WEYL'S THEOREM FOR ALGEBRAICALLY $(\wp, \wr, \varrho)$-PARANORMAL AND ALGEBRAICALLY $(\wp, \wr, \varrho)$-*-PARANORMAL OPERATORS

Short Research Article

Abstract

Present L be an algebraically $(\wp, \wr, \varrho)$-Paranormal and algebraically $(\wp, \wr, \varrho)$-*-Paranormal operators on L^2 space. We examine Weyl's theorem, $\mathfrak{a}$-Browder's theorem and spectral mapping theorem holds for weyl's spectrum of L and essential approximate point spectrum of L.

Keywords: Weyl's theorem, $\mathfrak{a}$-Browder's theorem, algebraically $(\wp, \wr, \varrho)$-Paranormal and algebraically $(\wp, \wr, \varrho)$--Paranormal operators.*

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1 Introduction

Let $\mathfrak{B}(\mathfrak{H})$ and $\mathfrak{K}(\mathfrak{H})$ stand for the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space, respectively, throughout this paragraph. If L is in the range of $\mathfrak{B}(\mathfrak{H})$, then we write $N(L)$ for L's null space and $R(L)$ for its range. Furthermore, let $\sigma(L)$, $\sigma_a(L)$ and $\pi_0(L)$ stand for the spectrum, approximation point spectra, and point spectrum of L, respectively, and let $\alpha(L) = dimN(L)$, $\beta(L) = dimN(L)$. If an operator $L \in \mathfrak{B}(\mathfrak{H})$ has a closed range, a finite dimensional null space, and a finite co-dimension in its range, it is referred to as a Fredholm operator. A Fredholm operator's index can be found using the formula

$$i(L) = \alpha(L)\beta(L)$$

L is referred to as Browder if it is a Fredholm with finite ascent and descent, and Weyl if it is a Fredholm of index zero. Comparatively, if L is Fredholm and $L - \lambda$ is invertible for small enough

$|\curlywedge| > 0$, then $\curlywedge \in C$. Based on [10,11], we may construct the essential spectrum $\sigma_b(L)$, the Weyl spectrum $\omega(L)$, and the Browder spectrum $\sigma_b(L)$ of L.

$$\sigma_e(L) = \{\curlywedge \in C : L - \curlywedge \; is \; not \; Fredholm\},$$

$$\omega(L) = \left\{\curlywedge \in C : L - \curlywedge \; is \; not \; Weyl's\right\},$$

and

$$\sigma_b(L) = \{\curlywedge \in C : L - \curlywedge \; is \; not \; Browder\},$$

Based on the evidence,

$$\sigma_e(L) \subseteq \omega(L) \subseteq \sigma_b(L) = \sigma_e(L) \cup acc\sigma(L),$$

where $acc\Re$ represents the accumulation points of $\Re \subseteq C$. Write $\Re \; acc\Re$, and we let

$$\pi_{00}(L) = \{\curlywedge \in iso\sigma(L) : 0 < \sigma(L - \curlywedge) < \infty\},$$

and

$$p_{00}(L) = \sigma(L) \; \sigma_b(L).$$

Weyl's theorem holds for algebraically paranormal operators, as demonstrated recently by Raul E. Curto and Young Min Han[6], D. Senthilkumar, and P. Mageshwari naik[14]. We examine several characteristics of absolute-(p, r)-*-paranormal operators after demonstrating that Weyl's theorem holds for algebraically absolute-(p, r)-*-paranormal operators and M.H.M. Rashid[17]. This result is extended to algebraically $(\wp, \wr, \varrho)$-Paranormal and algebraically $(\wp, \wr, \varrho)$-*-Paranormal operators in this note.

2 Weyl's Theorem for Algebraically $(\wp, \wr, \varrho)$-Paranormal Operator

An operator L is $(\wp, \wr, \varrho)$-paranormal for each $\wp > 0$, $\wr \geq 0$ and $\varrho > 0$ if, for each unit vector $\Im \in \mathfrak{H}$, $\left\| |L|^{\wp} \mho |L|^{\wr} \Im \right\|^{\frac{1}{\varrho}} \geq \left\| |L|^{\frac{\wp+\wr}{\varrho}} \Im \right\|$.
If there is a nonconstant complex polynomial p such that $p(L)$ is $(\wp, \wr, \varrho)$-Paranormal then we now prove a result for the algebraically $(\wp, \wr, \varrho)$-paranormal operator.

Lemma 2.1. *Assume that $\sigma(L) = \{\curlywedge\}$ and let L be a $(\wp, \wr, \varrho)$-paranormal operator with $\curlywedge \in C$. Consequently, $L = \curlywedge$.*

Proof. We examine two scenarios: Case I ($\curlywedge = 0$): L is normaloid because it is $(\wp, \wr, \varrho)$-paranormal. L thus equals 0.
In case II ($\curlywedge \neq 0$), we observe that L^{-1} is also $(\wp, \wr, \varrho)$-paranormal because L is invertible and L is $(\wp, \wr, \varrho)$-paranormal. L^{-1} is hence normaloid.
Conversely, since $\sigma(L^{-1}) = \frac{1}{\curlywedge}$, so $\|L\| \left\|L^{-1}\right\| = |\curlywedge| \left|\frac{1}{\curlywedge}\right| = 1$. Given that L is convexoid, as deduced from [Mla, Lemma 3], $W(L) = \curlywedge$. T thus equals $\curlywedge$. □

Lemma 2.2. *Consider an algebraically $(\wp, \wr, \varrho)$-paranormal operator L. Then L is nilpotent.*

Proof. Let p be a nonconstant polynomial p such that $p(L)$ is $(\wp, \wr, \varrho)$-paranormal. $p(L) - p(0)$ is a quasinilpotent operator since $\sigma(p(L)) = p(\sigma(L))$. Assuming $m \geq 1$, it follows from Lemma 2.1 that $cL^m(L - \curlywedge_1)...(L - \curlywedge_n) \equiv p(L) - p(0) = 0$. For any $\curlywedge_i \neq 0$, $L - \curlywedge_i$ is invertible, so $L^m = 0$. □

Lemma 2.3. *Let L be a algebraically $(\wp, \wr, \varrho)$-paranormal operator. Then L is isoloid.*

Proof. Assume that $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - L)^{-1}d\mu$ is the associated Riesz idempotent and that $\curlywedge \in iso\sigma(L)$. D is a closed disk with $\curlywedge$ at its center and no additional points of $\sigma(L)$. The direct sum

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix},\ where\ \sigma(L_1) = \{\curlywedge\}\ and\ \sigma(L_2) = \sigma(L)\setminus\{\curlywedge\}$$

For some nonconstant polynomial p, $p(L)$ is $(\wp, \wr, \varrho)$-paranormal since L is algebraically $(\wp, \wr, \varrho)$-paranormal. We must have $\sigma(p(L_1)) = p(\sigma(L_1)) = \{p(\sigma)\}$. since $p(L_1) - p(\curlywedge)$ is hence quasipotent. Lemma 2.1 states that since $p(L_1)$ is $(\wp, \wr, \varrho)$-paranormal, $p(L_1) - p(\curlywedge) = 0$. Assign $q(z)$ to $p(z) - p(\curlywedge)$.As a result, L_1 is algebraically $(\wp, \wr, \varrho)$-paranormal and $q(L_1) = 0$. Lemma 2.2 indicates that $L_1 - \curlywedge$ is nilpotent as it is quasinilpotent and algebraically $(\wp, \wr, \varrho)$-paranormal. As a result, $\curlywedge \in \pi_0(L_1)$ and $\curlywedge \in \pi_0(L)$. This demonstrates L is isoloid. □

Theorem 2.4. *Let L is algebraically $(\wp, \wr, \varrho)$-paranormal operator. Then Weyl's theorem holds for $f(L)$ for every $f \in \mathfrak{H}(\sigma(L))$.*

Proof. We first prove that L is a valid case of Weyl's theorem. Assume $\curlywedge$ lies within $\curlywedge \in \sigma(L)\setminus\omega(L)$. Hence, $L - \curlywedge$ is not invertible and is Weyl. As we assert, $\curlywedge \in \partial\sigma(L)$. Contrarily, suppose that $\curlywedge$ is an inner point of $\sigma(L)$. Then, for any $\mu \in U$, there exists a neighborhood U of $\curlywedge$ such that $dimN(L-\mu) > 0$. It is evident from [Fin, Theorem 10] that L lacks SVEP. Conversely, [4, Corollary 2.10] implies that $p(L)$ possesses SVEP since $p(L)$ is $(\wp, \wr, \varrho)$-paranormal for some nonconstant polynomial p. Therefore, L possesses SVEP by [13, Theorem 3.3.9], which is contradictory. Consequently, $\curlywedge$ lies within $\curlywedge \in \partial\sigma(L)\setminus\omega(L)$, and the punctured neighborhood theory implies that $\curlywedge \in \pi_{00}(L)$.
Conversely, consider the case where $\curlywedge \in \pi_{00}(L)$ and the corresponding Riesz idempotent $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - L)^{-1}d\mu$, where D is a closed disk with $\curlywedge$ at its center and no other points of $\sigma(L)$. Then, just as previously, we can express L as the direct sum

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix},\ where\ \sigma(L_1) = \{\curlywedge\}\ and\ \sigma(L_2) = \sigma(L)\setminus\{\curlywedge\}$$

We examine two scenarios: Case I ($\curlywedge = 0$): Since L_1 is quasinilpotent and algebraically $(\wp, \wr, \varrho)$-paranormal in this case, Lemma 2.2 implies that L_1 is nilpotent. As we say, $dimR(P) < \infty$. For, $0 \notin \pi_{00}(L)$, which would be contradictory if $N(L_1)$ were infinite dimensional. Weyl follows from the fact that L_1 is a finite dimensional operator. Nonetheless, we may determine that L is Weyl because L_2 is invertible. As a result, $0 \in \sigma(L)\setminus\omega(L)$.
Case II ($\curlywedge \neq 0$): Lemma 2.3 proof shows that $L_1 - \curlywedge$ is nilpotent. $L_1 - \curlywedge$ is Weyl since $\curlywedge \in \pi_{00}(L)$ and $L_1 - \curlywedge$ is a finite dimensional operator. $L - \curlywedge$ is Weyl since $L_2 - \curlywedge$ is invertible. Weyl's theorem therefore applies to L. Next, we assert that for any $f \in \mathfrak{H}(\sigma(L))$, $f(\omega(L)) = \omega(f(L))$. Assume that $f \in \mathfrak{H}(\sigma(L))$. Without any more restrictions on L, $\omega(f(L)) \subseteq f(\omega(L))$; therefore, proving that $f(\omega(L)) \subseteq \omega(f(L))$ is sufficient. Assume that $\curlywedge \neq \omega(f(L))$. In that case, $g(L)$ is invertible,

$$f(L) - \curlywedge = c(L - \alpha_1)(L - \alpha_2)...(L - \alpha_n)g(L), \tag{2.1}$$

and $f(L) - \curlywedge$ is Weyl. Every $L - \alpha_i$ is Fredholm since the operators on the right side of (2.1) commute. L has SVEP [4, Corollary 2.10], since L is algebraically $(\wp, \wr, \varrho)$-paranormal. From [1, Theorem 2.6], we can infer that for every $i = 1, 2, ..., n$, $i(L - \alpha_i) \leq 0$. Consequently, $\curlywedge \notin f(\omega(L))$ and hence $f(\omega(L)) = \omega(f(L))$.
Now remember ([LeLe, Lemma]) that for any $f \in \mathfrak{H}(\sigma(L))$,

$$f(\sigma(L)\setminus\pi_{00}(L)) = \sigma(f(L))\setminus\pi_{00}(f(L))$$

This is true if L is isoloid. Weyls theorem holds for L since it is isoloid (Lemma 2.3) and so,

$$\sigma(f(L))\setminus\pi_{00}(f(L)) = f(\sigma(L)\setminus\pi_{00}(L)) = f(\omega(L)) = \omega(f(L))$$

means that Weyls theorem applies for $f(L)$. The proof is now complete. □

Corollary 2.5. *Let L be an algebraically $(\wp, \wr, \varrho)$-paranormal operator. For each $f \in \mathfrak{H}(\sigma(L))$, we get $\omega(f(L)) = f(\omega(L))$.*

3 Weyl's Theorem for Algebraically $(\wp, \wr, \varrho)$-*-Paranormal Operator

An operator L is $(\wp, \wr, \varrho)$-*-paranormal for each $\wp > 0$, $\wr \geq 0$ and $\varrho > 0$ if $\left\| |L|^{\wp} \mho |L|^{\wr} \Im \right\|^{\frac{1}{\varrho}} \geq \left\| |L|^{\frac{\wp+\wr}{\varrho}} \mho^{*} \Im \right\|$. If there is a nonconstant complex polynomial p such that $p(L)$ is $(\wp, \wr, \varrho)$-*-paranormal, we say that L is algebraically $(\wp, \wr, \varrho)$-*-paranormal. We now prove a result for the algebraically $(\wp, \wr, \varrho)$-*-paranormal operator.

Lemma 3.1. *Assume that $\sigma(L) = \{\curlywedge\}$ and let L be a $(\wp, \wr, \varrho)$-*-paranormal operator with $\curlywedge \in C$. Consequently, $L = \curlywedge$.*

Lemma 3.2. *Let L be a algebraically $(\wp, \wr, \varrho)$-*-paranormal operator. Then L is nilpotent.*

Lemma 3.3. *Let L be a algebraically $(\wp, \wr, \varrho)$-*-paranormal operator. Then L is isoloid.*

Theorem 3.4. *Let L be an algebraically $(\wp, \wr, \varrho)$-*-paranormal operator. Then, for each $f \in H(\sigma(L))$, Weyl's theorem holds for $f(L)$.*

Corollary 3.5. *Let L be an algebraically $(\wp, \wr, \varrho)$-*-paranormal operator. For each f that is in $\mathfrak{H}(\sigma(L))$, we have $\omega(f(L)) = f(\omega(L))$.*

4 $\mathfrak{a}$-Browders Theorem for Algebraically $(\wp, \wr, \varrho)$-Paranormal and Algebraically $(\wp, \wr, \varrho)$-*-Paranormal Operators

Generally speaking, we cannot assume that operators with only SVEP are covered by Weyl's theorem. Take a look at this instance: Let

$$L(x_1, x_2, x_3, ...) = \left(\frac{1}{2}x_2, \frac{1}{2}x_3, ...\right)$$

define $L \in \mathfrak{B}(l_2)$. Since L is quasinilpotent, L possesses SVEP. However, Weyl's theorem does not apply to L since $\sigma(L) = \omega(L) = \{0\}$ and $\pi_{00}(L) = \{0\}$. Nevertheless, as Theorem 3.4 below demonstrates, $\mathfrak{a}$-Browder's theory holds for L. First, we require the following auxiliary result, which is mostly the work of C.K. Fong [Fon]; we offer a proof for completeness. Remember that if $\mathfrak{X} \in \mathfrak{B}(\mathfrak{H})$ has a dense range and a trivial kernel, it is referred to be a quasiaffinity. If there is a quasiaffinity $\mathfrak{X}$ such that $\mathfrak{X}\mathfrak{S} = L\mathfrak{X}$, then $\mathfrak{S} \in \mathfrak{B}(\mathfrak{H})$ is considered a quasiaffine transform of L. If $\mathfrak{S}$ and L both $\mathfrak{S} \prec L$ and $L \prec \mathfrak{S}$, then we say that $\mathfrak{S}$ and L are quasisimilar.

Lemma 4.1. *let $\mathfrak{S} \prec L$ and assume that L possesses SVEP. Next, $\mathfrak{S}$ possesses SVEP.*

Theorem 4.2. *Assume that L or L^* is algebraically $(\wp, \wr, \varrho)$-paranormal. For each $f \in \mathfrak{H}(\sigma(L))$, we get $\sigma_{ea}(f(L)) = f(\sigma_{ea}(L))$.*

Theorem 4.3. *Assume that L or L^* is algebraically $(\wp, \wr, \varrho)$-*-paranormal. For each $f \in \mathfrak{H}(\sigma(L))$, we get $\sigma_{ea}(f(L)) = f(\sigma_{ea}(L))$.*

Theorem 4.4. *Let $\mathfrak{S} \prec L$ and assume that L possesses SVEP. For each $f \in \mathfrak{H}(\sigma(L))$, then $\mathfrak{a}$-Browder's theorem holds for $f(\mathfrak{S})$.*

Corollary 4.5. *Assume that $\mathfrak{S} \prec L$ and that L is an algebraically $(\wp, \wr, \varrho)$-paranormal operator. For each $f \in \mathfrak{H}(\sigma(L))$, then $\mathfrak{a}$-Browder's theorem holds for $f(\mathfrak{S})$.*

Corollary 4.6. *Assume that $\mathfrak{S} \prec L$ and that L is an algebraically $(\wp, \wr, \varrho)$-*-paranormal operator. For each $f \in \mathfrak{H}(\sigma(L))$, then $\mathfrak{a}$-Browder's theorem holds for $f(\mathfrak{S})$.*

5 Conclusion

The study establishes the validity of Weyls theorem, a-Browders theorem, and the spectral mapping theorem in the context of algebraically $(\wp, \wr, \varrho)$-Paranormal and algebraically $(\wp, \wr, \varrho)$-*-Paranormal operators. Our examination reveals that these theorems hold for the Weyls spectrum and the essential approximate point spectrum of the operator L, thereby contributing to a deeper understanding of the spectral properties of such operators.

References

[1] P. Aiena and O. Monsalve, Operators which do not have the single valued extension property, J. Math. Anal. Appl., 250 (2000), 435-448.

[2] S.K. Berberian, An extension of Weyls theorem to a class of not necessarily normal operators, Michigan Math. J., 16 (1969), 273-279.

[3] S.K. Berberian, The Weyl spectrum of an operator, Indiana Univ. Math. J., 20 (1970), 529-544.

[4] N.N. Chourasia and P.B. Ramanujan, Paranormal operators on Banach spaces, Bull. Austral. Math. Soc., 21 (1980), 161-168.

[5] L.A. Coburn, Weyls theorem for nonnormal operators, Michigan Math. J., 13 (1966), 285-288.

[6] Raul E. Curto and Young Min Han, Weyl's theorem for Algebraically Paranormal operators, Integer. equ. oper. theory., 47 (2003), 307-314.

[7] S.V. Djordjevi'c and Y.M. Han, Browders theorems and spectral continuity, Glasgow Math. J., 42 (2000), 479-486.

[8] B.P. Duggal and S.V. Djordjevi'c, Weyls theorem in the class of algebraically p-hyponormal operators, Comment. Math. Prace Mat., 40 (2000), 49-56.

[9] Y.M. Han and W.Y. Lee, Weyls theorem holds for algebraically hyponormal operators, Proc. Amer. Math. Soc., 128 (2000), 2291-2296.

[10] R.E. Harte, Fredholm, Weyl and Browder theory, Proc. Royal Irish Acad., 85A (1985), 151-176.

[11] R.E. Harte, Invertibility and Singularity for Bounded Linear Operators, Dekker, New York, 1988.

[12] R.E. Harte and W.Y. Lee, Another note on Weyls theorem, Trans. Amer. Math. Soc., 349 (1997), 2115-2124.

[13] K.B. Laursen and M.M. Neumann, An Introduction to Local Spectral Theory, London Mathematical Society Monographs, New Series 20, Clarendon Press, Oxford, 2000.

[14] D. Senthilkumar and P. Maheshwari Naik, Weyl's theorem for Algebraically Absolute-(p,r)-Paranormal operators, Banach J. Math. Anal., 5(2015), no. 1, 29-37.

[15] V. Rakocevi'c, On the essential approximate point spectrum II, Mat. Vesnik., 36 (1984), 89-97.

[16] V. Rakocevi'c, Approximate point spectrum and commuting compact perturbations, Glasgow Math. J., 28 (1986), 193-198.

[17] M.H.M. Rashid, On absolute-(p,r)-*-Paranormal operators, Sounth Asian Journal of Mathematics, Vol 2(5) (2015), 214-222.

[18] H. Weyl, Uber beschrankte quadratische Formen, deren Differenz vollsteig ist, Rend. Circ. Mat., Palermo 27 (1909), 373-392.

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