THE LAPLACE TRANSFORM OF EXPONENTIATED LOGISTIC AND THE GENERALIZED INVERSE GAUSSIAN DISTRIBUTION.

Abstract

Raising a cumulative distribution function or survival function to a power is one way of generalizing a distribution known as exponentiated distribution. This work focuses on constructing the generalized exponentiated distributions for the logistic from a beta generated distribution. The generalized Exponentiated Type II logistic has been constructed and shown as obtained by [8]. The cdf and pdf of the standard logistic have been shown as special cases of the exponentiated distributions. The Exponentiated distributions have been expressed in terms of the Laplace transform. Moreover, the Laplace transform of the Generalized Inverse Gaussian (GIG), Inverse Gaussian(IG) and Gamma distributions have been derived. The reciprocal Inverse Gaussian is shown as a special case of GIG when $\lambda = \frac{1}{2}$. Also, we introduce two new distributions; the Generalized exponentiated logistic type I and type II respectively. We also show the behavior of the shapes of the resulting new distributions with varying parameter values.

Key words:
Cummulative distribution function, Generalized Exponentiatiated Logistic, Laplace Transform.

1 Introduction

Adding one or more parameters to a probability distribution is called generalization. The purpose of generalization is to make a distribution more flexible and tractable. It is well known, in general, that a generalized model is more flexible than the ordinary model and it is preferred by many data analysts in analyzing statistical data. [1] discusses various generalizations of the logistic distribution. One method of generalizing a distribution is to raise a cumulative distribution function (cdf) or a survival function to a power. The raised function is said to be exponentiated. [4] introduced a skewness parameter in generalization of logistic so that the generalized distribution can be used to model data exhibiting a unimodal density function.

[2] introduced the Exponentiated Weibull (EW) distribution that allowed for a non-monotone hazard rate. [3] studied exponentiated exponential (EE) distribution which they named Generalized Exponential distribution. [6] summarizes 8 well known generators for distributions. The generalized distributions nest various sub-families of distributions.

The concept of **EXPONENTIATION** was applied by [5] when solving a differentiation equation in cdf. The cdfs obtained were raised to powers. Out of the 12 cdfs obtained, 9 were raised to powers. The three cdfs not raised to powers and the other raised ones are known as Burr I, Burr II, Burr III up-to Burr XII.
Exponentiated distributions can be obtained as special cases of a beta generated distribution introduced by [7]. [9] investigated the shapes of density and hazard rate functions for the exponentiated half logistic family of distributions. The objective of this study is to: derive generalized exponentiated logistic distributions from the beta generator and show explicitly the Laplace transform of the exponentiated standard logistic, generalized inverse Gaussian, inverse Gaussian and Gamma distributions respectively.

The paper is organised as follows; Section two shows how an exponentiated distribution is deduced from a beta generated distribution and applied to the logistic distribution. Section three shows how an exponentiated mixture is expressed in terms of Laplace transform of a mixing distributions. In section four the Laplace transforms of the generalized inverse Gaussian, inverse Gaussian and Gamma distributions are derived. The concluding remarks are in section five.

2 Generalized Exponentiated Distributions.

The derivation of Exponentiated generalized logistic distributions follows;
Let the cdf and pdf of the logistic distribution be defined as:

$$S(x) = \frac{e^{-x}}{1+e^{-x}}, \quad -\infty < x < \infty \tag{2.1}$$

$$s(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty \tag{2.2}$$

[7] considered a cdf of the classical beta distribution as;

$$T(x) = \int_0^x \frac{w^{a-1}(1-w)^{b-1}}{B(a,b)} dw \tag{2.3}$$

Since $0 \leq x \leq 1$, [7] replaced it by a *cdf* $S(x)$. Thus we have;

$$V(x) = T[S(x)] = \int_0^{S(x)} \frac{w^{a-1}(1-w)^{b-1}}{B(a,b)} dw, \; 0 \leq x \leq 1 \tag{2.4}$$

Using equation (2.4), put $b = 1$ in (2.3) we obtain;

$$V_1(x) = \int_0^{S(x)} a w^{a-1} dw = \frac{a w^a}{a} \Big|_0^{S(x)}$$

$$= [S(x)]^a \tag{2.5}$$

$$v_1(x) = s(x) a \Big(S(x) \Big)^{a-1}, a > 0, \; -\infty < x < \infty \tag{2.6}$$

when values of the parameter a are varied Equations $(2.5)$and (2.6) are exponentiated generators. They can be used to generate new distributions with $S(x)$ as the parent *cdf*.
Using (2.1) and (2.2) in equations (2.5) and (2.6) we obtain;

$$V_1(x) = \Big(1 + e^{-x} \Big)^{-a}, a > 0, \; -\infty < x < \infty \tag{2.7}$$

$$v_1(x) = \frac{a e^{-x} (1 + e^{-x})^a}{(1 + e^{-x})^3}, a > 0, \; -\infty < x < \infty \tag{2.8}$$

which is the **GENERALIZED EXPONENTIATED TYPE I LOGISTIC** distribution. When $a = 1$ in (2.7) and (2.8) we obtain (2.1) and (2.2) as special cases. Using (2.3), when $a = 1$, we obtain;

$$V_2(x) = \int_0^{S(x)} b(1-w)^{b-1} dw \tag{2.9}$$

Put

$$y = 1 - w \implies dy = -dw$$

Therefore, we proceed as follows:

$$\begin{aligned} V_2(x) =& \int_1^{1-S(x)} by^{b-1}(-dw) \\ =& -\left[\frac{by^b}{b}\right]^{1-S(x)} \\ =& y^b\Big|_{1-S(x)}^{1} \\ =& 1-[1-S(x)]^b \qquad (2.10) \\ 1-V_2(x) =& [1-(x)]^b \\ v_2(x) =& b\Big(1-S(x)\Big)^{b-1} s(x) \qquad (2.11) \end{aligned}$$

Using the (2.1) and (2.2) in (2.10) and (2.11), we obtain the **Generalized Exponentiated Type II Logistic** as shown below;

$$V_2(x) = 1-\left[\frac{e^{-x}}{1+e^{-x}}\right]^b \; b>0, -\infty<x<\infty \qquad (2.12)$$

$$v_2(x) = \frac{be^{-xb}}{(1+e^{-x})^{b+1}}, \; b>0, -\infty<x<\infty \qquad (2.13)$$

[8] called (2.12) the Generalized logistic type I. [11] in his $Phd\ thesis$ summarized five different methods of constructing the Generalized logistic type I.
When $b = 1$ in (2.12) , (2.1) is obtained as a special case.

2.1 Graph of the Exponentiated Generator

When varying parameters of a and b from equation (2.6) are plotted against the density, the graph of the exponentiated generator is monotonically increasing resulting in a j- shape as shown in figure (1);

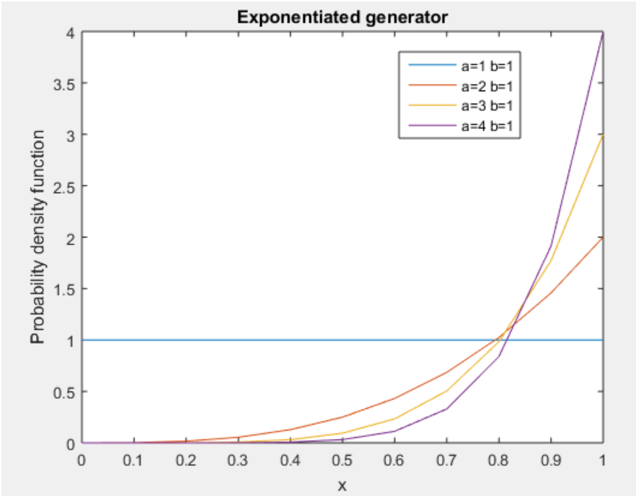

Figure 1:

2.2 Graph of the cdf of the standard logistic distribution

When varying parameters of a and b from equation (2.7) are plotted against the cdf, the graph of the cdf of standard logistic distribution is almost symmetrical with sharp peak indicating a higher concentration of data values hence the data is leptokurtic as shown in figure (2);

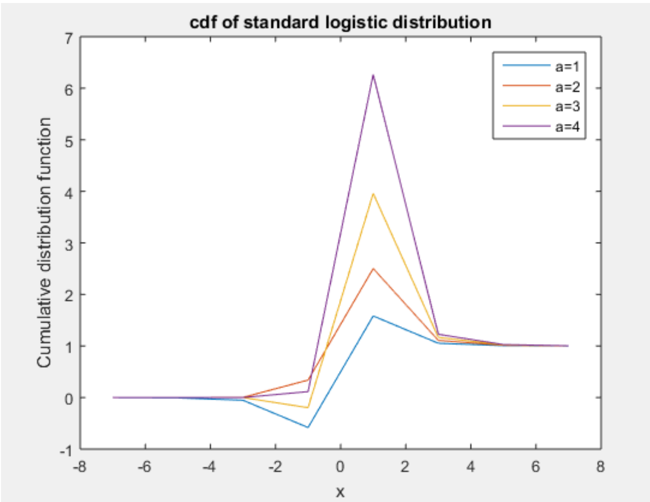

Figure 2:

2.3 Graph of the pdf of the standard logistic distribution

When $a = 1$ from equation (2.8) is plotted against the density, the graph of the pdf of standard logistic distribution is symmetrical as expected and is shown in figure (3);

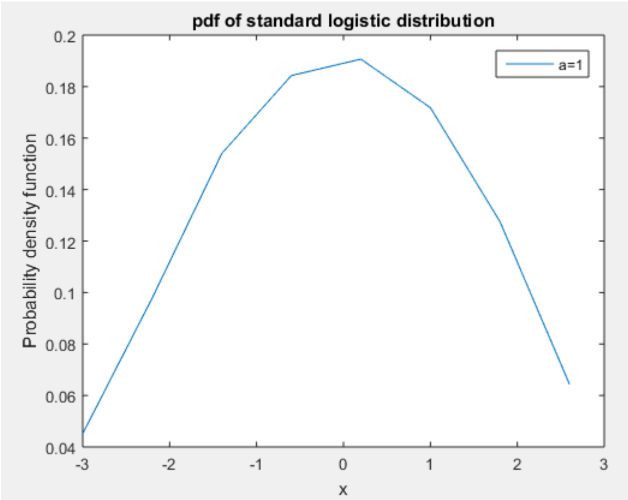

Figure 3:

3 Laplace Transform of Exponentiated Distributions.

Let $G(x)$ be a cdf of a random variable X. It's power is denoted by $[G(x)]^\theta$ where θ is varying parameter. Then a continuous mixture of the raised cdf is given by;

$$F(x) = \int_0^\infty [G(x)]^\theta g(\theta) d\theta \tag{3.1}$$

where $g(\theta)$ is a continuous mixing distribution. Equation (3.1) can be expressed as a Laplace transform of a mixing distribution as shown below;

$$
\begin{aligned}
F(x) &= \int_0^\infty \left[G(x)\right]^\theta g(\theta)d\theta && (3.2)\\
&= \int_0^\infty \exp(\ln[G(x)]^\theta)g(\theta)d\theta\\
&= \int_0^\infty \exp(\theta\ln(G(x)))g(\theta)d\theta\\
&= \int_0^\infty \exp[-(-\theta\ln(G(x)))]g(\theta)d\theta\\
&= \int_0^\infty e^{-[\theta(-\ln(G(x)))]}g(\theta)d\theta\\
&= \int_0^\infty e^{-\theta(-\ln(G(x))}g(\theta)d\theta\\
&=E[e^{-\theta(-\ln G(x))}]\\
&=L_\theta[-\ln G(x)] && (3.3)
\end{aligned}
$$

which is the Laplace transform of the mixing random variable θ evaluated at $-\ln(G(x))$. Putting (2.1) in (3.2) we obtain;

$$F(x) = L_\theta[\ln(1+e^{-x})] \tag{3.4}$$

which is the Laplace transform of the logistic with the mixing random variable θ.

4 Laplace Transform of The Generalized Inverse Gaussian, Inverse Gaussian and Gamma Distributions.

We now wish to consider Laplace transform of generalized inverse Gaussian distribution and Gamma distribution.

4.1 Generalized Inverse Gaussian distribution

Generalized inverse Gaussian distribution (GIG) distribution is based on modified Bessel function of the third kind with index λ evaluated at ω denoted by $K_\lambda(\omega)$ and defined as;

$$K_\lambda(\omega) = \frac{1}{2}\int_0^\infty x^{\lambda-1}e^{-\frac{\omega}{2}(\frac{1}{x}+x)}dx \tag{4.1}$$

For $-\infty < \lambda < \infty$ and $\omega > 0$ [10] used the following parametrization

$$\omega = \frac{\mu}{\beta} \tag{4.2}$$

Therefore,

$$K_\lambda(\frac{\mu}{\beta}) = \frac{1}{2}\int_0^\infty x^{\lambda-1}e^{-\frac{\mu}{2\beta(\frac{1}{x}+x)}}dx \tag{4.3}$$

And the transformation

$$x = \frac{z}{\mu} \tag{4.4}$$

Substituting(4.4) in (4.3) we obtain;

$$K_\lambda(\frac{\mu}{\beta}) = \frac{1}{2}\int_0^\infty (\frac{1}{\mu})^\lambda z^{\lambda-1}e^{-\frac{1}{2}(\frac{z}{\mu}+\frac{\mu}{z})}dz \tag{4.5}$$

$$= \frac{1}{2}\int_0^\infty (\frac{1}{\mu})^\lambda z^{\lambda-1}e^{-\frac{1}{2}(\frac{z}{\beta}+\frac{\mu^2}{\beta}\frac{1}{z})}dz \tag{4.6}$$

Therefore,

$$1 = \int_0^\infty \frac{(\frac{1}{\mu})^\lambda z^{\lambda-1}e^{-\frac{1}{2}(\frac{z}{\beta}+\frac{\mu^2}{\beta}\frac{1}{z})}}{K_\lambda(\frac{\mu}{\beta})}dz \tag{4.7}$$

Thus the pdf of the Generalized Inverse Gaussian distribution according to [10] and transform is given by;

$$g(z) = \frac{(\frac{1}{\mu})^\lambda z^{\lambda-1}e^{-\frac{1}{2}(\frac{z}{\beta}+\frac{\mu^2}{\beta}\frac{1}{z})}}{2K_\lambda(\frac{\mu}{\beta})} \tag{4.8}$$

For $z > 0; \quad -\infty < \lambda < \lambda, \;\; \mu > 0, \;\; \beta > 0$
Equation (4.8) can also be expressed as;

$$g(z) = \frac{(\frac{1}{\mu})^\lambda z^{\lambda-1}e^{-\frac{1}{2}(\frac{z}{\beta}+\frac{\mu^2}{\beta}\frac{1}{z})}dz}{\int_0^\infty (\frac{1}{\mu})^\lambda z^{\lambda-1}e^{-\frac{1}{2}(\frac{z}{\beta}+\frac{\mu^2}{\beta}\frac{1}{z})}dz} \tag{4.9}$$

$$= \frac{z^{\lambda-1}e^{-\frac{1}{2}(\frac{z}{\beta}+\frac{\mu^2}{\beta}\frac{1}{z})}}{\int_0^\infty z^{\lambda-1}e^{-\frac{1}{2}(\frac{z}{\beta}+\frac{\mu^2}{\beta}\frac{1}{z})}dz} \tag{4.10}$$

The Laplace transform of GIG distribution is given by;

$$L_z(s) = E(e^{-sz}) \tag{4.11}$$

$$= \int_0^\infty e^{-sz} g(z) dz \tag{4.12}$$

$$= \frac{\int_0^\infty e^{-sz} (\frac{1}{\mu})^\lambda z^{\lambda-1} e^{-\frac{1}{2}(\frac{z}{\beta}+\frac{\mu^2}{\beta}\frac{1}{z})} dz}{2K_\lambda(\frac{\mu}{\beta})}$$

$$= \frac{\frac{1}{2}\int_0^\infty (\frac{1}{\mu})^\lambda z^{\lambda-1} e^{-\frac{1}{2}(2s+\frac{1}{\beta})z+\frac{\mu^2}{\beta}\frac{1}{2}} dz}{K_\lambda(\frac{\mu}{\beta})}$$

$$= \frac{\frac{1}{2}\int_0^\infty (\frac{1}{\mu})^\lambda Z^{\lambda-1} e^{-\frac{1}{2}((\frac{2\beta s+1}{\beta})z)+\frac{\mu^2}{\beta}\frac{1}{2}} dz}{K_\lambda(\frac{\mu}{\beta})}$$

$$= \frac{\left(\frac{1}{\mu}\right)^\lambda}{K_\lambda(\frac{\mu}{\beta})} \frac{1}{2} \int_0^\infty z^{\lambda-1} e^{-\frac{1+2\beta s}{2\beta}(z+\frac{\mu^2}{1+2\beta s})\frac{1}{z}} dz \tag{4.13}$$

Let

$$z = \frac{\mu}{\sqrt{1+2\beta s}} t \implies dz = \frac{\mu}{\sqrt{1+2\beta s}} dt$$

Therefore,

$$L_Z(s) = \frac{(\mu^{-1})^\lambda}{K_\lambda(\frac{\mu}{\beta})} \left(\frac{\mu}{\sqrt{1+2\beta s}}\right)^\lambda \frac{1}{2} \int_0^\infty t^{\lambda-1} e^{-\frac{\mu}{2\beta}\sqrt{1+2\beta s}(t+\frac{1}{t})} dt \tag{4.14}$$

$$= \frac{(\mu^{-1})^\lambda}{K_\lambda(\frac{\mu}{\beta})} \left(\frac{\mu}{\sqrt{1+2\beta s}}\right)^\lambda K_\lambda\left(\frac{\mu}{\beta}\sqrt{1+2\beta s}\right)$$

$$= \left(\sqrt{1+2\beta s}\right)^{-\lambda} \frac{K_\lambda(\frac{\mu}{\beta}\sqrt{1+2\beta s})}{K_\lambda(\frac{\mu}{\beta})}$$

$$= (1+2\beta s)^{-\frac{\lambda}{2}} \frac{K_\lambda(\mu\beta^{-1}(1+2\beta s))^{\frac{1}{2}}}{K_\lambda(\mu\beta-1)} \tag{4.15}$$

which is the Laplace transform of the GIG.

4.2 Laplace Transform of the Inverse Gaussian Distribution

The Laplace transform of inverse Gaussian distribution is obtained by putting

$$\lambda = -\frac{1}{2}$$

Therefore,

$$L_z(s) = \left(\sqrt{1+2\beta s}\right)^{\frac{1}{2}} \frac{K_{-\frac{1}{2}}(\mu\beta^{-1}(1+2\beta s)^{\frac{1}{2}})}{K_{-\frac{1}{2}}(\mu\beta^{-1})} \tag{4.16}$$

Therefore,

$$L_z(s) = \frac{\left(\sqrt{1+2\beta s}\right)^{\frac{1}{2}} \left[\frac{\pi}{2\mu\beta^{-1}(1+2\beta s)^{\frac{1}{2}}}\right]^{\frac{1}{2}} e^{-\mu\beta^{-1}}(1+2\beta s)^{\frac{1}{2}}}{\left(\frac{\pi}{2\mu\beta^{-1}}\right)^{\frac{1}{2}} e^{-\mu\beta^{-1}}} \tag{4.17}$$

Therefore,

$$\begin{aligned} L_z(s) =& e^{\mu\beta^{-1} - \mu\beta^{-1}(1+2\beta s)^{\frac{1}{2}}} \\ =& e^{\frac{\mu}{\beta}\left[1-\sqrt{1+2\beta s}\right]} \\ =& e^{-\frac{\mu}{\beta}\left[\sqrt{1+2\beta s}-1\right]} \end{aligned} \tag{4.18}$$

4.3 Laplace transform of Gamma distribution

Using equation (4.15)i.e. the Laplace transform of GIG, we have

$$L_z(s) = \frac{\int_0^\infty \left(\frac{1}{\mu}\right)^\lambda z^{\lambda-1} e^{-\frac{1}{2}(2s+\frac{1}{\beta})z} dz}{\int_0^\infty \left(\frac{1}{\mu}\right)^\lambda z^{\lambda-1} e^{-\frac{z}{2\beta}} dz} \tag{4.19}$$

$$\begin{aligned} =& \frac{\int_0^\infty z^{\lambda-1} e^{-\frac{(1+2\beta s)}{2\beta}} dz}{\int_0^\infty Z^{\lambda-1} e^{-\frac{z}{2\beta}} dz} \\ =& \frac{\Gamma(\lambda)}{\left(\frac{1+2\beta s}{2\beta}\right)^\lambda} \frac{\left(\frac{1}{2\beta}\right)^\lambda}{\Gamma(\lambda)} \\ =& \left(\frac{1}{1+2\beta s}\right)^\lambda \\ =& \left(\frac{\frac{1}{2\beta}}{s+\frac{1}{2\beta}}\right)^\lambda \end{aligned} \tag{4.20}$$

which is the Laplace transform of a gamma distribution with parameters λ and $\frac{1}{2\beta}$.

4.4 Laplace Transform of The Inverse Gaussian

The Laplace transform of reciprocal inverse Gaussian distribution is obtained by putting $\lambda = \frac{1}{2}$ in equation (4.15) as follows;

$$L_z(s) =(\sqrt{1+2\beta s})^{-\frac{1}{2}} \frac{K_{\frac{1}{2}}(\mu\beta^{-1}(1+2\beta s)^{\frac{1}{2}})}{K_{\frac{1}{2}(\ \mu\beta^{-1})}} \tag{4.21}$$

$$=\frac{(\sqrt{1+2\beta s})^{-\frac{1}{2}} \left[\frac{\pi}{2\mu\beta^{-1}(\sqrt{1+2\beta s})}\right]^{\frac{1}{2}} e^{-\mu\beta^{-1}\sqrt{1+2\beta s}}}{(\frac{\pi}{\mu\beta^{-1}})^{\frac{1}{2}} e^{-\mu\beta^{-1}}}$$

$$=\frac{1}{\sqrt{1+2\beta s}} e^{-\mu\beta^{-1}[\sqrt{1+2\beta s}-1]}$$

$$=\left(\frac{1}{1+2\beta s}\right)^{\frac{1}{2}} e^{-\mu\beta^{-}(\sqrt{1+2\beta s}-1)} \tag{4.22}$$

Thus the Laplace transform of reciprocal inverse Gaussian distribution is the product of a Laplace transform of a gamma distribution with parameters $\frac{1}{2}$ and $\frac{1}{2\beta}$ and the Laplace transform of inverse Gaussian distribution.

5 Conclusion

In this work, we explicitly derived the generalized Exponentiated logistic type I and II distributions. We also showed how the logistic distribution can be obtained as a special case from the generalized distribution. If the cdf of a distribution is known, it's exponentiated distributions can be obtained. The generalized logistic type I of Johnson 1995 has been derived and shown as the Exponentiated logistic type II. The graphs of the exponentiated generators are symmetrical and have a sharp peak Further work can be done on generalizations by control the location and scale parameters. The generalized distributions obtained can also be applied to data.

We have also shown that, the Laplace transform of reciprocal inverse Gaussian distribution is the product of a Laplace transform of a gamma distribution with parameters $\frac{1}{2}$and $\frac{1}{2\beta}$ and the Laplace transform of inverse Gaussian distribution. Therefore the generalized inverse Gaussian distribution nests other distributions which are obtained as special cases.

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