

SMARANDACHE CURVES OF NATURAL CURVES PAIR ACCORDING TO FRENET FRAME

ABSTRACT. Smarandache curves play a significant role in differential geometry. They extend conventional differential geometry concepts and help uncover novel geometric properties. In this work, we define the Smarandache curves of the Natural mate of any given curve α and calculate their Frenet apparatus. As a particular case, we present the Frenet apparatus when α is a helix. Additionally, we illustrate an example for the slant helix.

Keywords: Smarandache curves. Natural mate curves. Frenet apparatus.

1. INTRODUCTION

A starting point in the study of differential geometry is the analysis of curves in space. By introducing special geometric properties, it is possible to produce new curves from a given curve. In this context, there are some special curves such as Bertrand, Mannheim, Natural mate, Smarandache, involute, evolute, and pedal curves which have important applications ranging from physics to surface modeling in engineering and computer graphics [15, 17, 12, 26, 7, 18, 22, 11]. For any given unit speed curve α in \mathbb{E}^3 there is a unique unit speed curve $\bar{\alpha}$ whose tangent vector coincides with the principal normal vector of α , called *principal-direction curve* or *Natural mate curve* [9]. Some authors have studied this class of curves: Abdel-Baky and Naghi [1] study sweeping surfaces with Natural mate curves, Deshmukh et al [10] prove some relationships between a Frenet curve and its natural mate, Camci et al [6] extend the natural mate $\bar{\alpha}$ to sequential natural mates $\{\alpha_1, \alpha_2, \dots, \alpha_{n_\alpha}\}$ with $\alpha_1 = \bar{\alpha}$, Mak [13] introduce the natural mate curves in a three dimensional Lie group with bi-invariant metric and give some relationships between a Frenet curve and its natural mate in this group.

In differential geometry, the Smarandache geometry has a significant role in the theory of relativity and parallel universes [3]. In Smarandache geometry, regular curves that are defined as having the position vector being a combination of the tangent, normal and binormal vectors of another regular curve are called *Smarandache curves*.

Smarandache curves are used to expand traditional concepts of differential geometry and discover new geometric properties. These curves allow for detailed analyses in areas such as curve theory and surface theory. Their applications span various fields, including physics, engineering, robotics, computer graphics, and biomechanics. In physics, they are especially used in space-time geometry and general relativity theory [19]. In engineering, they find application in materials science and structural analysis. In computer graphics and robotics, they play a crucial role in motion planning and object recognition.

Special Smarandache curves have been studied by some authors. Savas et al [20] studied special Smarandache curves according to the Sabban frame in hyperbolic space and new Smarandache partners in de Sitter space, Ali [2] studied some special Smarandache curves in the Euclidean space, Şenyurt et al [23] studied the Smarandache curves of Bertrand curves pair according to Frenet frame and expresses its curvature and torsion in terms of the curvature and torsion of the Bertrand curve and Şenyurt and Kaya studied the NC- Smarandache curve and NW- Smarandache curve according to alternative frame [21]. These curves have been also studied widely [25, 4, 5, 8].

In this paper, we start by mentioning the Frenet apparatus of any regular curve α parametrized by arc-length in Euclidean space \mathbb{E}^3 . Then, we relate the Frenet frame of the Natural pair and we give the definitions of the tn, tb, nb, tnb -Smarandache curves in \mathbb{E}^3 for Natural mate curves and calculate the Frenet apparatus of these curves using the curvature, torsion, the unit Darboux vector and the function angle between the Darboux vector and the binormal vector of α . We calculate the Frenet apparatus for the case where α is a helix and close with an example where α is a slant helix.

2. PRELIMINARIES

Consider the Euclidean space \mathbb{E}^3 with inner product

$$\langle \cdot, \cdot \rangle = dx^2 + dy^2 + dz^2,$$

where $(x, y, z) \in \mathbb{E}^3$ is a rectangular coordinate system. Let $\alpha : I \rightarrow \mathbb{E}^3$ be a differentiable curve in the Euclidean space defined on an open interval I , parametrized by arc-length and let $\{t = \alpha', n, b\}$ be Frenet frame satisfying [14]

$$\begin{cases} t' = \kappa n, \\ n' = -\kappa t + \tau b, \\ b' = -\tau n, \end{cases} \quad (2.1)$$

where κ and τ are differentiable functions on I called the *curvature* and the *torsion* of α , respectively, t is the tangent vector, n is the principal normal vector and b is the binormal vector of α . The 5-uple (t, n, b, κ, τ) is called a *Frenet apparatus*.

Definition 2.1. A curve $\bar{\alpha} : J \rightarrow \mathbb{E}^3$ is called *Natural mate curve* of $\alpha : I \rightarrow \mathbb{E}^3$ if $\bar{\alpha}(\bar{s})$ is the integral curve of the principal normal vector of $\alpha(s^*)$, and the pair $(\alpha(s^*), \bar{\alpha}(\bar{s}))$ is called the *Natural pair* [10].

The existence of Natural mate curves is guaranteed by existence theorem for differential equation and $\bar{\alpha}$ is given by $\bar{\alpha} = \int n ds^*$. It can be also shown that the arc-length parameters of the curve $\bar{\alpha}$ can be the same of α , that is, $s^* = \bar{s}$ [16].

The relations between the Frenet frames $\{t, n, b\}$ and $\{\bar{t}, \bar{n}, \bar{b}\}$ are given by [1]

$$\begin{cases} \bar{t} = t, \\ \bar{n} = -\cos \psi t + \sin \psi b, \\ \bar{b} = \sin \psi t + \cos \psi b, \end{cases} \quad (2.2)$$

where $\psi = \angle(\bar{b}, b)$,

$$\cos \psi = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \quad \text{and} \quad \sin \psi = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}. \quad (2.3)$$

Let the Darboux vector defined by

$$W = \tau t + \kappa b.$$

The Darboux vector represents the Frenet frame's angular momentum. Its direction determines the frame's momentary axis of motion (its centroid) and its length the angular speed, $\|W\| = \sqrt{\kappa^2 + \tau^2}$.

If we consider the normalization of the Darboux $C = \frac{1}{\|W\|}W$ we have,

$$\sin \varphi = \frac{\tau}{\|W\|} = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \quad \text{and} \quad \cos \varphi = \frac{\kappa}{\|W\|} = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}, \quad (2.4)$$

where $\varphi = \angle(W, b)$. Therefore, by the equations (2.3) and (2.4), we have $\psi = \varphi$, and consequently, the binormal vector of the Natural mate curve $\bar{\alpha}$ is equal to the Darboux vector of the curve α .

From equation (2.4), we obtain

$$\varphi' = \frac{\kappa\tau' - \kappa'\tau}{\|W\|^2} = \left(\frac{\tau}{\kappa} \right)' \frac{\kappa^2}{\|W\|^2}. \quad (2.5)$$

Remark 2.1. A regular curve α is called a helix if the tangent lines of the curve make a constant angle with a fixed direction and a helix is characterized by the property that $\frac{\tau}{\kappa}$ is constant. From equation (2.5), we have in this case $\varphi = \text{constant}$.

3. SMARANDACHE CURVES OF NATURAL CURVES PAIR ACCORDING TO FRENET FRAME

In this section, we investigate the Smarandache curves of Natural pair according to Frenet frame in Euclidean 3-space and we give the Frenet apparatus for these curves.

Definition 3.2. A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [27].

In the light of the above definition, Ali adapt it to regular curves in the Euclidean space the definition of Smarandache curves [2]. For Natural pair we have

Definition 3.3. Let $(\alpha, \bar{\alpha})$ be a Natural pair in \mathbb{E}^3 and $\{\bar{t}, \bar{n}, \bar{b}\}$ be the Frenet frame of the Natural mate curve $\bar{\alpha}$. The $\bar{t}\bar{n}$ -Smarandache curve is defined by

$$\beta_1(s) = \frac{1}{\sqrt{2}}(\bar{t}(\bar{s}) + \bar{n}(\bar{s})). \quad (3.6)$$

Theorem 3.1. *The Frenet apparatus of the $\bar{t}\bar{n}$ -Smarandache curve is given by*

$$T_{\beta_1} = \frac{(\varphi' \sin \varphi - \kappa)t - \|W\|n + (\varphi' \cos \varphi + \tau)b}{\sqrt{(\varphi')^2 + 2\|W\|^2}}, \quad (3.7)$$

$$N_{\beta_1} = \frac{\lambda_1 t + \lambda_2 n + \lambda_3 b}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}, \quad (3.8)$$

$$B_{\beta_1} = \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{\sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)((\varphi')^2 + 2\|W\|^2)}}, \quad (3.9)$$

$$\kappa_{\beta_1} = \frac{\sqrt{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}}{[(\varphi')^2 + 2\|W\|^2]^2}, \quad (3.10)$$

$$\tau_{\beta_1} = \frac{\sqrt{2}[(\varphi')^2 + 2\|W\|^2](\sigma_1 \eta_1 + \sigma_2 \eta_2 + \sigma_3 \eta_3)}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}, \quad (3.11)$$

where $W = \tau t + \kappa b$ and the coefficients are given by

$$\begin{aligned} \lambda_1 &= [(\varphi')^2 \cos \varphi - \kappa' + \kappa \|W\|] \cdot [(\varphi')^2 + 2\|W\|^2] + 2\varphi'' \|W\| \tau \\ &\quad + \kappa(\varphi' \varphi'' + 2\|W\| \|W\|'), \\ \lambda_2 &= -\|W\|^2 \cdot [(\varphi')^2 + 2\|W\|^2] + \varphi'(\|W\| \varphi'' - \|W\|' \varphi'), \\ \lambda_3 &= [-(\varphi')^2 \sin \varphi + \tau' - \tau \|W\|] \cdot [(\varphi')^2 + 2\|W\|^2] + 2\varphi'' \|W\| \kappa \\ &\quad - \tau(\varphi' \varphi'' + 2\|W\| \|W\|'), \end{aligned}$$

$$\begin{aligned} \sigma_1 &= -\lambda_2(\varphi' \cos \varphi + \tau) - \lambda_3 \|W\|, \\ \sigma_2 &= \lambda_1(\varphi' \cos \varphi + \tau) - \lambda_3(\varphi' \sin \varphi - \kappa), \\ \sigma_3 &= \lambda_1 \|W\| + \lambda_2(\varphi' \sin \varphi - \kappa), \end{aligned}$$

$$\begin{aligned} \eta_1 &= \left(\frac{\lambda_1}{[(\varphi')^2 + 2\|W\|^2]^2} \right)' - \frac{\kappa \lambda_2}{[(\varphi')^2 + 2\|W\|^2]^2}, \\ \eta_2 &= \left(\frac{\lambda_2}{[(\varphi')^2 + 2\|W\|^2]^2} \right)' + \frac{\kappa \lambda_1 - \tau \lambda_3}{[(\varphi')^2 + 2\|W\|^2]^2}, \\ \eta_3 &= \left(\frac{\lambda_3}{[(\varphi')^2 + 2\|W\|^2]^2} \right)' + \frac{\tau \lambda_2}{[(\varphi')^2 + 2\|W\|^2]^2}. \end{aligned}$$

Proof. Substituting the equation (2.2) into equation (3.6), we obtain

$$\beta_1 = \frac{-\cos \varphi t + n + \sin \varphi b}{\sqrt{2}}. \quad (3.12)$$

Taking the derivative of the equation (3.12) with respect to \bar{s} , we get

$$T_{\beta_1} \frac{ds}{d\bar{s}} = \frac{(\varphi' \sin \varphi - \kappa)t - \|W\|n + (\varphi' \cos \varphi + \tau)b}{\sqrt{2}}.$$

Thus

$$T_{\beta_1} = \frac{(\varphi' \sin \varphi - \kappa)t - \|W\|n + (\varphi' \cos \varphi + \tau)b}{\sqrt{(\varphi')^2 + 2\|W\|^2}}, \quad (3.13)$$

where

$$\frac{ds}{d\bar{s}} = \frac{\sqrt{(\varphi')^2 + 2\|W\|^2}}{\sqrt{2}}. \quad (3.14)$$

Taking the derivative of the equation (3.13) with respect to \bar{s} and use (3.14), we obtain

$$T'_{\beta_1} = \frac{\sqrt{2}(\lambda_1 t + \lambda_2 n + \lambda_3 b)}{[(\varphi')^2 + 2\|W\|^2]^2},$$

where

$$\begin{aligned} \lambda_1 &= ((\varphi')^2 \cos \varphi - \kappa' + \kappa \|W\|)[(\varphi')^2 + 2\|W\|^2] + 2\varphi'' \|W\| \tau \\ &\quad + \kappa(\varphi' \varphi'' + 2\|W\| \|W'\|), \\ \lambda_2 &= -\|W\|^2 [(\varphi')^2 + 2\|W\|^2] + \varphi'(\|W\| \varphi'' - \|W'\| \varphi'), \\ \lambda_3 &= (-\varphi')^2 \sin \varphi + \tau' - \tau \|W\| [(\varphi')^2 + 2\|W\|^2] + 2\varphi'' \|W\| \kappa \\ &\quad - \tau(\varphi' \varphi'' + 2\|W\| \|W'\|), \end{aligned}$$

Therefore, the curvature, the principal normal vector and the binormal vector of the curve β_1 are given by

$$\kappa_{\beta_1} = \frac{\sqrt{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}}{[(\varphi')^2 + 2\|W\|^2]^2}, \quad N_{\beta_1} = \frac{\lambda_1 t + \lambda_2 n + \lambda_3 b}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}},$$

$$\begin{aligned} B_{\beta_1} &= T_{\beta_1} \times N_{\beta_1} \\ &= \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{\sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)((\varphi')^2 + 2\|W\|^2)}}, \end{aligned}$$

where

$$\begin{aligned} \sigma_1 &= -\lambda_2(\varphi' \cos \varphi + \tau) - \lambda_3 \|W\|, \\ \sigma_2 &= \lambda_1(\varphi' \cos \varphi + \tau) - \lambda_3(\varphi' \sin \varphi - \kappa), \\ \sigma_3 &= \lambda_1 \|W\| + \lambda_2(\varphi' \sin \varphi - \kappa). \end{aligned}$$

The torsion of curve β_1 is given by

$$\tau_{\beta_1} = \frac{\det(\beta'_1, \beta''_1, \beta'''_1)}{\|\beta'_1 \times \beta''_1\|^2} = \frac{\sqrt{2}[(\varphi')^2 + 2\|W\|^2](\sigma_1 \eta_1 + \sigma_2 \eta_2 + \sigma_3 \eta_3)}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2},$$

where

$$\begin{aligned}\eta_1 &= \left(\frac{\lambda_1}{[(\varphi')^2 + 2 \|W\|^2]^2} \right)' - \frac{\kappa\lambda_2}{[(\varphi')^2 + 2 \|W\|^2]^2}, \\ \eta_2 &= \left(\frac{\lambda_2}{[(\varphi')^2 + 2 \|W\|^2]^2} \right)' + \frac{\kappa\lambda_1 - \tau\lambda_3}{[(\varphi')^2 + 2 \|W\|^2]^2}, \\ \eta_3 &= \left(\frac{\lambda_3}{[(\varphi')^2 + 2 \|W\|^2]^2} \right)' + \frac{\tau\lambda_2}{[(\varphi')^2 + 2 \|W\|^2]^2}.\end{aligned}$$

□

From Remark (2.1) and Theorem (3.1), we get

Corollary 3.1. *If α is a helix, then the Frenet apparatus of the $\bar{t}\bar{n}$ -Smarandache curve of its Natural mate curve is given by*

$$\begin{aligned}T_{\beta_1} &= -\frac{1}{\sqrt{2}C_1}t - \frac{1}{\sqrt{2}}n + \frac{C}{\sqrt{2}C_1}b, & N_{\beta_1} &= \frac{1}{\sqrt{2}C_1}t - \frac{1}{\sqrt{2}}n - \frac{C}{\sqrt{2}C_1}b, \\ B_{\beta_1} &= \frac{C}{C_1}t + \frac{1}{C_1}b, & \kappa_{\beta_1} &= 1, \quad \tau_{\beta_1} = 0,\end{aligned}$$

where $\frac{\tau}{\kappa} = C = \text{constant}$ and $C_1 = \sqrt{1 + C^2}$.

Definition 3.4. *Let $(\alpha, \bar{\alpha})$ be a Natural pair in \mathbb{E}^3 and $\{\bar{t}, \bar{n}, \bar{b}\}$ be the Frenet frame of the Natural mate curve $\bar{\alpha}$. The $\bar{n}\bar{b}$ -Smarandache curve is defined by*

$$\beta_2(s) = \frac{1}{\sqrt{2}}(\bar{n}(\bar{s}) + \bar{b}(\bar{s})). \quad (3.15)$$

Theorem 3.2. *The Frenet apparatus of the $\bar{n}\bar{b}$ -Smarandache curve is given by*

$$T_{\beta_2} = \frac{\varphi'(\sin \varphi + \cos \varphi)t - \|W\|n + \varphi'(\cos \varphi - \sin \varphi)b}{\sqrt{2(\varphi')^2 + \|W\|^2}}, \quad (3.16)$$

$$N_{\beta_2} = \frac{\lambda_1 t + \lambda_2 n + \lambda_3 b}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}, \quad (3.17)$$

$$B_{\beta_2} = \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{\sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(2(\varphi')^2 + \|W\|^2)}}, \quad (3.18)$$

$$\kappa_{\beta_2} = \frac{\sqrt{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}}{[2(\varphi')^2 + \|W\|^2]^2}, \quad (3.19)$$

$$\tau_{\beta_2} = \frac{\sqrt{2}[2(\varphi')^2 + \|W\|^2](\sigma_1\eta_1 + \sigma_2\eta_2 + \sigma_3\eta_3)}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}, \quad (3.20)$$

where $W = \tau t + \kappa b$ and the coefficients are given by

$$\begin{aligned}\lambda_1 &= ((\varphi')^2(\cos \varphi - \sin \varphi) + \kappa \|W\|)[2(\varphi')^2 + \|W\|^2] \\ &\quad + \|W\|(\sin \varphi + \cos \varphi)(\varphi'' \|W\| - \varphi' \|W\|'), \\ \lambda_2 &= \varphi'(\|W\|[2(\varphi')^2 + \|W\|^2] + 2(\varphi'' \|W\| - \varphi' \|W\|')), \\ \lambda_3 &= -((\varphi')^2(\sin \varphi + \cos \varphi) + \tau \|W\|)[2(\varphi')^2 + \|W\|^2] \\ &\quad + \|W\|(\cos \varphi - \sin \varphi)(\varphi'' \|W\| + \varphi' \|W\|'),\end{aligned}$$

$$\begin{aligned}\sigma_1 &= -\lambda_2 \varphi'(\cos \varphi - \sin \varphi) - \lambda_3 \|W\|, \\ \sigma_2 &= \lambda_1 \varphi'(\cos \varphi - \sin \varphi) - \lambda_3 \varphi'(\sin \varphi + \cos \varphi), \\ \sigma_3 &= \lambda_1 \|W\| + \lambda_2 \varphi'(\sin \varphi + \cos \varphi),\end{aligned}$$

$$\begin{aligned}\eta_1 &= \left(\frac{\lambda_1}{[2(\varphi')^2 + \|W\|^2]^2} \right)' - \frac{\kappa \lambda_2}{[2(\varphi')^2 + \|W\|^2]^2}, \\ \eta_2 &= \left(\frac{\lambda_2}{[2(\varphi')^2 + \|W\|^2]^2} \right)' + \frac{\kappa \lambda_1 - \tau \lambda_3}{[2(\varphi')^2 + \|W\|^2]^2}, \\ \eta_3 &= \left(\frac{\lambda_3}{[2(\varphi')^2 + \|W\|^2]^2} \right)' + \frac{\tau \lambda_2}{[2(\varphi')^2 + \|W\|^2]^2}.\end{aligned}$$

Proof. The proof is similar to proof of Theorem 3.1. □

Corollary 3.2. *If α is a helix, then the Frenet apparatus of the $\bar{n}b$ -Smarandache curve of its Natural mate curve is given by*

$$\begin{aligned}T_{\beta_2} &= -n, & N_{\beta_2} &= \frac{1}{C_1}t - \frac{C}{C_1}b, & B_{\beta_2} &= \frac{C}{C_1}t + \frac{1}{C_1}b, \\ & & & & \kappa_{\beta_2} &= \sqrt{2}, & \tau_{\beta_2} &= 0,\end{aligned}$$

where $\frac{\tau}{\kappa} = C = \text{constant}$ and $C_1 = \sqrt{1 + C^2}$.

Definition 3.5. *Let $(\alpha, \bar{\alpha})$ be a Natural pair in \mathbb{E}^3 and $\{\bar{t}, \bar{n}, \bar{b}\}$ be the Frenet frame of the Natural mate curve $\bar{\alpha}$. The $\bar{t}b$ -Smarandache curve is defined by*

$$\beta_3(s) = \frac{1}{\sqrt{2}}(\bar{t}(\bar{s}) + \bar{b}(\bar{s})). \quad (3.21)$$

Theorem 3.3. *The Frenet apparatus of the $\bar{t}\bar{b}$ -Smarandache curve is given by*

$$T_{\beta_3} = \frac{(\varphi' \cos \varphi - \kappa)t + (-\varphi' \sin \varphi + \tau)b}{|\varphi' - \|W\||}, \quad (3.22)$$

$$N_{\beta_3} = \frac{\lambda_1 t + \|W\| (\varphi' - \|W\|)^2 n + \lambda_2 b}{\sqrt{\lambda_1^2 + \|W\|^2} (\varphi' - \|W\|)^4 + \lambda_2^2}, \quad (3.23)$$

$$B_{\beta_3} = \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{(\varphi' - \|W\|) \sqrt{(\lambda_1^2 + \|W\|^2) (\varphi' - \|W\|)^4 + \lambda_2^2}}, \quad (3.24)$$

$$\kappa_{\beta_3} = \frac{\sqrt{2(\lambda_1^2 + \|W\|^2) (\varphi' - \|W\|)^4 + \lambda_2^2}}{|\varphi' - \|W\||^3}, \quad (3.25)$$

$$\tau_{\beta_3} = \frac{\sqrt{2}(\varphi' - \|W\|)(\sigma_1 \eta_1 + \sigma_2 \eta_2 + \sigma_3 \eta_3)}{\lambda_1^2 + \|W\|^2 (\varphi' - \|W\|)^4 + \lambda_2^2}, \quad (3.26)$$

where $W = \tau t + \kappa b$ and the coefficients are given by

$$\begin{aligned} \lambda_1 &= -(\varphi')^3 \sin \varphi + \kappa'(\|W\| - \varphi') + \tau(\varphi')^2 + \|W\|' (\varphi' \cos \varphi - \kappa), \\ \lambda_2 &= -(\varphi')^3 \cos \varphi - \tau'(\|W\| - \varphi') + \kappa(\varphi')^2 - \|W\|' (\varphi' \sin \varphi - \tau), \end{aligned}$$

$$\begin{aligned} \sigma_1 &= -\|W\| (\varphi' - \|W\|)^2 (-\varphi' \sin \varphi + \tau), \\ \sigma_2 &= \lambda_1 (-\varphi' \sin \varphi + \tau) - \lambda_2 (\varphi' \cos \varphi - \kappa), \\ \sigma_3 &= \|W\| (\varphi' - \|W\|)^2 (\varphi' \cos \varphi - \kappa), \end{aligned}$$

$$\begin{aligned} \eta_1 &= \left(\frac{\lambda_1}{(\varphi' - \|W\|)^3} \right)' - \frac{\kappa \|W\|}{\varphi' - \|W\|}, \\ \eta_2 &= \left(\frac{\|W\|}{\varphi' - \|W\|} \right)' + \frac{\kappa \lambda_1 - \tau \lambda_2}{(\varphi' - \|W\|)^3}, \\ \eta_3 &= \left(\frac{\lambda_2}{(\varphi' - \|W\|)^3} \right)' + \frac{\tau \|W\|}{\varphi' - \|W\|}. \end{aligned}$$

Proof. The proof is similar to proof of Theorem 3.1. □

Corollary 3.3. *If α is a helix, then the Frenet apparatus of the $\bar{t}\bar{b}$ -Smarandache curve of its Natural mate curve is given by*

$$\begin{aligned} T_{\beta_3} &= -\frac{1}{C_1} t + \frac{C}{C_1} b, & N_{\beta_3} &= n, & B_{\beta_3} &= -\frac{C}{C_1} t - \frac{1}{C_1} b, \\ & & & & \kappa_{\beta_3} &= \sqrt{2}, & \tau_{\beta_3} &= 0, \end{aligned}$$

where $\frac{\tau}{\kappa} = C = \text{constant}$ and $C_1 = \sqrt{1 + C^2}$.

Definition 3.6. Let $(\alpha, \bar{\alpha})$ be a Natural pair in \mathbb{E}^3 and $\{\bar{t}, \bar{n}, \bar{b}\}$ be the Frenet frame of the Natural mate curve $\bar{\alpha}$. The $\bar{t}\bar{n}\bar{b}$ -Smarandache curve is defined by

$$\beta_4(s) = \frac{1}{\sqrt{3}}(\bar{t}(\bar{s}) + \bar{n}(\bar{s}) + \bar{b}(\bar{s})). \quad (3.27)$$

Theorem 3.4. The Frenet apparatus of the $\bar{t}\bar{n}\bar{b}$ -Smarandache curve is given by

$$T_{\beta_4} = \frac{[\varphi'(\cos \varphi + \sin \varphi) - \kappa]t - \|W\|n + [\varphi'(\cos \varphi - \sin \varphi) + \tau]b}{\sqrt{(\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2}}, \quad (3.28)$$

$$N_{\beta_4} = \frac{\lambda_1 t + \lambda_2 n + \lambda_3 b}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}, \quad (3.29)$$

$$B_{\beta_4} = \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{\sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)[(\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2]}}, \quad (3.30)$$

$$\kappa_{\beta_4} = \frac{\sqrt{3}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}{[(\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2]^2}, \quad (3.31)$$

$$\tau_{\beta_4} = \frac{\sqrt{3}[(\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2](\sigma_1 \eta_1 + \sigma_2 \eta_2 + \sigma_3 \eta_3)}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}, \quad (3.32)$$

where $W = \tau t + \kappa b$ and the coefficients are given by

$$\lambda_1 = [\cos \varphi(\varphi'' + (\varphi')^2) + \sin \varphi(\varphi'' - (\varphi')^2) - \kappa' + \kappa \|W\|] \cdot [(\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2] - (\varphi'(\cos \varphi + \sin \varphi) - \kappa)((\varphi')^2 + \|W\|^2 - \varphi' \|W\|)',$$

$$\lambda_2 = [\|W\|(\varphi' - \|W\|) - \|W\|'] \cdot [(\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2] + \|W\|((\varphi')^2 + \|W\|^2 - \varphi' \|W\|)',$$

$$\lambda_3 = [\cos \varphi(\varphi'' - (\varphi')^2) - \sin \varphi(\varphi'' + (\varphi')^2) + \tau' - \tau \|W\|] \cdot [(\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2] - (\varphi'(\cos \varphi - \sin \varphi) + \tau)((\varphi')^2 + \|W\|^2 - \varphi' \|W\|)',$$

$$\sigma_1 = -\lambda_2[\varphi'(\cos \varphi - \sin \varphi) + \tau] - \lambda_3 \|W\|,$$

$$\sigma_2 = \lambda_1[\varphi'(\cos \varphi - \sin \varphi) + \tau] - \lambda_3[\varphi'(\cos \varphi + \sin \varphi) - \kappa],$$

$$\sigma_3 = \lambda_1 \|W\| + \lambda_2[\varphi'(\cos \varphi + \sin \varphi) - \kappa],$$

$$\eta_1 = \left(\frac{\lambda_1}{((\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2)^2} \right)' - \frac{\kappa \lambda_2}{((\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2)^2},$$

$$\eta_2 = \left(\frac{\lambda_2}{((\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2)^2} \right)' + \frac{\kappa \lambda_1 - \tau \lambda_3}{((\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2)^2},$$

$$\eta_3 = \left(\frac{\lambda_3}{((\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2)^2} \right)' + \frac{\tau \lambda_2}{((\varphi')^2 + \|W\|^2 + (\varphi' - \|W\|)^2)^2}.$$

Proof. The proof is similar to proof of Theorem 3.1. □

Corollary 3.4. *If α is a helix, then the Frenet apparatus of the $\bar{t}\bar{n}\bar{b}$ -Smarandache curve of its Natural mate curve is given by*

$$\begin{aligned} T_{\beta_4} &= -\frac{1}{\sqrt{2}C_1}t - \frac{1}{\sqrt{2}}n + \frac{C}{\sqrt{2}C_1}b, & N_{\beta_4} &= \frac{1}{\sqrt{2}C_1}t - \frac{1}{\sqrt{2}}n - \frac{C}{\sqrt{2}C_1}b, \\ B_{\beta_4} &= \frac{C}{C_1}t + \frac{1}{C_1}b, & \kappa_{\beta_4} &= \frac{\sqrt{3}}{\sqrt{2}}, & \tau_{\beta_4} &= 0, \end{aligned}$$

where $\frac{\tau}{\kappa} = C = \text{constant}$ and $C_1 = \sqrt{1 + C^2}$.

Example 1. *Given the slant helix*

$$\alpha(\bar{s}) = \left(\frac{3}{4} \cos(\bar{s}) + \frac{\cos(3\bar{s})}{12}, \frac{3}{4} \sin(\bar{s}) + \frac{\sin(3\bar{s})}{12}, -\frac{\sqrt{3}}{2} \cos(\bar{s}) \right), \quad \bar{s} \in [0, 2\pi].$$

After simple computation, we get

$$\begin{aligned} t &= \left(-\frac{3}{4} \sin(\bar{s}) - \frac{\sin(3\bar{s})}{4}, \frac{3}{4} \cos(\bar{s}) + \frac{\cos(3\bar{s})}{4}, \frac{\sqrt{3}}{2} \sin(\bar{s}) \right), \\ n &= \left(-\frac{\sqrt{3}}{2} \cos(2\bar{s}), -\frac{\sqrt{3}}{2} \sin(2\bar{s}), \frac{1}{2} \right), \\ b &= \left(\frac{1}{4}(3 \cos(\bar{s}) - \cos(3\bar{s})), \sin^3(\bar{s}), \frac{\sqrt{3}}{2} \cos(\bar{s}) \right), \\ \kappa &= \sqrt{3} \cos(\bar{s}), \quad \tau = \sqrt{3} \sin(\bar{s}). \end{aligned}$$

The Natural mate curve of α is the helix

$$\bar{\alpha}(\bar{s}) = \left(-\frac{\sqrt{3}}{4} \sin(2\bar{s}), \frac{\sqrt{3}}{4} \cos(2\bar{s}), \frac{\bar{s}}{2} \right), \quad \bar{s} \in [0, 2\pi].$$

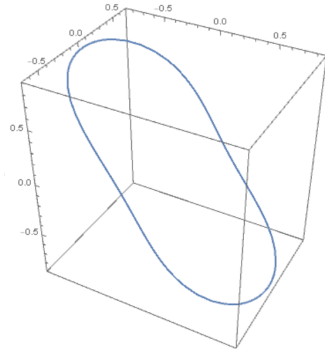
From equation (2.5), we get $\varphi(\bar{s}) = \bar{s} + \varphi_0$. If we choose $\varphi_0 = 0$, we have that

$$\begin{aligned} \bar{t} &= \left(-\frac{\sqrt{3}}{2} \cos(2\bar{s}), -\frac{\sqrt{3}}{2} \sin(2\bar{s}), \frac{1}{2} \right), \\ \bar{n} &= \left(\sin(2\bar{s}), -\cos(2\bar{s}), 0 \right), \\ \bar{b} &= \left(\frac{1}{2} \cos(2\bar{s}), \frac{1}{2} \sin(2\bar{s}), \frac{\sqrt{3}}{2} \right), \\ \bar{\kappa} &= \sqrt{3}, \quad \bar{\tau} = 1. \end{aligned}$$

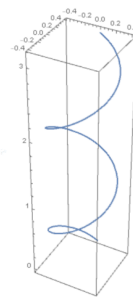
The $\bar{t}\bar{n}$, $\bar{n}\bar{b}$, $\bar{t}\bar{b}$, $\bar{t}\bar{n}\bar{b}$ -Smarandache curves are, respectively

$$\begin{aligned} \beta_1(s) &= \frac{1}{\sqrt{2}} \left(-\frac{\sqrt{3}}{2} \cos\left(\frac{2\sqrt{2}}{\sqrt{7}}s\right) + \sin\left(\frac{2\sqrt{2}}{\sqrt{7}}s\right), -\frac{\sqrt{3}}{2} \sin\left(\frac{2\sqrt{2}}{\sqrt{7}}s\right) - \cos\left(\frac{2\sqrt{2}}{\sqrt{7}}s\right), \frac{1}{2} \right), \\ \beta_2(s) &= \frac{1}{\sqrt{2}} \left(\frac{1}{2} \cos\left(\frac{2\sqrt{2}}{\sqrt{5}}s\right) + \sin\left(\frac{2\sqrt{2}}{\sqrt{5}}s\right), \frac{1}{2} \sin\left(\frac{2\sqrt{2}}{\sqrt{5}}s\right) - \cos\left(\frac{2\sqrt{2}}{\sqrt{5}}s\right), \frac{\sqrt{3}}{2} \right), \\ \beta_3(s) &= \frac{1}{\sqrt{2}} \left(\frac{\sqrt{3}-1}{2} \cos\left(\frac{2\sqrt{2}}{\sqrt{3}-1}s\right), -\frac{\sqrt{3}-1}{2} \sin\left(\frac{2\sqrt{2}}{\sqrt{3}-1}s\right), \frac{\sqrt{3}+1}{2} \right), \end{aligned}$$

$$\beta_4(s) = \frac{1}{\sqrt{3}} \left(\frac{1 - \sqrt{3}}{2} \cos \left(\frac{2\sqrt{3}}{\sqrt{8 - 2\sqrt{3}}} s \right) + \sin \left(\frac{2\sqrt{3}}{\sqrt{8 - 2\sqrt{3}}} s \right), \right. \\ \left. \frac{1 - \sqrt{3}}{2} \sin \left(\frac{2\sqrt{3}}{\sqrt{8 - 2\sqrt{3}}} s \right) - \cos \left(\frac{2\sqrt{3}}{\sqrt{8 - 2\sqrt{3}}} s \right), \frac{1 + \sqrt{3}}{2} \right).$$

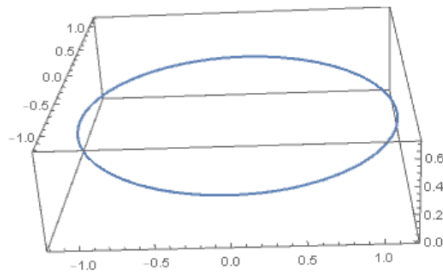


(A) Natural curve α

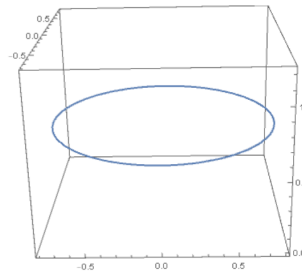


(B) Natural mate curve $\bar{\alpha}$

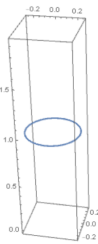
FIGURE 1. Natural Curves Pair $(\alpha, \bar{\alpha})$



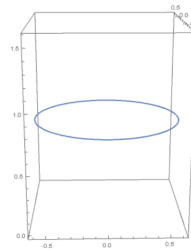
(A) $\bar{t}\bar{n}$ -Smarandache curve



(B) $\bar{n}\bar{b}$ -Smarandache curve



(C) $\bar{t}\bar{b}$ -Smarandache curve



(D) $\bar{t}\bar{n}\bar{b}$ -Smarandache curve

4. CONCLUSION

Smarandache curves have been receiving increasing attention from researchers due to their utility in differential geometry and their applications in various fields such as physics, engineering, robotics, computer graphics, and others. In this work, we study a special class of curves called Smarandache curves of curves, which are the natural mate of a curve α , according to the Frenet frame in three-dimensional Euclidean space. We calculate the Frenet apparatus for these curves and provide as a corollary the Frenet apparatus for the case where the curve α is a helix. Additionally, we present an example in which we calculate the Smarandache curves of the natural mate for the case where α is a slant helix. We hope that the results presented here contribute to the study of these curves and to the development of their applications.

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