On Some Properties of Square normal operators

Abstract

The study of operators in Hilbert spaces is an important concept due to its wide application in areas like computer programming, financial mathematics and quantum physics. This paper focused on a class of square normal operators in a Hilbert space. Let H be a complex Hilbert space and $B(H)$ be a bounded linear operator acting on H. Then an operator T in $B(H)$ is a square normal if $T^2(T^*)^2 = (T^*)^2T^2$. This paper studied the commutation relations and properties of this class of operators and showed that for any square normal operator T, then T^* and T^{-1} if it exists is square normal. Furthermore, the sum $T + S$ and product TS of two square normal operators which commute with the adjoint of each other is square normal. To achieve this, the properties of normal operators and other operators related to normal operators were extended to square normal operators.

Keywords: Normal operators, Square normal operators, commutation relations and adjoint.

1 Introduction

Throughout this paper $B(H)$ denotes the algebra of all bounded linear operators on Hilbert space H. A linear operator T on a Hilbert space H is said to be bounded if there exist a constant $c > 0$ such that $\|Tx\| \leq c\,\|x\|\,\forall x \in H$. T is called self-adjoint if $T = T^*$, invertible with inverse S if there exists $S \in B(H)$ such that $ST = I = TS$ where $I \in B(H)$ is the identity operator. An operator $T \in B(H)$ is said to be normal if it commutes with its adjoint i.e ($T^*T = TT^*$), equivalently $T^*T - TT^* = 0$. An operator $T \in B(H)$ is said to be positive if $T^* = T$ and $\langle Tx, x\rangle \geq 0\ \forall x \in H$. An operator $T \in B(H)$ is said to be n-power normal if $T^nT^* = T^*T^n$ for $n \in \mathbb{N}$, class Q operator if for any $T \in Q$, $T^{*2}T^2 = (T^*T)^2$. $T \in B(H)$ is called a class Q^* if $T^{*2}T^2 = (TT^*)^2$ and Quasi-class Q if $T^{*3}T^3 - 2T^{*2}T^2 + T^*T \geq 0$. An operator $T \in B(H)$ is in class μ if $T^2 =-T^{*2}$ An operator $T \in B(H)$ is called an n-power-hyponormal operator if $T^nT^* \leq T^*T^n$. This class includes all normal, all n-normal and all hyponormal operators. An operator $T \in B(H)$ is Binormal if T^*T commutes with TT^*, That is $(T^*T)(TT^*) = (TT^*)(T^*T)$. An operator $T \in B(H)$ is square normal if $T^2(T^*)^2 = (T^*)^2T^2$

2 Methodology

In [8], the author extended the class of normal operators to a class of square normal operators and established that every normal operator is a square normal operator, but the reverse is not

necessarily true. This study extends the properties of normal operators and their related operators to square normal operators. These properties include adjoint, inverse, unitary equivalence, scalar addition and multiplication, as well as the product and sum of operators.
Many authors have investigated these properties on other operators in Hilbert spaces. For instance, [11] investigated the above properties on n-binomial operators. [7] investigated the concept of unitary equivalence and noted that any operator S unitarily equivalent to operator T in μ is also in μ. [3] proved that if $T \in B(H)$ is of quasi-class Q, then T, if it exists, is of quasi-class Q. As seen in [9], the authors proved that any operator unitarily equivalent to an n-hyponormal operator is also an n-hyponormal operator. This concept of unitary equivalence is also extended to n-class Q operators. Also, according to [12], for any T in n-class Q whose T exists, T is an operator of n-class Q. [13] also showed that the above three properties hold for class Q^* operators.

The concept of scalar multiplication and scalar addition on operators in the Hilbert space has been investigated by many authors. [1] established that n-normal operators are closed under scalar multiplication but not closed under scalar subtraction unless the scalar is zero. According to [6], operators whose squares are 2-normal are closed under scalar multiplication. However, [6] also noted that the concept of scalar addition does not hold for (SN) operators. That is, if $T \in (SN)$, then it is not necessary that $T + \lambda I$ is in (SN). According to [11], n-power class Q operators are closed under scalar multiplication for any scalar $\lambda \in \mathbb{R}$. [12] noted that if $T \in B(H)$ is an n-class Q operator, then λT is also an n-class Q operator.

[2] studied the sum and product of two normal operators that commute with each other and obtained the following results.

Lemma 2.1: Conway, (1985)
If T and S are normal operators on H with the property that each commutes with the adjoint of the other, then

(i) $T + S$ is normal.

(ii) TS is normal.

In [1], the authors extended this concept to n-normal operators and noted that the commutativity condition of n-normal operators should not be ignored as seen in lemma 2.2.

Lemma 2.2: Alzuraiqi and Patel, (2010)
If S and T are commuting n-normal operators, then ST is an n-normal operator.
However, if operators S and T are non-commuting, then ST are not necessarily n-normal. According to [1], the sum of two commuting n-normal operators need not be n-normal as seen in the following example.
Given operators T and S as,

$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$TS = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$ST = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Now

$$S+T=\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}$$

and

$$(S+T)^2=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}1&2\\0&1\end{bmatrix}$$

$\begin{bmatrix}1&2\\0&1\end{bmatrix}$ is not normal and hence S+T is not an n-normal operator.

[6] studied the sum and product of operators in SN and noted that the product of two operators in SN is also in SN provided that the operators commute with each other. However, the sum of two operators in SN is not necessarily in SN regardless of the commutation property of the two operators being satisfied. According to [11], the product of two non-commuting 2-normal operators is not 2-normal. These also proved that the sum and difference of two commuting n-normal operators need not be n-normal. According to [7], μ operators are not closed under addition or multiplication. [9] noted that if T and S 2-power-hyponormal operators, such that $TS^* = S^*T$ and $ST + TS = 0$, then $T + S$ and ST are 2-power-hyponormal.

3 Properties of Square normal operators

Proposition 3.1.
If an operator $T \in B(H)$ is square normal, then

(i) T^* is a square normal operator.

(ii) If T is invertible, that is T^{-1} exist, then T^{-1} is a square normal operator.

(iii) Any operator $S \in B(H)$ that is unitarily equivalent to T is a square normal operator.

***Proof* (i)**
Since T is a square normal operator, we have

$$T^2T^{*2} = T^{*2}T^2 \tag{1}$$

Replacing T with T^* on the left and right of equation (1) we have,

$$(T^*)^2(T^*)^{*2} = (T^*)^{*2}(T^*)^2 \tag{2}$$

Simplifying equation (2) we obtain

$$T^{*2}T^2 = T^2T^{*2} \tag{3}$$

a square normal. Therefore, T^* is square normal. □

***Proof* (ii)**

$$\begin{aligned}(T^{-1})^2(T^{-1})^{*2} &= (T^2)^{-1}(T^{*2})^{-1}\\ &= (T^2T^{*2})^{-1}\\ &= (T^{*2})^{-1}(T^2)^{-1}\\ &= (T^{-1})^{*2}(T^{-1})^2\end{aligned}$$

Hence T^{-1} is square normal. □

***Proof* (iii)**

Let $S \in B(H)$ which is unitarily equivalent to T. There is a unitary operator $U \in B(H)$ such that $S = U^*TU$ which implies that $S^* = (U^*TU)^* = U^*T^*U$.

$$\begin{aligned} S^2S^{*2} &= U^*TUU^*TUU^*T^*UU^*T^*U \\ &= U^*T^2T^{*2}U \end{aligned} (UU^* = I) \tag{4}$$

$$\begin{aligned} S^{*2}S^2 &= U^*T^*UU^*T^*UU^*TUU^*TU \\ &= U^*T^*T^*TTU \\ &= U^*T^{*2}T^2U \end{aligned} \tag{5}$$

Since T is square normal, we have $U^*T^2T^{*2}U = U^*T^{*2}T^2U$. This implies that $S^2S^{*2} = S^{*2}S^2$ and so S is a square normal operator. □

Square normal operators are closed under scalar multiplication and addition.

Theorem 3.2.
Let $T \in B(H)$ be a normal operator, then, for any scalar $\lambda \in (\mathbb{C})$,

(i) λT is square normal.

(ii) $T + \lambda$ is square normal.

***Proof* (i)** Since T is a normal operator, from [8], T is square normal.

$$\begin{aligned} (\lambda T)^2(\lambda T)^{*2} &= (\lambda)^2(T)^2(\lambda^*)^2(T^*)^2 \\ &= (\lambda)^2(\lambda^*)^2(T)^2(T^*)^2 \\ &= (\lambda^*)^2(\lambda)^2(TT^*)^2 \\ &= (\lambda^*\lambda)^2(T^*T)^2 \\ &= (\lambda^*)^2(\lambda)^2(T^*)^2(T)^2 \\ &= (\lambda^*)(T^*)^2(\lambda)^2(T)^2 \\ &= (\lambda^*T^*)^2(\lambda)^2(T)^2 \\ &= (\lambda T)^{*2}(\lambda T)^2 \end{aligned}$$

Hence (λT) is square normal. □

***Proof* (ii)**

Suppose on the contrary $(T + \lambda)$ is not square normal.Then,

$$(T + \lambda)^2(T + \lambda)^{*2} - (T + \lambda)^{*2}(T + \lambda)^2 \neq 0 \tag{6}$$

Then

$$T^2T^{*2} + \lambda T^2T^* + \lambda TT^{*2} + \lambda TT^* - T^{*2}T^2 - \lambda T^{*2}T - \lambda T^*T^2 - \lambda T^*T \neq 0 \tag{7}$$

Since T is square normal,

$$\lambda T^2T^* + \lambda TT^{*2} + \lambda TT^* - \lambda T^{*2}T - \lambda T^*T^2 - \lambda T^*T \neq 0 \tag{8}$$

Since T is normal,

$$\lambda T^2T^* + \lambda TT^{*2} - \lambda T^{*2}T - \lambda T^*T^2 \neq 0 \tag{9}$$

Multiplying equation (9) with T^* from left on both sides we get

$$\lambda T^*T^2T^* + \lambda T^*TT^{*2} - \lambda T^*T^{*2}T - \lambda T^*T^*T^2 \neq 0 \tag{10}$$

Multiply equation (10) with T from right on both sides we get

$$\lambda T^*T^2T^*T + \lambda T^*TT^{*2}T - \lambda T^*T^{*2}TT - \lambda T^*T^*T^2T \neq 0 \tag{11}$$

Since T is normal, equation (11) becomes

$$\lambda T^*TT^*TT + \lambda T^*TT^{*2}T - \lambda T^*T^{*2}TT - \lambda T^*T^*T^2T \neq 0 \tag{12}$$

$$\lambda T^*T^*TTT + \lambda T^*TT^{*2}T - \lambda T^*T^{*2}TT - \lambda T^*T^*T^2T \neq 0 \tag{13}$$

Equation (13) yields

$$\lambda T^*T^*T^2T + \lambda T^*TT^{*2}T - \lambda T^*T^{*2}TT - \lambda T^*T^*T^2T \neq 0 \tag{14}$$

By simplifying equation (14) we obtain

$$\lambda T^*TT^{*2}T - \lambda T^*T^{*2}TT \neq 0 \tag{15}$$

Since T is normal,

$$\lambda T^*TT^{*2}T - \lambda T^*T^{*2}T^2 \neq 0 \tag{16}$$

$$\lambda T^*TT^*T^*T - \lambda T^*T^{*2}T^2 \neq 0$$

$$\lambda T^*T^*TT^*T - \lambda T^*T^{*2}T^2 \neq 0$$

$$\lambda T^*T^*T^*TT - \lambda T^*T^{*2}T^2 \neq 0$$

$$\lambda T^*T^{*2}T^2 - \lambda T^*T^{*2}T^2 \neq 0$$

A contradiction.
This implies that the assumption that $T + \lambda$ is not square normal was wrong. So $T + \lambda$ is square normal and therefore

$$(T + \lambda)^2(T + \lambda)^{*2} = (T + \lambda)^{*2}(T + \lambda)^2$$

□

Note that the condition T is normal should not be ignored otherwise $T + \lambda$ will not be square normal.
Example 3.3 illustrates the existence of a square normal operator that is not normal as seen in Manhood, (2016). As noted, scalar addition does not hold for such operators.

Example 3.3: Manhood, (2016)
Let T be given by $T = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix}$ where $i \in \mathbb{C}$, then $T^* = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix}$

We have

$$TT^* = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$T^*T = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Since $TT^* \neq T^*T$, T is not normal.

$$T^2 = \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} \begin{bmatrix} i & 0 \\ i & -i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$T^{*2} = \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} \begin{bmatrix} -i & -i \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$T^2T^{*2} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$T^{*2}T^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$T^2T^{*2} = T^{*2}T^2$ and hence square normal.
Therefore, T is square normal but not normal.
Let our scalar $\lambda = i$ be represented as λI where I is a 2×2 identity matrix.

Now $\lambda I = \begin{bmatrix} i & o \\ o & i \end{bmatrix}$
Then

$$T + \lambda I = \begin{bmatrix} 2i & 0 \\ i & 0 \end{bmatrix}$$

$$(T + \lambda I)^2 = \begin{bmatrix} -4 & 0 \\ -2 & 0 \end{bmatrix}$$

$$(T + \lambda I)^* = \begin{bmatrix} -2i & -i \\ 0 & 0 \end{bmatrix}$$

$$(T + \lambda I)^{*2} = \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix}$$

$$(T + \lambda I)^2(T + \lambda I)^{*2} = \begin{bmatrix} -4 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 8 & 4 \end{bmatrix}$$

$$(T + \lambda I)^{*2}(T + \lambda I)^2 = \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 0 \\ 8 & 0 \end{bmatrix}$$

Hence $T + \lambda I$ is not a square normal.

The theorem 3.4 shows that the sum and product of two commuting square normal operators is a square normal operator.

Theorem 3.4.
Let H be a Hilbert space and T and S be normal operators on $B(H)$. Then,

(i) $T + S$ is square normal.

(ii) TS is square normal

Proof **(i)**

$$\begin{aligned}(T+S)^2(T+S)^{*2} &= (T+S)(T+S)(T^*+S^*)(T^*+S^*)\\ &= (T+S)(T^*+S^*)(T+S)(T^*+S^*)\\ &= (T^*+S^*)(T+S)(T^*+S^*)(T+S)\\ &= (T^*+S^*)(T^*+S^*)(T+S)(T+S)\\ &= (T+S)^*(T+S)^*(T+S)(T+S)\\ &= (T+S)^{*2}(T+S)^2\end{aligned}$$

Hence square normal. □

Proof **(ii)**

$$\begin{aligned}(TS)^2(TS)^{*2} &= (TS)(TS)(TS)^*(TS)^*\\ &= (TS)(TS)^*(TS)(TS)^*\\ &= (TS)^*(TS)(TS)^*(TS)\\ &= (TS)^*(TS)^*(TS)(TS)\\ &= (TS)^{*2}(TS)^2\end{aligned}$$

Hence square normal. □

It is worth noting that if the operators T and S are non-commutative, then $T+S$ and TS are not necessarily square normal operators.
Example 3.5 shows that the sum two commuting square normal operators is a square normal operator.

Example 3.5

Let $T = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ and $S = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$ be two square normal operators which are normal.

Then we have the following

$$TS = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix} = \begin{bmatrix} i & -1 \\ 1 & 2i \end{bmatrix},$$

$$ST = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i & -1 \\ 1 & 2i \end{bmatrix}$$

Therefore S and T commutes.
Now,

$$T+S = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} + \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix} = \begin{bmatrix} i+1 & i \\ -i & i+2 \end{bmatrix}$$

$$(T+S)^* = \begin{bmatrix} -i+1 & i \\ -i & -i+2 \end{bmatrix}$$

$$(T+S)^{*2} = \begin{bmatrix} -i+1 & i \\ -i & -i+2 \end{bmatrix}\begin{bmatrix} -i+1 & i \\ -i & -i+2 \end{bmatrix} = \begin{bmatrix} -2i+1 & 2+3i \\ -2-3i & -4i+4 \end{bmatrix}$$

$$(T+S)^2 = \begin{bmatrix} i+1 & i \\ -i & i+2 \end{bmatrix} \begin{bmatrix} i+1 & i \\ -i & i+2 \end{bmatrix} = \begin{bmatrix} 2i+1 & -2+3i \\ 2-3i & 4i+4 \end{bmatrix}$$

To show that $T+S$ is Square normal,

$$(T+S)^2(T+S)^{*2} = \begin{bmatrix} 2i+1 & -2+3i \\ 2-3i & 4i+4 \end{bmatrix} \begin{bmatrix} -2i+1 & 2+3i \\ -2-3i & -4i+4 \end{bmatrix} = \begin{bmatrix} 18 & 27i \\ -27i & 45 \end{bmatrix}$$

$$(T+S)^{*2}(T+S)^2 = \begin{bmatrix} -2i+1 & 2+3i \\ -2-3i & -4i+4 \end{bmatrix} \begin{bmatrix} 2i+1 & -2+3i \\ 2-3i & 4i+4 \end{bmatrix} = \begin{bmatrix} 18 & 27i \\ -27i & 45 \end{bmatrix}$$

Therefore, $(T+S)^2(T+S)^{*2} = (T+S)^{*2}(T+S)^2$ and hence $T+S$ is Square normal.

Next, we give an example to show that if two square normal operators T and S do not commute, then $S+T$ is not necessarily a square normal operator.

Example 3.6
Let two operators T and S be two operators in $\mathbb{R}^2$.

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, S = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then,

$$T^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, S^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$TT^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$T^*T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since T is normal, it is square normal.
On the other hand, S=S^*. This is a Hermitian operator which is normal implying square normal.

$$TS = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$ST = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Clearly, $TS \neq ST$ hence non-commutative.
We now have,

$$T+S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$(T+S)^2 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$(T+S)^* = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$(T+S)^{*2} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(T+S)^2(T+S)^{*2} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$(T+S)^{*2}(T+S)^2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

This shows that $(T+S)^2(T+S)^{*2} \neq (T+S)^{*2}(T+S)^2$ and therefore $T+S$ is not square normal.

4 Conclusion

This research focused on investigating square normal operators and their properties in Hilbert spaces. Several properties, including sum and product, scalar addition and multiplication, adjoint, inverse, and unitarily equivalence relations have been investigated. Future research plans to explore the spectral properties of this class of operators.

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