0.950 | 86.516 | 1.404 | 1.224 | 0.188 | 24.467 | 0.033 | 0.032 | 0.012 |
| | | 0.990 | 308.820 | 7.369 | 6.113 | 0.911 | 35.699 | 0.182 | 0.065 | 0.014 |
| | | 0.850 | 54.261 | 1.357 | 1.355 | 0.063 | 18.454 | 0.056 | 0.138 | 0.053 |
| 20% | 0.95 | 0.950 | 104.599 | 4.259 | 4.451 | 0.173 | 34.586 | 0.074 | 0.122 | 0.011 |
| | | 0.990 | 371.003 | 22.282 | 23.383 | 0.830 | 48.796 | 0.433 | 0.316 | 0.012 |
| | | 0.850 | 67.573 | 2.267 | 10.716 | 0.072 | 21.568 | 0.090 | 0.936 | 0.053 |
| | 0.99 | 0.950 | 138.918 | 7.037 | 40.476 | 0.196 | 39.657 | 0.127 | 1.827 | 0.012 |
| | | 0.990 | 532.447 | 36.671 | 220.367 | 0.942 | 55.551 | 0.727 | 9.034 | 0.013 |
| | | 0.850 | 474.990 | 1.916 | 0.608 | 0.078 | 69.733 | 0.285 | 0.044 | 0.057 |
| | 0.85 | 0.950 | 1205.431 | 4.757 | 1.742 | 0.215 | 148.007 | 0.236 | 0.034 | 0.014 |
| 30% | | 0.990 | 5438.885 | 21.543 | 8.584 | 1.029 | 458.181 | 0.643 | 0.089 | 0.014 |
| | 0.95 | 0.850 | 696.372 | 0.753 | 1.531 | 0.074 | 99.619 | 0.340 | 0.140 | 0.058 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.950 | 1712.754 | 1.362 | 6.097 | 0.205 | 194.927 | 0.159 | 0.146 | 0.013 |
| | | 0.990 | 7547.278 | 5.282 | 28.265 | 1.000 | 445.363 | 0.127 | 0.349 | 0.013 |
| | | 0.850 | 793.216 | 0.431 | 8.685 | 0.087 | 121.058 | 0.426 | 1.183 | 0.060 |
| | 0.99 | 0.950 | 1911.602 | 0.450 | 47.519 | 0.239 | 230.985 | 0.194 | 1.239 | 0.015 |
| | | 0.990 | 8291.114 | 1.555 | 213.142 | 1.157 | 454.620 | 0.095 | 3.456 | 0.015 |

**Table 4: Estimated MSE values when the error variance = 1, n = 10**

| % of Outlier | ρ | ▪ | OLS | HL | LAD | HLLAD | KL | HLKL | LADKL | HLLADKL |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.850 | 343.699 | 334.626 | 7.896 | 5.958 | 114.240 | 105.132 | 1.707 | 1.090 |
| | 0.85 | 0.950 | 1,042.108 | 1,031.215 | 22.190 | 17.502 | 340.283 | 319.193 | 4.053 | 2.752 |
| | | 0.990 | 5,324.892 | 5,281.187 | 112.411 | 92.432 | 1,708.318 | 1,611.434 | 18.658 | 14.742 |
| | | 0.850 | 345.564 | 335.488 | 21.505 | 14.449 | 113.469 | 103.879 | 6.717 | 4.703 |
| 10% | 0.95 | 0.950 | 1,048.632 | 1,028.104 | 53.772 | 39.124 | 340.060 | 311.489 | 14.045 | 11.928 |
| | | 0.990 | 5,365.874 | 5,257.699 | 266.435 | 191.402 | 1,719.973 | 1,575.373 | 59.892 | 57.132 |
| | | 0.850 | 347.890 | 345.036 | 106.766 | 59.435 | 107.250 | 103.371 | 43.366 | 28.428 |
| | 0.99 | 0.950 | 1,057.246 | 1,046.818 | 254.757 | 160.300 | 327.486 | 305.704 | 90.253 | 75.521 |
| | | 0.990 | 5,441.413 | 5,321.595 | 1,089.635 | 781.176 | 1,703.759 | 1,519.877 | 335.657 | 370.736 |
| | | 0.850 | 1,375.904 | 395.653 | 8.609 | 3.317 | 446.291 | 57.524 | 1.605 | 0.302 |
| | 0.85 | 0.950 | 4,006.481 | 1,017.617 | 24.553 | 10.201 | 1,245.067 | 119.224 | 4.086 | 0.665 |
| | | 0.990 | 19,891.582 | 4,566.275 | 122.033 | 50.196 | 5,973.754 | 446.043 | 18.922 | 2.526 |
| | | 0.850 | 1,516.990 | 466.806 | 23.134 | 3.203 | 494.613 | 70.777 | 5.835 | 0.291 |
| 20% | 0.95 | 0.950 | 4,428.459 | 1,222.494 | 62.086 | 9.638 | 1,387.484 | 153.742 | 14.332 | 0.631 |
| | | 0.990 | 22,029.640 | 5,557.644 | 300.884 | 46.732 | 6,686.747 | 596.904 | 65.301 | 2.558 |
| | | 0.850 | 1,575.449 | 496.346 | 112.015 | 4.041 | 507.479 | 76.482 | 37.155 | 0.427 |
| | 0.99 | 0.950 | 4,608.731 | 1,308.003 | 291.356 | 12.209 | 1,438.502 | 168.853 | 90.037 | 0.784 |
| | | 0.990 | 22,986.168 | 5,973.235 | 1,404.542 | 58.368 | 7,014.128 | 663.914 | 419.684 | 3.122 |
| | | 0.850 | 2,843.630 | 782.519 | 9.506 | 5.091 | 832.366 | 173.200 | 2.031 | 0.471 |
| 30% | 0.85 | 0.950 | 8,471.523 | 2,368.125 | 24.798 | 24.127 | 2,434.076 | 523.278 | 4.492 | 1.608 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.990 | 42,833.768 | 12,016.768 | 122.743 | 185.220 | 12,112.544 | 2,634.218 | 20.599 | 11.230 |
| | | 0.850 | 3,448.516 | 997.576 | 25.577 | 4.688 | 1,028.840 | 229.585 | 6.765 | 0.463 |
| | 0.95 | 0.950 | 10,269.144 | 2,995.660 | 60.936 | 26.132 | 3,005.284 | 682.777 | 14.036 | 2.248 |
| | | 0.990 | 51,901.567 | 15,108.369 | 286.936 | 197.259 | 14,942.728 | 3,393.912 | 63.037 | 16.852 |
| | | 0.850 | 3,717.893 | 1,096.313 | 121.788 | 5.315 | 1,117.439 | 255.503 | 38.309 | 0.611 |
| | 0.99 | 0.950 | 11,078.386 | 3,283.840 | 283.502 | 48.104 | 3,279.493 | 755.863 | 80.027 | 6.280 |
| | | 0.990 | 56,040.075 | 16,527.954 | 1,306.606 | 344.620 | 16,389.203 | 3,741.034 | 360.849 | 51.762 |

**Table 5: Estimated MSE values when the error variance = 1, n = 30**

| % of Outlier | ρ | ▪ | OLS | HL | LAD | HLLAD | KL | HLKL | LADKL | HLLADKL |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.850 | 213.438 | 16.216 | 2.531 | 0.310 | 71.912 | 2.343 | 0.224 | 0.045 |
| | 0.85 | 0.950 | 403.777 | 46.018 | 6.747 | 0.823 | 91.903 | 6.612 | 0.472 | 0.044 |
| | | 0.990 | 1504.819 | 221.720 | 31.731 | 3.887 | 240.585 | 31.446 | 1.761 | 0.155 |
| | | 0.850 | 251.919 | 20.458 | 10.509 | 0.299 | 81.791 | 3.143 | 2.250 | 0.043 |
| 10% | 0.95 | 0.950 | 494.780 | 59.440 | 28.151 | 0.803 | 117.188 | 9.774 | 4.767 | 0.039 |
| | | 0.990 | 1902.636 | 291.319 | 131.642 | 3.831 | 361.867 | 50.517 | 19.410 | 0.141 |
| | | 0.850 | 276.160 | 22.762 | 71.570 | 0.360 | 93.018 | 4.025 | 34.020 | 0.048 |
| | 0.99 | 0.950 | 561.625 | 66.600 | 181.466 | 0.987 | 151.764 | 12.666 | 72.052 | 0.050 |
| | | 0.990 | 2223.741 | 328.216 | 795.224 | 4.755 | 502.086 | 65.842 | 289.352 | 0.187 |
| | | 0.850 | 633.040 | 22.909 | 2.576 | 0.370 | 188.579 | 3.223 | 0.140 | 0.050 |
| | 0.85 | 0.950 | 951.440 | 67.370 | 6.805 | 0.970 | 212.799 | 8.844 | 0.242 | 0.040 |
| | | 0.990 | 3339.255 | 337.412 | 32.433 | 4.601 | 473.558 | 44.984 | 0.851 | 0.126 |
| | | 0.850 | 983.224 | 29.311 | 11.008 | 0.358 | 293.548 | 3.679 | 0.647 | 0.049 |
| 20% | 0.95 | 0.950 | 1521.572 | 87.926 | 28.626 | 0.940 | 335.536 | 10.038 | 1.126 | 0.037 |
| | | 0.990 | 5330.629 | 444.663 | 137.049 | 4.459 | 790.943 | 51.614 | 4.211 | 0.115 |
| | | 0.850 | 1183.860 | 33.483 | 66.828 | 0.457 | 359.325 | 4.180 | 9.106 | 0.055 |
| | 0.99 | 0.950 | 1873.974 | 101.334 | 179.193 | 1.218 | 410.880 | 11.331 | 19.729 | 0.047 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.990 | 6631.106 | 514.889 | 788.841 | 5.776 | 992.094 | 58.359 | 65.757 | 0.151 |
| | | 0.850 | 1994.133 | 5.892 | 4.293 | 0.405 | 914.695 | 0.451 | 0.313 | 0.038 |
| | 0.85 | 0.950 | 5015.932 | 15.277 | 12.717 | 1.074 | 1343.140 | 0.988 | 0.464 | 0.036 |
| | | 0.990 | 25969.863 | 72.216 | 62.059 | 5.073 | 6208.221 | 3.233 | 1.145 | 0.122 |
| | | 0.850 | 3128.382 | 15.056 | 12.697 | 0.387 | 1476.102 | 0.884 | 1.302 | 0.037 |
| 30% | 0.95 | 0.950 | 7208.544 | 41.244 | 37.846 | 1.023 | 2104.096 | 2.554 | 1.797 | 0.034 |
| | | 0.990 | 36034.887 | 196.538 | 181.795 | 4.848 | 6709.366 | 11.386 | 3.626 | 0.114 |
| | | 0.850 | 3852.314 | 24.414 | 56.250 | 0.464 | 1820.974 | 1.388 | 9.461 | 0.039 |
| | 0.99 | 0.950 | 8541.566 | 68.493 | 204.511 | 1.242 | 2576.137 | 4.127 | 19.562 | 0.042 |
| | | 0.990 | 41853.273 | 329.916 | 1000.313 | 5.893 | 8080.052 | 19.435 | 67.831 | 0.148 |

**Table 6: Estimated MSE values when the error variance = 1, n = 50**

| % of Outlier | ρ | ▪ | OLS | HL | LAD | HLLAD | KL | HLKL | LADKL | HLLADKL |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.850 | 77.247 | 2.977 | 0.826 | 0.119 | 7.802 | 0.440 | 0.085 | 0.056 |
| | 0.85 | 0.950 | 219.262 | 7.923 | 2.223 | 0.323 | 21.931 | 0.527 | 0.073 | 0.021 |
| | | 0.990 | 1070.621 | 37.689 | 11.063 | 1.556 | 106.265 | 1.415 | 0.163 | 0.035 |
| | | 0.850 | 114.955 | 4.060 | 2.756 | 0.109 | 13.459 | 0.577 | 0.290 | 0.059 |
| 10% | 0.95 | 0.950 | 326.817 | 10.876 | 8.243 | 0.296 | 37.791 | 0.695 | 0.415 | 0.019 |
| | | 0.990 | 1598.636 | 52.339 | 43.335 | 1.435 | 183.673 | 1.929 | 1.597 | 0.027 |
| | | 0.850 | 139.269 | 4.679 | 22.097 | 0.126 | 18.941 | 0.648 | 5.194 | 0.061 |
| | 0.99 | 0.950 | 395.405 | 12.595 | 75.282 | 0.335 | 52.658 | 0.787 | 16.743 | 0.021 |
| | | 0.990 | 1936.026 | 60.974 | 403.667 | 1.588 | 256.412 | 2.248 | 90.047 | 0.033 |
| | | 0.850 | 88.035 | 0.878 | 0.900 | 0.138 | 26.097 | 0.028 | 0.076 | 0.056 |
| | 0.85 | 0.950 | 173.032 | 2.808 | 2.448 | 0.376 | 49.383 | 0.075 | 0.065 | 0.017 |
| 20% | | 0.990 | 617.640 | 14.738 | 12.226 | 1.822 | 71.509 | 0.370 | 0.132 | 0.027 |
| | | 0.850 | 108.521 | 2.713 | 2.711 | 0.126 | 37.989 | 0.077 | 0.246 | 0.056 |
| | 0.95 | 0.950 | 209.197 | 8.517 | 8.902 | 0.346 | 69.706 | 0.153 | 0.249 | 0.016 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.990 | 742.007 | 44.565 | 46.766 | 1.659 | 97.721 | 0.871 | 0.635 | 0.023 |
| | | 0.850 | 135.147 | 4.534 | 21.432 | 0.143 | 44.280 | 0.135 | 1.865 | 0.057 |
| | 0.99 | 0.950 | 277.836 | 14.074 | 80.951 | 0.392 | 79.873 | 0.257 | 3.660 | 0.017 |
| | | 0.990 | 1064.894 | 73.342 | 440.733 | 1.884 | 111.238 | 1.460 | 18.072 | 0.026 |
| | | 0.850 | 949.980 | 3.832 | 1.216 | 0.156 | 140.636 | 0.471 | 0.060 | 0.063 |
| | 0.85 | 0.950 | 2410.863 | 9.514 | 3.484 | 0.430 | 296.561 | 0.446 | 0.072 | 0.020 |
| | | 0.990 | 10877.771 | 43.086 | 17.168 | 2.058 | 916.494 | 1.280 | 0.181 | 0.028 |
| | | 0.850 | 1392.745 | 1.506 | 3.063 | 0.148 | 201.126 | 0.540 | 0.256 | 0.063 |
| 30% | 0.95 | 0.950 | 3425.508 | 2.724 | 12.193 | 0.410 | 390.759 | 0.283 | 0.306 | 0.019 |
| | | 0.990 | 15094.555 | 10.564 | 56.530 | 1.999 | 890.946 | 0.249 | 0.703 | 0.025 |
| | | 0.850 | 1586.432 | 0.863 | 17.370 | 0.175 | 244.427 | 0.690 | 2.336 | 0.067 |
| | 0.99 | 0.950 | 3823.204 | 0.900 | 95.038 | 0.478 | 463.092 | 0.347 | 2.525 | 0.022 |
| | | 0.990 | 16582.227 | 3.110 | 426.284 | 2.314 | 909.514 | 0.185 | 6.927 | 0.030 |

**Table 7: Estimated MSE values when the error variance = 4, n = 10**

| % of Outlier | ρ | ▪ | OLS | HL | LAD | HLLAD | KL | HLKL | LADKL | HLLADKL |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.850 | 1374.797 | 1338.503 | 31.584 | 23.831 | 456.955 | 420.378 | 6.827 | 4.358 |
| | 0.85 | 0.950 | 4168.430 | 4124.858 | 88.758 | 70.010 | 1361.132 | 1276.700 | 16.214 | 11.011 |
| | | 0.990 | 21299.568 | 21124.750 | 449.642 | 369.730 | 6833.273 | 6445.699 | 74.632 | 58.972 |
| | | 0.850 | 1382.255 | 1341.952 | 86.020 | 57.797 | 453.872 | 415.369 | 26.871 | 18.822 |
| 10% | 0.95 | 0.950 | 4194.528 | 4112.418 | 215.087 | 156.496 | 1360.239 | 1245.887 | 56.181 | 47.722 |
| | | 0.990 | 21463.498 | 21030.794 | 1065.738 | 765.607 | 6879.894 | 6301.458 | 239.567 | 228.536 |
| | | 0.850 | 1391.561 | 1380.145 | 427.066 | 237.742 | 428.995 | 413.336 | 173.476 | 113.733 |
| | 0.99 | 0.950 | 4228.984 | 4187.274 | 1019.029 | 641.201 | 1309.946 | 1222.746 | 361.017 | 302.098 |
| | | 0.990 | 21765.654 | 21286.381 | 4358.542 | 3124.703 | 6815.038 | 6079.475 | 1342.628 | 1482.957 |
| | | 0.850 | 5503.615 | 1582.610 | 34.437 | 13.269 | 1785.234 | 230.089 | 6.429 | 1.218 |
| 20% | 0.85 | 0.950 | 16025.923 | 4070.466 | 98.211 | 40.805 | 4980.290 | 476.886 | 16.347 | 2.662 |
| | | 0.990 | 79566.328 | 18265.099 | 488.131 | 200.783 | 23895.019 | 1784.165 | 75.690 | 10.105 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.850 | 6067.959 | 1867.223 | 92.536 | 12.811 | 1978.524 | 283.076 | 23.360 | 1.175 |
| | 0.95 | 0.950 | 17713.837 | 4889.978 | 248.343 | 38.551 | 5549.956 | 614.946 | 57.332 | 2.530 |
| | | 0.990 | 88118.561 | 22230.575 | 1203.538 | 186.927 | 26.746.992 | 2387.602 | 261.206 | 10.232 |
| | | 0.850 | 6301.794 | 1985.386 | 448.061 | 16.165 | 2029.989 | 305.887 | 148.663 | 1.728 |
| | 0.99 | 0.950 | 18434.925 | 5232.012 | 1165.422 | 48.836 | 5754.030 | 675.386 | 360.158 | 3.141 |
| | | 0.990 | 91944.672 | 23892.941 | 5618.168 | 233.472 | 28056.516 | 2655.638 | 1678.738 | 12.488 |
| | | 0.850 | 11374.521 | 3130.077 | 38.024 | 20.363 | 3329.553 | 692.588 | 8.127 | 1.863 |
| | 0.85 | 0.950 | 33886.093 | 9472.502 | 99.193 | 96.509 | 9736.329 | 2092.980 | 17.967 | 6.417 |
| | | 0.990 | 171335.071 | 48067.071 | 490.973 | 740.880 | 48450.183 | 10536.789 | 82.397 | 44.910 |
| | | 0.850 | 13794.064 | 3990.305 | 102.306 | 18.753 | 4115.452 | 918.107 | 27.073 | 1.834 |
| 30% | 0.95 | 0.950 | 41076.574 | 11982.639 | 243.745 | 104.527 | 12021.163 | 2730.966 | 56.145 | 8.974 |
| | | 0.990 | 207606.268 | 60433.476 | 1147.746 | 789.034 | 59770.918 | 13575.558 | 252.147 | 67.394 |
| | | 0.850 | 14871.570 | 4385.251 | 487.151 | 21.259 | 4469.855 | 1021.769 | 153.275 | 2.432 |
| | 0.99 | 0.950 | 44313.543 | 13135.361 | 1134.008 | 192.417 | 13117.999 | 3023.303 | 320.116 | 25.098 |
| | | 0.990 | 224160.301 | 66111.817 | 5226.425 | 1378.480 | 65556.817 | 14964.044 | 1443.396 | 207.034 |

**Table 8: Estimated MSE values when the error variance = 4, n = 30**

| % of Outlier | ρ | ▪ | OLS | HL | LAD | HLLAD | KL | HLKL | LADKL | HLLADKL |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.850 | 853.753 | 64.865 | 10.123 | 1.239 | 288.966 | 9.360 | 0.945 | 0.116 |
| | 0.85 | 0.950 | 1615.107 | 184.074 | 26.988 | 3.292 | 368.180 | 26.490 | 1.917 | 0.169 |
| | | 0.990 | 6019.277 | 886.882 | 126.923 | 15.547 | 962.463 | 125.826 | 7.049 | 0.621 |
| | | 0.850 | 1007.677 | 81.834 | 42.038 | 1.195 | 328.586 | 12.528 | 9.135 | 0.108 |
| 10% | 0.95 | 0.950 | 1979.121 | 237.762 | 112.603 | 3.210 | 469.336 | 39.149 | 19.125 | 0.151 |
| | | 0.990 | 7610.545 | 1165.275 | 526.567 | 15.326 | 1447.590 | 202.134 | 77.649 | 0.567 |
| | | 0.850 | 1104.639 | 91.048 | 286.278 | 1.441 | 373.532 | 16.042 | 136.371 | 0.128 |
| | 0.99 | 0.950 | 2246.501 | 266.401 | 725.865 | 3.946 | 607.619 | 50.719 | 288.306 | 0.193 |
| | | 0.990 | 8894.966 | 1312.862 | 3180.895 | 19.021 | 2008.460 | 263.440 | 1157.422 | 0.751 |
| 20% | 0.85 | 0.850 | 2532.158 | 91.638 | 10.305 | 1.480 | 759.294 | 12.513 | 0.597 | 0.123 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.950 | 3805.760 | 269.479 | 27.218 | 3.880 | 852.919 | 35.221 | 0.993 | 0.150 |
| | | 0.990 | 13357.019 | 1349.648 | 129.733 | 18.402 | 1894.552 | 179.857 | 3.410 | 0.502 |
| | | 0.850 | 3932.897 | 117.245 | 44.031 | 1.430 | 1179.838 | 14.306 | 2.748 | 0.118 |
| | 0.95 | 0.950 | 6086.290 | 351.705 | 114.505 | 3.758 | 1344.155 | 40.023 | 4.568 | 0.138 |
| | | 0.990 | 21322.518 | 1778.653 | 548.195 | 17.835 | 3164.150 | 206.410 | 16.851 | 0.462 |
| | | 0.850 | 4735.439 | 133.934 | 267.313 | 1.829 | 1443.153 | 16.290 | 36.851 | 0.138 |
| | 0.99 | 0.950 | 7495.895 | 405.337 | 716.773 | 4.874 | 1645.641 | 45.208 | 79.079 | 0.176 |
| | | 0.990 | 26524.426 | 2059.555 | 3155.366 | 23.102 | 3968.776 | 233.411 | 263.044 | 0.605 |
| | | 0.850 | 7976.531 | 23.566 | 17.174 | 1.621 | 3665.370 | 2.107 | 1.472 | 0.097 |
| | 0.85 | 0.950 | 20063.727 | 61.106 | 50.866 | 4.296 | 5374.728 | 4.090 | 1.938 | 0.140 |
| | | 0.990 | 103879.450 | 288.865 | 248.235 | 20.294 | 24833.214 | 12.988 | 4.597 | 0.488 |
| | | 0.850 | 12513.530 | 60.223 | 50.789 | 1.550 | 5913.133 | 3.780 | 5.710 | 0.093 |
| 30% | 0.95 | 0.950 | 28834.174 | 164.975 | 151.384 | 4.093 | 8419.261 | 10.373 | 7.369 | 0.132 |
| | | 0.990 | 144139.548 | 786.150 | 727.181 | 19.392 | 26838.000 | 45.630 | 14.537 | 0.459 |
| | | 0.850 | 15409.257 | 97.654 | 224.999 | 1.856 | 7293.708 | 5.745 | 39.187 | 0.108 |
| | 0.99 | 0.950 | 34166.264 | 273.973 | 818.043 | 4.968 | 10307.789 | 16.672 | 78.719 | 0.165 |
| | | 0.990 | 167413.091 | 1319.664 | 4001.251 | 23.572 | 32320.817 | 77.843 | 271.415 | 0.594 |

**Table 9: Estimated MSE values when the error variance = 4, n = 50**

| % of Outlier | ρ | ▪ | OLS | HL | LAD | HLLAD | KL | HLKL | LADKL | HLLADKL |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.850 | 308.986 | 11.906 | 3.303 | 0.478 | 31.183 | 1.376 | 0.228 | 0.086 |
| | 0.85 | 0.950 | 877.050 | 31.691 | 8.893 | 1.293 | 87.718 | 1.982 | 0.287 | 0.060 |
| | | 0.990 | 4282.485 | 150.756 | 44.251 | 6.223 | 425.057 | 5.638 | 0.653 | 0.140 |
| | | 0.850 | 459.819 | 16.240 | 11.025 | 0.437 | 53.712 | 1.882 | 1.082 | 0.084 |
| 10% | 0.95 | 0.950 | 1307.266 | 43.502 | 32.974 | 1.184 | 151.123 | 2.642 | 1.663 | 0.052 |
| | | 0.990 | 6394.542 | 209.355 | 173.342 | 5.742 | 734.682 | 7.696 | 6.390 | 0.107 |
| | | 0.850 | 557.076 | 18.715 | 88.386 | 0.502 | 75.620 | 2.149 | 20.858 | 0.090 |
| | 0.99 | 0.950 | 1581.620 | 50.379 | 301.128 | 1.342 | 210.583 | 3.009 | 66.985 | 0.061 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 0.990 | 7744.104 | 243.897 | 1614.668 | 6.352 | 1025.636 | 8.976 | 360.192 | 0.128 |
| | | | 0.850 | 352.139 | 3.512 | 3.599 | 0.550 | 108.772 | 0.070 | 0.217 | 0.076 |
| | | 0.85 | 0.950 | 692.130 | 11.233 | 9.793 | 1.504 | 199.716 | 0.356 | 0.269 | 0.048 |
| | | | 0.990 | 2470.559 | 58.952 | 48.906 | 7.288 | 286.574 | 1.505 | 0.535 | 0.109 |
| | | | 0.850 | 434.085 | 10.854 | 10.843 | 0.505 | 157.267 | 0.208 | 0.928 | 0.074 |
| 20% | | 0.95 | 0.950 | 836.789 | 34.068 | 35.608 | 1.385 | 281.413 | 0.649 | 1.029 | 0.044 |
| | | | 0.990 | 2968.028 | 178.258 | 187.066 | 6.636 | 391.511 | 3.509 | 2.551 | 0.091 |
| | | | 0.850 | 540.588 | 18.135 | 85.730 | 0.573 | 182.732 | 0.399 | 7.513 | 0.078 |
| | | 0.99 | 0.950 | 1111.345 | 56.295 | 323.806 | 1.568 | 322.203 | 1.050 | 14.677 | 0.050 |
| | | | 0.990 | 4259.578 | 293.370 | 1762.933 | 7.535 | 445.604 | 5.866 | 72.310 | 0.104 |
| | | | 0.850 | 3799.922 | 15.326 | 4.864 | 0.623 | 568.242 | 1.470 | 0.196 | 0.090 |
| | | 0.85 | 0.950 | 9643.451 | 38.055 | 13.936 | 1.719 | 1188.895 | 1.667 | 0.321 | 0.055 |
| | | | 0.990 | 43511.083 | 172.345 | 68.672 | 8.231 | 3666.614 | 5.092 | 0.735 | 0.109 |
| | | | 0.850 | 5570.979 | 6.025 | 12.250 | 0.593 | 813.682 | 1.577 | 1.002 | 0.088 |
| 30% | | 0.95 | 0.950 | 13702.031 | 10.895 | 48.773 | 1.641 | 1567.413 | 0.976 | 1.309 | 0.051 |
| | | | 0.990 | 60378.220 | 42.256 | 226.121 | 7.996 | 3564.847 | 0.971 | 2.838 | 0.099 |
| | | | 0.850 | 6345.727 | 3.451 | 69.480 | 0.699 | 988.942 | 2.077 | 9.293 | 0.099 |
| | | 0.99 | 0.950 | 15292.814 | 3.600 | 380.153 | 1.911 | 1857.800 | 1.207 | 10.343 | 0.062 |
| | | | 0.990 | 66328.908 | 12.439 | 1705.136 | 9.258 | 3639.382 | 0.713 | 27.781 | 0.116 |

**Table 10.**

| $\alpha$ | K | $n$ | $p$ | $F_{(3,12)}$ | $R^2$ | $\hat{\sigma}^2$ | $dl$ | $du$ | $DW$ | $VIF_1$ | $VIF_2$ | $VIF_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 3.248 | 16 | 3 | 19.950 | 0.833 | 7564.42 | 0.860 | 1.730 | 0.803 | 133.709 | 37.380 | 41.928 |

**Table 11. Coefficient estimates, MSE, and ranks of the proposed and considered estimators**

| Estimator | Coefficients Estimates | | | | AMSE | Rank |
|---|---|---|---|---|---|---|
| | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_3$ | | |
| OLS | 151.916796 | 1.9391019 | -0.3409638 | 0.09901955 | 3223.63517 | 8 |

| | | | | | | |
|---|---|---|---|---|---|---|
| HL | 44.0793292 | 0.27069863 | 0.40093283 | 0.0379899 | 57.8128197 | 3 |
| KL | 132.205018 | 2.65452496 | -0.6078002 | -0.9097097 | 2439.3762 | 7 |
| LAD | 53.91659 | 5.992997 | -1.63648 | -6.89525 | 481.907 | 6 |
| HLKL | 42.8024969 | 0.28411587 | 0.39876059 | 0.03093152 | 213.532204 | 4 |
| LADHL | 33.78182 | 0.110542 | 0.469087 | 0.398365 | 49.703 | 2 |
| LADKL | 19.7427914 | 7.22685362 | -2.0970903 | -8.631091 | 352.453154 | 5 |
| HLLADKL | 32.5811165 | 0.11802228 | 0.46912318 | 0.39889461 | 40.1687232 | 1 |

The correlation matrix of the predictors:

$$r_X = \begin{pmatrix} 1 & 0.9617 & 0.9660 \\ & 1 & 0.8721 \\ & & 1 \end{pmatrix}$$