Original Research Article

SMARANDACHE CURVES OF NATURAL CURVES PAIR ACCORDING TO FRENET FRAME

Abstract. In this paper, we investigate the Smarandache curves obtained by the vector of the Natural mate curve and their Frenet apparatus is calculated and expressed depending of the Natural curve. Moreover, we illustrate example with our main results.

Keywords: Smarandache curves. Natural mate curves. Frenet apparatus.

1. Introduction

A starting point in the study of differential geometry is the analysis of curves in space. By introducing special geometric properties, it is possible to produce new curves from a given curve. In this context, there are some special curves such as Bertrand, Mannheim, Natural mate, Smarandache, involute, evolute, and pedal curves.

For any given unit speed curve α in $\mathbb{E}^3$ there is a unique unit speed curve $\bar{\alpha}$ whose tangent vector coincides with the principal normal vector of α, called *principal-direction curve* or *Natural mate curve* [8]. Some authors have studied this class of curves: Abdel-Baky and Naghi [1] study sweeping surfaces with Natural mate curves, Deshmukh et al [9] prove some relationships between a Frenet curve and its natural mate, Camci et al [6] extend the natural mate $\bar{\alpha}$ to sequential natural mates $\{\alpha_1, \alpha_2, \cdots, \alpha_{n_\alpha}\}$ with $\alpha_1 = \bar{\alpha}$, Mak [10] introduce the natural mate curves in a three dimensional Lie group with bi-invariant metric and give some relationships between a Frenet curve and its natural mate in this group.

In differential geometry, the Smarandache geometry has a significant role in the theory of relativity and parallel universes [3]. In Smarandache geometry, regular curves that are defined as having the position vector being a combination of the tangent, normal and binormal vectors of another regular curve are called *Smarandache curves*. Special Smarandache curves have been studied by some authors. Savas et al studied special Smarandache curves according to the Sabban frame in hyperbolic space and new Smarandache partners in de Sitter space [13], Ali studied some special Smarandache curves in the Euclidean space [2], Şenyurt studied the Smarandache curves of Bertrand curves pair according to Frenet frame and expresses its curvature and torsion in terms of the curvature and torsion of the bertrand curve [14]. These curves have been also studied widely [16, 4, 5, 7].

In this paper, we start by mentioning the Frenet apparatus of any regular curve parametrized by arc-length in Euclidean space $\mathbb{E}^3$. Then, we give the definitions of the $tn, tb, nb, tnb-$Smarandache curves in $\mathbb{E}^3$ for Natural mate curves and calculate the

Frenet apparatus of these curves using the unit Darboux vector. We close with an example.

2. Preliminaries

Consider the Euclidean space $\mathbb{E}^3$ with inner product

$$\langle \cdot, \cdot \rangle = dx^2 + dy^2 + dz^2$$

where $(x, y, z) \in \mathbb{E}^3$ is a rectangular coordinate system. Let $\alpha : I \to \mathbb{E}^3$ be a differentiable curve in the Euclidean space defined on an open interval I, parametrized by arc-length and let $\{t = \alpha', n, b\}$ be Frenet frame satisfying [11]

$$\begin{cases} t' = \kappa n, \\ n' = -\kappa t + \tau b, \\ b' = -\tau n, \end{cases} \tag{2.1}$$

where κ e τ are differentiable functions on I called the *curvature* and the *torsion* of α, respectively, t is the tangent vector, n is the principal normal vector and b is the binormal vector of α. The 5-uple (t, n, b, κ, τ) is called a *Frenet apparatus.*

Definition 2.1. *A curve $\bar{\alpha} : J \to \mathbb{E}^3$ is called Natural mate curve of $\alpha : I \to \mathbb{E}^3$ if $\bar{\alpha}(\bar{s})$ is the integral curve of the principal normal vector of $\alpha(s^*)$, and the pair $(\alpha(s^*), \bar{\alpha}(\bar{s}))$ is called the Natural pair* [9].

The existence of Natural mate curves is guaranteed by existence theorem for differential equation and $\bar{\alpha}$ is given by $\bar{\alpha} = \int n ds^*$. It can be also shown that the arc-lenght parameters of the curve $\bar{\alpha}$ can be the same of α, that is, $s^* = \bar{s}$ [12].

The relations between the Frenet frames $\{t, n, b\}$ and $\{\bar{t}, \bar{n}, \bar{b}\}$ are given by [1]

$$\begin{cases} \bar{t} &= n, \\ \bar{n} &= -\cos\psi t + \sin\psi b, \\ \bar{b} &= \sin\psi t + \cos\psi b, \end{cases} \tag{2.2}$$

where $\psi = \angle(\bar{b}, b)$ is

$$\cos\psi = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \quad \text{and} \quad \sin\psi = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}. \tag{2.3}$$

Let the Darboux vector defined by

$$W = \tau t + \kappa b.$$

The Darboux vector represents the Frenet frame's angular momentum. Its direction determines the frame's momentary axis of motion (its centroid) and its length the angular speed, $\parallel W \parallel = \sqrt{\kappa^2 + \tau^2}$.

If we consider the normalization of the Darboux $C = \dfrac{1}{\parallel W \parallel} W$ we have,

$$\sin\varphi = \frac{\tau}{\parallel W \parallel} = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \quad \text{and} \quad \cos\varphi = \frac{\kappa}{\parallel W \parallel} = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}, \tag{2.4}$$

where $\varphi = \angle(W, b)$. Thus, by the equations (2.3) and (2.4), we have $\psi = \varphi + 2n\pi$, $n \in \mathbb{N}$.

From equation (2.4), we obtain

$$\varphi' = \frac{\kappa\tau' - \kappa'\tau}{\| W \|^2}. \tag{2.5}$$

3. Smarandache Curves of Natural Curves Pair According to Frenet Frame

In this section, we investigate the Smarandache curves of Natural pair according to Frenet frame in Euclidean 3-space and we give the Frenet apparatus for these curves.

Definition 3.2. *A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve* [17].

In the light of the above definition, Ali adapt it to regular curves in the Euclidean space the definition of Smarandache curves [2]. For Natural pair we have

Definition 3.3. *Let $(\alpha, \bar{\alpha})$ be a Natural pair in $\mathbb{E}^3$ and $\{\bar{t}, \bar{n}, \bar{b}\}$ be the Frenet frame of the Natural mate curve $\bar{\alpha}$. The $\bar{t}\bar{n}-$Smarandache curve is defined by*

$$\beta_1(s) = \frac{1}{\sqrt{2}}(\bar{t}(\bar{s}) + \bar{n}(\bar{s})). \tag{3.6}$$

Theorem 3.1. *The Frenet apparatus of the $\bar{t}\bar{n}-$Smarandache curve is given by:*

$$T_{\beta_1} = \frac{(\varphi'\sin\varphi - \kappa)t - \| W \| n + (\varphi'\cos\varphi + \tau)b}{\sqrt{(\varphi')^2 + 2 \| W \|^2}} \tag{3.7}$$

$$N_{\beta_1} = \frac{\lambda_1 t + \lambda_2 n + \lambda_3 b}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \tag{3.8}$$

$$B_{\beta_1} = \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{\sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)((\varphi')^2 + 2 \| W \|^2)}} \tag{3.9}$$

$$\kappa_{\beta_1} = \frac{\sqrt{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}}{[(\varphi')^2 + 2 \| W \|^2]^2} \tag{3.10}$$

$$\tau_{\beta_1} = \frac{\sqrt{2}[(\varphi')^2 + 2 \| W \|^2](\sigma_1\eta_1 + \sigma_2\eta_2 + \sigma_3\eta_3)}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \tag{3.11}$$

where $W = \tau t + \kappa b$ and the coefficients are given by

$$\begin{aligned}
\lambda_1 &= ((\varphi')^2\cos\varphi - \kappa' + \kappa \| W \|)[(\varphi')^2 + 2 \| W \|^2] + 2\varphi'' \| W \| \tau \\
&+ \kappa(\varphi'\varphi'' + 2 \| W \|\| W \|'), \\
\lambda_2 &= - \| W \|^2 [(\varphi')^2 + 2 \| W \|^2] + \varphi'(\| W \| \varphi'' - \| W \|' \varphi'), \\
\lambda_3 &= (-(\varphi')^2\sin\varphi + \tau' - \tau \| W \|)[(\varphi')^2 + 2 \| W \|^2] + 2\varphi'' \| W \| \kappa \\
&- \tau(\varphi'\varphi'' + 2 \| W \|\| W \|'),
\end{aligned}$$

$$\begin{aligned}
\sigma_1 &= -\lambda_2(\varphi'\cos\varphi + \tau) - \lambda_3 \| W \|, \\
\sigma_2 &= \lambda_1(\varphi'\cos\varphi + \tau) - \lambda_3(\varphi'\sin\varphi - \kappa), \\
\sigma_3 &= \lambda_1 \| W \| + \lambda_2(\varphi'\sin\varphi - \kappa),
\end{aligned}$$

$$\eta_1 = \left(\frac{\lambda_1}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}\right)' - \frac{\kappa\lambda_2}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2},$$

$$\eta_2 = \left(\frac{\lambda_2}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}\right)' + \frac{\kappa\lambda_1 - \tau\lambda_3}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2},$$

$$\eta_3 = \left(\frac{\lambda_3}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}\right)' + \frac{\tau\lambda_2}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}.$$

Proof. Substituting the equation (2.2) into equation (3.6), we obtain

$$\beta_1 = \frac{-\cos\varphi t + n + \sin\varphi b}{\sqrt{2}}. \tag{3.12}$$

Taking the derivative of the equation (3.12) with respect to $\bar{s}$, we get

$$T_{\beta_1}\frac{ds}{d\bar{s}} = \frac{(\varphi'\sin\varphi - \kappa)t - \parallel W \parallel n + (\varphi'\cos\varphi + \tau)b}{\sqrt{2}}.$$

Thus

$$T_{\beta_1} = \frac{(\varphi'\sin\varphi - \kappa)t - \parallel W \parallel n + (\varphi'\cos\varphi + \tau)b}{\sqrt{(\varphi')^2 + 2 \parallel W \parallel^2}}, \tag{3.13}$$

where

$$\frac{ds}{d\bar{s}} = \frac{\sqrt{(\varphi')^2 + 2 \parallel W \parallel^2}}{\sqrt{2}}. \tag{3.14}$$

Taking the derivative of the equation (3.13) with respect to $\bar{s}$ and use (3.14), we obtain

$$T'_{\beta_1} = \frac{\sqrt{2}(\lambda_1 t + \lambda_2 n + \lambda_3 b)}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2},$$

where

$$\begin{aligned}
\lambda_1 &= ((\varphi')^2\cos\varphi - \kappa' + \kappa \parallel W \parallel)[(\varphi')^2 + 2 \parallel W \parallel^2] + 2\varphi'' \parallel W \parallel \tau \\
&+ \kappa(\varphi'\varphi'' + 2 \parallel W \parallel \parallel W \parallel'), \\
\lambda_2 &= - \parallel W \parallel^2 [(\varphi')^2 + 2 \parallel W \parallel^2] + \varphi'(\parallel W \parallel \varphi'' - \parallel W \parallel' \varphi'), \\
\lambda_3 &= (-(\varphi')^2\sin\varphi + \tau' - \tau \parallel W \parallel)[(\varphi')^2 + 2 \parallel W \parallel^2] + 2\varphi'' \parallel W \parallel \kappa \\
&- \tau(\varphi'\varphi'' + 2 \parallel W \parallel \parallel W \parallel'),
\end{aligned}$$

Therefore, the curvature, the principal normal vector and the binormal vector of the curve β_1 are given by

$$\kappa_{\beta_1} = \frac{\sqrt{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}, \qquad N_{\beta_1} = \frac{\lambda_1 t + \lambda_2 n + \lambda_3 b}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}},$$

$$\begin{aligned}
B_{\beta_1} &= T_{\beta_1} \times N_{\beta_1} \\
&= \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{\sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)((\varphi')^2 + 2 \parallel W \parallel^2)}},
\end{aligned}$$

where

$$\begin{aligned}
\sigma_1 &= -\lambda_2(\varphi' \cos\varphi + \tau) - \lambda_3 \parallel W \parallel, \\
\sigma_2 &= \lambda_1(\varphi' \cos\varphi + \tau) - \lambda_3(\varphi' \sin\varphi - \kappa), \\
\sigma_3 &= \lambda_1 \parallel W \parallel + \lambda_2(\varphi' \sin\varphi - \kappa).
\end{aligned}$$

The torsion of curve β_1 is given by

$$\tau_{\beta_1} = \frac{\det(\beta_1', \beta_1'', \beta_1''')}{\parallel \beta_1' \times \beta_1'' \parallel^2} = \frac{\sqrt{2}[(\varphi')^2 + 2 \parallel W \parallel^2](\sigma_1\eta_1 + \sigma_2\eta_2 + \sigma_3\eta_3)}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2},$$

where

$$\begin{aligned}
\eta_1 &= \left(\frac{\lambda_1}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}\right)' - \frac{\kappa\lambda_2}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}, \\
\eta_2 &= \left(\frac{\lambda_2}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}\right)' + \frac{\kappa\lambda_1 - \tau\lambda_3}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}, \\
\eta_3 &= \left(\frac{\lambda_3}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}\right)' + \frac{\tau\lambda_2}{[(\varphi')^2 + 2 \parallel W \parallel^2]^2}.
\end{aligned}$$

□

Definition 3.4. *Let $(\alpha, \bar{\alpha})$ be a Natural pair in $\mathbb{E}^3$ and $\{\bar{t}, \bar{n}, \bar{b}\}$ be the Frenet frame of the Natural mate curve $\bar{\alpha}$. The $\bar{n}\bar{b}-$Smarandache curve is defined by*

$$\beta_2(s) = \frac{1}{\sqrt{2}}(\bar{n}(\bar{s}) + \bar{b}(\bar{s})). \tag{3.15}$$

Theorem 3.2. *The Frenet apparatus of the $\bar{n}\bar{b}-$Smarandache curve is given by:*

$$T_{\beta_2} = \frac{\varphi'(\sin\varphi + \cos\varphi)t - \parallel W \parallel n + \varphi'(\cos\varphi - \sin\varphi)b}{\sqrt{2(\varphi')^2 + \parallel W \parallel^2}} \tag{3.16}$$

$$N_{\beta_2} = \frac{\lambda_1 t + \lambda_2 n + \lambda_3 b}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \tag{3.17}$$

$$B_{\beta_2} = \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{\sqrt{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(2(\varphi')^2 + \parallel W \parallel^2)}} \tag{3.18}$$

$$\kappa_{\beta_2} = \frac{\sqrt{2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}}{[2(\varphi')^2 + \parallel W \parallel^2]^2} \tag{3.19}$$

$$\tau_{\beta_2} = \frac{\sqrt{2}[2(\varphi')^2 + \parallel W \parallel^2](\sigma_1\eta_1 + \sigma_2\eta_2 + \sigma_3\eta_3)}{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \tag{3.20}$$

where $W = \tau t + \kappa b$ *and the coefficients are given by*

$$\begin{aligned}
\lambda_1 &= ((\varphi')^2(\cos\varphi - \sin\varphi) + \kappa \parallel W \parallel)[2(\varphi')^2 + \parallel W \parallel^2] \\
&+ \parallel W \parallel (\sin\varphi + \cos\varphi)(\varphi'' \parallel W \parallel - \varphi' \parallel W \parallel'), \\
\lambda_2 &= \varphi'(\parallel W \parallel [2(\varphi')^2 + \parallel W \parallel^2] + 2(\varphi'' \parallel W \parallel - \varphi' \parallel W \parallel')), \\
\lambda_3 &= -((\varphi')^2(\sin\varphi + \cos\varphi) + \tau \parallel W \parallel)[2(\varphi')^2 + \parallel W \parallel^2] \\
&+ \parallel W \parallel (\cos\varphi - \sin\varphi)(\varphi'' \parallel W \parallel + \varphi' \parallel W \parallel'),
\end{aligned}$$

$$\begin{aligned}
\sigma_1 &= -\lambda_2\varphi'(\cos\varphi - \sin\varphi) - \lambda_3 \parallel W \parallel, \\
\sigma_2 &= \lambda_1\varphi'(\cos\varphi - \sin\varphi) - \lambda_3\varphi'(\sin\varphi + \cos\varphi), \\
\sigma_3 &= \lambda_1 \parallel W \parallel + \lambda_2\varphi'(\sin\varphi + \cos\varphi),
\end{aligned}$$

$$\begin{aligned}
\eta_1 &= \left(\frac{\lambda_1}{[2(\varphi')^2 + \parallel W \parallel^2]^2}\right)' - \frac{\kappa\lambda_2}{[2(\varphi')^2 + \parallel W \parallel^2]^2}, \\
\eta_2 &= \left(\frac{\lambda_2}{[2(\varphi')^2 + \parallel W \parallel^2]^2}\right)' + \frac{\kappa\lambda_1 - \tau\lambda_3}{[2(\varphi')^2 + \parallel W \parallel^2]^2}, \\
\eta_3 &= \left(\frac{\lambda_3}{[2(\varphi')^2 + \parallel W \parallel^2]^2}\right)' + \frac{\tau\lambda_2}{[2(\varphi')^2 + \parallel W \parallel^2]^2}.
\end{aligned}$$

Proof. The proof is similar to proof of Theorem 3.1. □

Definition 3.5. *Let* $(\alpha, \bar{\alpha})$ *be a Natural pair in* $\mathbb{E}^3$ *and* $\{\bar{t}, \bar{n}, \bar{b}\}$ *be the Frenet frame of the Natural mate curve* $\bar{\alpha}$*. The* $\bar{t}\bar{b}-$*Smarandache curve is defined by*

$$\beta_3(s) = \frac{1}{\sqrt{2}}(\bar{t}(\bar{s}) + \bar{b}(\bar{s})). \tag{3.21}$$

Theorem 3.3. *The Frenet apparatus of the* $\bar{t}\bar{b}-$*Smarandache curve is given by:*

$$T_{\beta_3} = \frac{(\varphi'\cos\varphi - \kappa)t + (-\varphi'\sin\varphi + \tau)b}{\varphi' - \parallel W \parallel} \tag{3.22}$$

$$N_{\beta_3} = \frac{\lambda_1 t + \parallel W \parallel (\varphi' - \parallel W \parallel)^2 n + \lambda_2 b}{\sqrt{\lambda_1^2 + \parallel W \parallel^2 (\varphi' - \parallel W \parallel)^4 + \lambda_2^2}} \tag{3.23}$$

$$B_{\beta_3} = \frac{\sigma_1 t + \sigma_2 n + \sigma_3 b}{(\varphi' - \parallel W \parallel)\sqrt{(\lambda_1^2 + \parallel W \parallel^2 (\varphi' - \parallel W \parallel)^4 + \lambda_2^2)}} \tag{3.24}$$

$$\kappa_{\beta_3} = \frac{\sqrt{2(\lambda_1^2 + \parallel W \parallel^2 (\varphi' - \parallel W \parallel)^4 + \lambda_2^2)}}{(\varphi' - \parallel W \parallel)^3} \tag{3.25}$$

$$\tau_{\beta_3} = \frac{\sqrt{2}(\varphi' - \parallel W \parallel)(\sigma_1\eta_1 + \sigma_2\eta_2 + \sigma_3\eta_3)}{\lambda_1^2 + \parallel W \parallel^2 (\varphi' - \parallel W \parallel)^4 + \lambda_2^2} \tag{3.26}$$

where $W=\tau t+\kappa b$ *and the coefficients are given by*

$$\begin{aligned}\lambda_1&=-(\varphi')^3\sin\varphi+\kappa'(\parallel W\parallel-\varphi')+\tau(\varphi')^2+\parallel W\parallel'(\varphi'\cos\varphi-\kappa),\\ \lambda_2&=-(\varphi')^3\cos\varphi-\tau'(\parallel W\parallel-\varphi')+\kappa(\varphi')^2-\parallel W\parallel'(\varphi'\sin\varphi-\tau),\end{aligned}$$

$$\begin{aligned}\sigma_1&=-\parallel W\parallel(\varphi'-\parallel W\parallel)^2(-\varphi'\sin\varphi+\tau),\\ \sigma_2&=\lambda_1(-\varphi'\sin\varphi+\tau)-\lambda_2(\varphi'\cos\varphi-\kappa),\\ \sigma_3&=\parallel W\parallel(\varphi'-\parallel W\parallel)^2(\varphi'\cos\varphi-\kappa),\end{aligned}$$

$$\begin{aligned}\eta_1&=\left(\frac{\lambda_1}{(\varphi'-\parallel W\parallel)^3}\right)'-\frac{\kappa\parallel W\parallel}{\varphi'-\parallel W\parallel},\\ \eta_2&=\left(\frac{\parallel W\parallel}{\varphi'-\parallel W\parallel}\right)'+\frac{\kappa\lambda_1-\tau\lambda_2}{(\varphi'-\parallel W\parallel)^3},\\ \eta_3&=\left(\frac{\lambda_2}{(\varphi'-\parallel W\parallel)^3}\right)'+\frac{\tau\parallel W\parallel}{\varphi'-\parallel W\parallel}.\end{aligned}$$

Proof. The proof is similar to proof of Theorem 3.1. □

Definition 3.6. *Let* $(\alpha,\bar{\alpha})$ *be a Natural pair in* $\mathbb{E}^3$ *and* $\{\bar{t},\bar{n},\bar{b}\}$ *be the Frenet frame of the Natural mate curve* $\bar{\alpha}$. *The* $\bar{t}\bar{n}\bar{b}-$*Smarandache curve is defined by*

$$\beta_4(s)=\frac{1}{\sqrt{3}}(\bar{t}(\bar{s})+\bar{n}(\bar{s})+\bar{b}(\bar{s})). \tag{3.27}$$

Theorem 3.4. *The Frenet apparatus of the* $\bar{t}\bar{n}\bar{b}-$*Smarandache curve is given by:*

$$T_{\beta_4}=\frac{[\varphi'(\cos\varphi+\sin\varphi)-\kappa]t-\parallel W\parallel n+[\varphi'(\cos\varphi-\sin\varphi)+\tau]b}{\sqrt{(\varphi')^2+\parallel W\parallel^2+(\varphi'-\parallel W\parallel)^2}} \tag{3.28}$$

$$N_{\beta_4}=\frac{\lambda_1 t+\lambda_2 n+\lambda_3 b}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}} \tag{3.29}$$

$$B_{\beta_4}=\frac{\sigma_1 t+\sigma_2 n+\sigma_3 b}{\sqrt{(\lambda_1^2+\lambda_2^2+\lambda_3^2)[(\varphi')^2+\parallel W\parallel^2+(\varphi'-\parallel W\parallel)^2]}} \tag{3.30}$$

$$\kappa_{\beta_4}=\frac{\sqrt{3(\lambda_1^2+\lambda_2^2+\lambda_3^2)}}{[(\varphi')^2+\parallel W\parallel^2+(\varphi'-\parallel W\parallel)^2]^2} \tag{3.31}$$

$$\tau_{\beta_4}=\frac{\sqrt{3}[(\varphi')^2+\parallel W\parallel^2+(\varphi'-\parallel W\parallel)^2](\sigma_1\eta_1+\sigma_2\eta_2+\sigma_3\eta_3)}{\lambda_1^2+\lambda_2^2+\lambda_3^2} \tag{3.32}$$

where $W = \tau t + \kappa b$ *and the coefficients are given by*

$$\begin{aligned}
\lambda_1 &= (\cos\varphi(\varphi'' + (\varphi')^2) + \sin\varphi(\varphi'' - (\varphi')^2) - \kappa' + \kappa \parallel W \parallel)((\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2) \\
&- (\varphi'(\cos\varphi + \sin\varphi) - \kappa)(\varphi^2 + \parallel W \parallel^2 - \varphi' \parallel W \parallel)', \\
\lambda_2 &= [\parallel W \parallel (\varphi' - \parallel W \parallel) - \parallel W \parallel'][(\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2] \\
&+ \parallel W \parallel (\varphi^2 + \parallel W \parallel^2 - \varphi' \parallel W \parallel)', \\
\lambda_3 &= (\cos\varphi(\varphi'' - (\varphi')^2) - \sin\varphi(\varphi'' + (\varphi')^2) + \tau' - \tau \parallel W \parallel)((\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2) \\
&- (\varphi'(\cos\varphi - \sin\varphi) + \tau)(\varphi^2 + \parallel W \parallel^2 - \varphi' \parallel W \parallel)',
\end{aligned}$$

$$\begin{aligned}
\sigma_1 &= -\lambda_2[\varphi'(\cos\varphi - \sin\varphi) + \tau] - \lambda_3 \parallel W \parallel, \\
\sigma_2 &= \lambda_1[\varphi'(\cos\varphi - \sin\varphi) + \tau] - \lambda_3[\varphi'(\cos\varphi + \sin\varphi) - \kappa], \\
\sigma_3 &= \lambda_1 \parallel W \parallel + \lambda_2[\varphi'(\cos\varphi + \sin\varphi) - \kappa],
\end{aligned}$$

$$\begin{aligned}
\eta_1 &= \left(\frac{\lambda_1}{(\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2)^2}\right)' - \frac{\kappa\lambda_2}{((\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2)^2}, \\
\eta_2 &= \left(\frac{\lambda_2}{((\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2)^2}\right)' + \frac{\kappa\lambda_1 - \tau\lambda_3}{((\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2)^2}, \\
\eta_3 &= \left(\frac{\lambda_3}{((\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2)^2}\right)' + \frac{\tau\lambda_2}{((\varphi')^2 + \parallel W \parallel^2 + (\varphi' - \parallel W \parallel)^2)^2}.
\end{aligned}$$

Proof. The proof is similar to proof of Theorem 3.1. □

Example 1. *Given the slant helix*

$$\alpha(\bar{s}) = \left(\frac{3}{4}\cos(\bar{s}) + \frac{\cos(3\bar{s})}{12}, \frac{3}{4}\sin(\bar{s}) + \frac{\sin(3\bar{s})}{12}, -\frac{\sqrt{3}}{2}\cos(\bar{s})\right), \quad \bar{s} \in [0, 2\pi].$$

After simple computation, we get

$$t = \left(-\frac{3}{4}\sin(\bar{s}) - \frac{\sin(3\bar{s})}{4}, \frac{3}{4}\cos(\bar{s}) + \frac{\cos(3\bar{s})}{4}, \frac{\sqrt{3}}{2}\sin(\bar{s})\right),$$

$$n = \left(-\frac{\sqrt{3}}{2}\cos(2\bar{s}), -\frac{\sqrt{3}}{2}\sin(2\bar{s}), \frac{1}{2}\right),$$

$$b = \left(\frac{1}{4}(3\cos(\bar{s}) - \cos(3\bar{s})), \sin^3(\bar{s}), \frac{\sqrt{3}}{2}\cos(\bar{s})\right),$$

$$\kappa = \sqrt{3}\cos(\bar{s}), \quad \tau = \sqrt{3}\sin(\bar{s}).$$

The Natural mate curve of α *is the helix*

$$\bar{\alpha}(\bar{s}) = \left(-\frac{\sqrt{3}}{4}\sin(2\bar{s}), \frac{\sqrt{3}}{4}\cos(2\bar{s}), \frac{\bar{s}}{2}\right), \quad \bar{s} \in [0, 2\pi].$$

From equation (2.5), we get $\varphi(\bar{s}) = \bar{s} + \varphi_0$*. If we choose* $\varphi_0 = 0$*, we have that*

$$\bar{t} = \left(-\frac{\sqrt{3}}{2}\cos(2\bar{s}), -\frac{\sqrt{3}}{2}\sin(2\bar{s}), \frac{1}{2} \right),$$

$$\bar{n} = \left(\sin(2\bar{s}), -\cos(2\bar{s}), 0 \right),$$

$$\bar{b} = \left(\frac{1}{2}\cos(2\bar{s}), \frac{1}{2}\sin(2\bar{s}), \frac{\sqrt{3}}{2} \right),$$

$$\bar{\kappa} = \sqrt{3}, \qquad \bar{\tau} = 1.$$

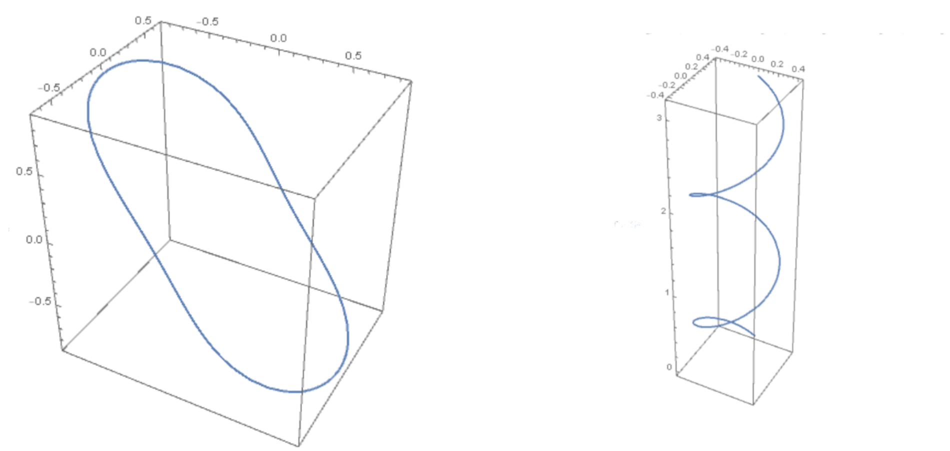
The $\bar{t}\bar{n}, \bar{n}\bar{b}, \bar{t}\bar{b}, \bar{t}\bar{n}\bar{b}-$*Smarandache curves are, respectively*

$$\beta_1(s) = \frac{1}{\sqrt{2}}\left(-\frac{\sqrt{3}}{2}\cos\left(\frac{2\sqrt{2}}{\sqrt{7}}s\right) + \sin\left(\frac{2\sqrt{2}}{\sqrt{7}}s\right), -\frac{\sqrt{3}}{2}\sin\left(\frac{2\sqrt{2}}{\sqrt{7}}s\right) - \cos\left(\frac{2\sqrt{2}}{\sqrt{7}}s\right), \frac{1}{2} \right),$$

$$\beta_2(s) = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\cos\left(\frac{2\sqrt{2}}{\sqrt{5}}s\right) + \sin\left(\frac{2\sqrt{2}}{\sqrt{5}}s\right), \frac{1}{2}\sin\left(\frac{2\sqrt{2}}{\sqrt{5}}s\right) - \cos\left(\frac{2\sqrt{2}}{\sqrt{5}}s\right), \frac{\sqrt{3}}{2} \right),$$

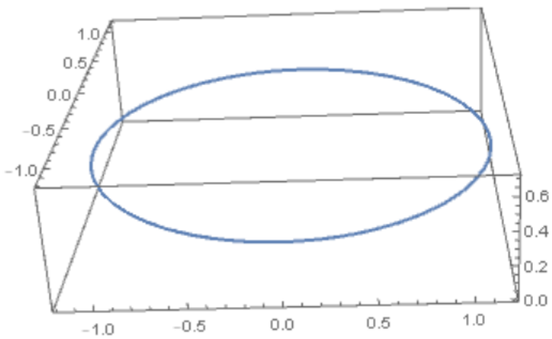
$$\beta_3(s) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}-1}{2}\cos\left(\frac{2\sqrt{2}}{\sqrt{3}-1}s\right), -\frac{\sqrt{3}-1}{2}\sin\left(\frac{2\sqrt{2}}{\sqrt{3}-1}s\right), \frac{\sqrt{3}+1}{2} \right),$$

$$\beta_4(s) = \frac{1}{\sqrt{3}}\left(\frac{1-\sqrt{3}}{2}\cos\left(\frac{2\sqrt{3}}{\sqrt{8-2\sqrt{3}}}s\right) + \sin\left(\frac{2\sqrt{3}}{\sqrt{8-2\sqrt{3}}}s\right), \right.$$
$$\left. \frac{1-\sqrt{3}}{2}\sin\left(\frac{2\sqrt{3}}{\sqrt{8-2\sqrt{3}}}s\right) - \cos\left(\frac{2\sqrt{3}}{\sqrt{8-2\sqrt{3}}}s\right), \frac{1+\sqrt{3}}{2} \right).$$

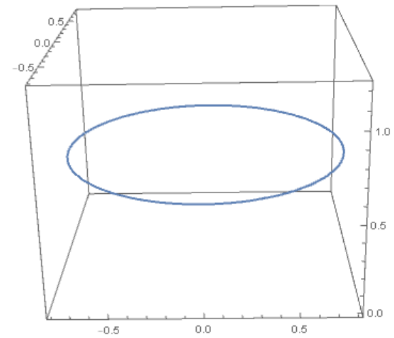


(A) Natural curve α (B) Natural mate curve $\bar{\alpha}$

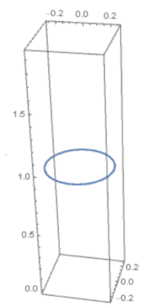
FIGURE 1. Natural Curves Pair $(\alpha, \bar{\alpha})$



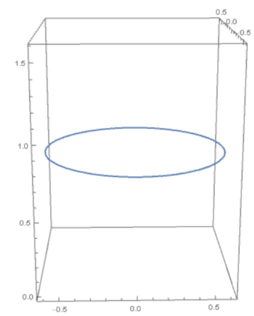
(A) $\bar{t}\bar{n}-$Smarandache curve



(B) $\bar{n}\bar{b}-$Smarandache curve



(C) $\bar{t}\bar{b}-$Smarandache curve



(D) $\bar{t}\bar{n}\bar{b}-$Smarandache curve

Fig. 2. Smarandache curve

4. Conclusion

In the present paper, we have studied special curves called Smarandache curves of Natural mate curves pair according to Frenet frame in the Euclidean 3-space $\mathbb{E}^3$. These curves are composed using Frenet frame vectors of Natural mate curve. Moreover, the Frenet apparatus for the meaning curves are obtained. Finally, a computational example is given and plotted.

References

[1] Abdel-Baky, R. A., Naghi, M. F., *Sweeping surfaces with Natural mate curve of a spatial curve in Euclidean 3-space*, WSEAS Transactions on Mathematics, 19, 581-588, (2020).

[2] Ali, A. T., *Smarandache Curves in the Euclidean Space*, International Journal of Mathematical Combinatorics, 2, 30-36, (2010).

[3] Ashbacher, C., *Smarandache geometries*, Smarandache Notions Journal, 8, 13, 212-215, (1997).

[4] Çalışkan, A., Şenyurt, S., *Smarandache Curves In Terms of Sabban Frame of Fixed Pole Curve*, Boletim da Sociedade Paranaense de Matemática, 34, 2, 53-62, (2016).

[5] Çalışkan, A., Şenyurt, S., *N^*C^*- Smarandache Curves of Mannheim Curve Couple According to Frenet Frame*, International J.Math. Combin. 1, 1-13, (2015).

[6] Camci, Ç., Chen, BY., Ilarslan, K., Uçum, A., *Sequential natural mates of Frenet curves in Euclidean 3-space*, Journal of Geometry, 112, 46, (2021).

[7] Cetin, M., Tuncer, Y., Karacan, M. K., *Smarandache Curves According to Bishop Frame in Euclidean 3-Space*, Gen. Math. Notes, 20, 2, 50-66, (2014).

[8] Choi, J., Kim, Y., *Associated curves of a Frenet curve and their Applications*, Applied Mathematics and Computation, 218, 9116-9124, (2012).

[9] Deshmukh, S., Chen, B.Y. and Alghanemi, A. *Natural mates of Frenet curves in Euclidean 3-space*, Turkish Journal of Mathematics, 42, 2826-2840, (2018).

[10] Mak, M., *Natural and Conjugate Mates of Frenet Curves in Three-Dimensinal Lie Group*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 70, 1, 522-540, (2021).

[11] do Carmo, M. P., *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, NJ, (1976).

[12] Nurkan, S. K., Güven, *İ*. A., Karakan, M. K., *Characterizations of Adjoint Curves in Euclidean 3-space*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 89, 155-161, (2018).

[13] Savas. M., Yakut, A. T., Tamirci, T., *The Smarandache Curves on* $\mathbb{H}_0^2$, Gazi University Journal of Science, 29, 1, 69-77, (2016).

[14] Şenyurt, S., Çalişkan, A., Çelik, *Ü*., *Smarandache Curves of Bertrand Curves Pair According to Frenet Frame*, Boletim da Sociedade Paranaense de Matemática, 39, 5, 163-173, (2021).

[15] Solouma, E. M., Wageeda, M. M., *Special Smarandache Curves According to Bishop Frame in Euclidean Spacetime*, International Journal of Mathematical Combinatorics, 1, 01-09, (2017).

[16] Taşköprü, K., Tosun, M., *Smarandache Curves According to Sabban Frame on* $\mathbb{S}^2$, Boletim da Sociedade Paranaense de Matemática, 32, 1, 51-59, (2014).

[17] Turgut, M., Yilmaz, S., *Smarandache Curves in Minkowski Space-time*, International Journal of Mathematical Combinatorics, 3, 51-55, (2008).