
WEYL'S THEOREM FOR ALGEBRAICALLY (\wp, λ, ϱ) -PARANORMAL AND ALGEBRAICALLY (\wp, λ, ϱ) -*-PARANORMAL OPERATORS

Abstract

Present L be an algebraically (\wp, λ, ϱ) -Paranormal and algebraically (\wp, λ, ϱ) -*-Paranormal operators on L^2 space. We examine Weyl's theorem, a-Browder's theorem and spectral mapping theorem holds for weyl's spectrum of L and essential approximate point spectrum of L .

Keywords: Weyl's theorem, a-Browder's theorem, algebraically (\wp, λ, ϱ) -Paranormal and algebraically (\wp, λ, ϱ) -*-Paranormal operators.

2010 Mathematics Subject Classification: 47A10, 47A13, 47A53, 47B37

1 Introduction

Throughout this note let $B(H)$ and $K(H)$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space H . If $L \in B(H)$ we shall write $N(L)$ and $R(L)$ for the null space and range of L , respectively. Also, let $\alpha(L) = \dim N(L)$, $\beta(L) = \dim R(L)$, and let $\sigma(L)$, $\sigma_a(L)$ and $\pi_0(L)$ denote the spectrum, approximate point spectrum and point spectrum of L , respectively. An operator $L \in B(H)$ is called Fredholm if it has closed range, finite dimensional null space, and its range has finite co-dimension. The index of a Fredholm operator is given by

$$i(L) = \alpha(L) - \beta(L).$$

L is called Weyl if it is Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. equivalently if L is Fredholm and $L - \lambda$ is invertible for sufficiently small $|\lambda| > 0$, $\lambda \in C$. The essential spectrum $\sigma_e(L)$, the Weyl spectrum $\omega(L)$ and the Browder spectrum $\sigma_b(L)$ of L are defined by [10,11]

$$\sigma_e(L) = \{\lambda \in C : L - \lambda \text{ is not Fredholm}\},$$

$$\omega(L) = \{\lambda \in C : L - \lambda \text{ is not Weyl}\},$$

and

$$\sigma_b(L) = \{\lambda \in C : L - \lambda \text{ is not Browder}\},$$

respectively. Evidently

$$\sigma_e(L) \subseteq \omega(L) \subseteq \sigma_b(L) = \sigma_e(L) \cup acc\sigma(L),$$

where we write $accK$ for the accumulation points of $K \subseteq C$. If we write $K accK$ then we let

$$\pi_{00}(L) = \{\lambda \in iso\sigma(L) : 0 < \sigma(L - \lambda) < \infty\},$$

and

$$p_{00}(L) = \sigma(L) \sigma_b(L).$$

We say that Weyl's theorem holds for L if

$$\sigma(L) \setminus \omega(L) = \pi_{00}(L), \tag{1.1}$$

and that Browder's theorem holds for L if

$$\sigma(L) \setminus \omega(L) = p_{00}(L), \tag{1.2}$$

In this note we investigate the validity of Weyl's theorem and of Browder's theorem for algebraically paranormal operators.

We consider the sets

$$\Phi_+(H) = \{L \in B(H) : R(L) \text{ is closed and } \alpha(L) < \infty\},$$

$$\Phi_-(H) = \{L \in B(H) : R(L) \text{ is closed and } \beta(L) < \infty\},$$

and

$$\Phi_+^-(H) = \{L \in B(H) : L \in \Phi_+(H) \text{ and } i(L) \leq 0\},$$

By definition,

$$\sigma_{ab}(L) = \cap \{\sigma_a(L + K) : K \in K(H)\},$$

is the essential approximate point spectrum and

$$\sigma_{ab}(L) = \cap \{\sigma_a(L + K) : LK = KL \text{ and } K \in K(H)\},$$

is the Browder essential approximate point spectrum.

In [15, Theorem 3.1], it was shown that $\sigma_{ea}(L) = \{\lambda \in C : L - \lambda \notin \Phi_+^-(H)\}$.

Example 1.1. Let $U_+ \in B(l_2)$ be the unilateral shift. Then $\sigma_{ea}(U_+^*)$ is the closed unit disk.

Proof. Since $\sigma_{ea}(U_+^* - \lambda) = -i(U_+ - \lambda) = 1$ for all $|\lambda| < 1$, $U_+ - \lambda \notin \Phi_+^-(l_2)$ whenever $|\lambda| < 1$. Since $\sigma_{ea}(U_+^*) \subseteq \sigma(U_+^*)$ and $\sigma_{ea}(U_+^*)$ is closed, $\sigma_{ea}(U_+^*)$ must be the closed unit disk. \square

We say that a-Browder's theorem holds for L if

$$\sigma_{ea}(L) = \sigma_{ab}(L). \tag{1.3}$$

It is known ([7],[12]) that if $L \in B(H)$ then

$$\text{Weyl's theorem} \Rightarrow \text{Browder's theorem} \Leftrightarrow \text{a-Browder's theorem}$$

In [18], H. Weyl proved that (1.1) holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal and Toeplitz operators ([5]) and to several classes of operators including seminormal operators ([2],[3]). Recently, Raul E. Curto and Young Min Han[6] Showed that Weyl's theorem holds for algebraically paranormal operators, D. Senthilkumar and P. Mageshwari naik[14] Showed that Weyl's theorem holds for algebraically Absolute- (p, r) -paranormal operators and M.H.M. Rashid[17], we discuss some properties of absolute- (p, r) -*paranormal operators. In this note, we extend this result to algebraically $(\varphi, \lambda, \varrho)$ -Paranormal and algebraically $(\varphi, \lambda, \varrho)$ -*-Paranormal operators.

2 Weyl's Theorem for Algebraically (\wp, ι, ϱ) -Paranormal Operator

For each $\wp > 0$, $\iota \geq 0$ and $\varrho > 0$, an operator L is (\wp, ι, ϱ) -paranormal if $\left\| |L|^\wp \cup |L|^\iota \mathfrak{S} \right\|^{\frac{1}{\wp}} \geq \left\| |L|^{\frac{\wp+\iota}{\wp}} \mathfrak{S} \right\|$, for every unit vector $\mathfrak{S} \in \mathfrak{H}$.

We say that L is algebraically (\wp, ι, ϱ) -paranormal if there exists a nonconstant complex polynomial p such that $p(L)$ is (\wp, ι, ϱ) -paranormal.

We now establish a result for algebraically (\wp, ι, ϱ) -paranormal operator.

Lemma 2.1. *Let L be a (\wp, ι, ϱ) -paranormal operator, $\lambda \in C$ and assume that $\sigma(L) = \{\lambda\}$. Then $L = \lambda$.*

Proof. We consider two cases:

Case I ($\lambda = 0$): Since L is (\wp, ι, ϱ) -paranormal, L is normaloid. Therefore $L = 0$.

Case II ($\lambda \neq 0$): Here L is invertible and since L is (\wp, ι, ϱ) -paranormal, we see that T^{-1} is also (\wp, ι, ϱ) -paranormal. Therefore L^{-1} is normaloid. On the other hand, $\sigma(L^{-1}) = \frac{1}{\lambda}$, so $\|L\| \|L^{-1}\| = \left| \lambda \right|^{-1} = 1$. It follows from [Mla, Lemma 3] that L is convexoid, so $W(L) = \lambda$. Therefore $T = \lambda$. \square

Lemma 2.2. *Let L be a algebraically (\wp, ι, ϱ) -paranormal operator. Then L is nilpotent.*

Proof. Suppose that $p(L)$ is (\wp, ι, ϱ) -paranormal for some nonconstant polynomial p . Since $\sigma(p(L)) = p(\sigma(L))$, the operator $p(L) - p(0)$ is quasinilpotent. It follows from Lemma 2.1 that $cL^m(L - \lambda_1) \dots (L - \lambda_n) \equiv p(L) - p(0) = 0$ (where $m \geq 1$). Since $L - \lambda_i$ is invertible for every $\lambda_i \neq 0$, we must have $L^m = 0$. \square

Lemma 2.3. *Let L be a algebraically (\wp, ι, ϱ) -paranormal operator. Then L is isoloid.*

Proof. Let $\lambda \in \text{iso}\sigma(L)$ and let $P = \frac{1}{2\pi i} \int_{\partial D} (\mu - L)^{-1} d\mu$ be the associated Riesz idempotent, where D is a closed disk centered at λ which contains no other points of $\sigma(L)$. We can then represent L as the direct sum

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \text{ where } \sigma(L_1) = \{\lambda\} \text{ and } \sigma(L_2) = \sigma(L) \setminus \{\lambda\}$$

Since L is algebraically (\wp, ι, ϱ) -paranormal, $p(L)$ is (\wp, ι, ϱ) -paranormal for some nonconstant polynomial p . Since $\sigma(L_1) = \{\lambda\}$, we must have $\sigma(p(L_1)) = p(\sigma(L_1)) = \{p(\lambda)\}$. Therefore $p(L_1) - p(\lambda)$ is quasinilpotent. Since $p(L_1)$ is (\wp, ι, ϱ) -paranormal, it follows from Lemma 2.1 that $p(L_1) - p(\lambda) = 0$. Put $q(z) = p(z) - p(\lambda)$. Then $q(L_1) = 0$ and hence L_1 is algebraically (\wp, ι, ϱ) -paranormal. Since $L_1 - \lambda$ is quasinilpotent and algebraically (\wp, ι, ϱ) -paranormal, it follows from Lemma 2.2 that $L_1 - \lambda$ is nilpotent. Therefore $\lambda \in \pi_0(L_1)$ and hence $\lambda \in \pi_0(L)$. This shows that L is isoloid. \square

In the following theorem, recall that $H(\sigma(L))$ is the space of functions analytic in an open neighborhood of $\sigma(L)$. Also, we say that $L \in B(H)$ has the single valued extension property (SVEP) if for every open set $U \subseteq C$ the only analytic function $f : U \rightarrow H$ which satisfies the equation

$$(L_1 - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$.

Theorem 2.4. *Let L is algebraically (\wp, ι, ϱ) -paranormal operator. Then Weyl's theorem holds for $f(L)$ for every $f \in H(\sigma(L))$.*

Proof. We first show that Weyl's theorem holds for L . Suppose that $\lambda \in \sigma(L) \setminus \omega(L)$. Then $L - \lambda$ is Weyl and not invertible. We claim that $\lambda \in \partial\sigma(L)$. Assume to the contrary that λ is an interior point of $\sigma(L)$. Then there exists a neighborhood U of λ such that $\dim N(L - \mu) > 0$ for all $\mu \in U$. It

follows from [Fin, Theorem 10] that L does not have SVEP. On the other hand, since $p(L)$ is (\wp, λ, ϱ) -paranormal for some nonconstant polynomial p , it follows from [4, Corollary 2.10] that $p(L)$ has SVEP. Hence by [13, Theorem 3.3.9], L has SVEP, a contradiction. Therefore $\lambda \in \partial\sigma(L) \setminus \omega(L)$, and it follows from the punctured neighborhood theorem that $\lambda \in \pi_{00}(L)$.

Conversely, suppose that $\lambda \in \pi_{00}(L)$, with associated Riesz idempotent $P = \frac{1}{2\pi i} \int_{\partial D} (\mu - L)^{-1} d\mu$, where D is a closed disk centered at λ which contains no other points of $\sigma(L)$. As before, we can then represent L as the direct sum

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \text{ where } \sigma(L_1) = \{\lambda\} \text{ and } \sigma(L_2) = \sigma(L) \setminus \{\lambda\}$$

We consider two cases:

Case I ($\lambda = 0$): Here L_1 is algebraically (\wp, λ, ϱ) -paranormal and quasinilpotent, so from Lemma 2.2 it follows that L_1 is nilpotent. We claim that $\dim R(P) < \infty$. For, if $N(L_1)$ were infinite dimensional, then $0 \notin \pi_{00}(L)$, a contradiction. Therefore L_1 is a finite dimensional operator, therefore Weyl. But since L_2 is invertible, we can conclude that L is Weyl. Thus $0 \in \sigma(L) \setminus \omega(L)$.

Case II ($\lambda \neq 0$): By the proof of Lemma 2.3, $L_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(L)$, $L_1 - \lambda$ is a finite dimensional operator, so $L_1 - \lambda$ is Weyl. Since $L_2 - \lambda$ is invertible, $L_2 - \lambda$ is Weyl.

Thus Weyl's theorem holds for L . Next we claim that $f(\omega(L)) = \omega(f(L))$ for all $f \in H(\sigma(L))$. Let $f \in H(\sigma(L))$. Since $\omega(f(L)) \subseteq f(\omega(L))$ with no other restriction on L , it suffices to show that $f(\omega(L)) \subseteq \omega(f(L))$. Suppose $\lambda \neq \omega(f(L))$. Then $f(L) - \lambda$ is Weyl and

$$f(L) - \lambda = c(L - \alpha_1)(L - \alpha_2) \dots (L - \alpha_n)g(L), \tag{2.1}$$

where $c, \alpha_1, \alpha_2, \dots, \alpha_n \in C$ and $g(L)$ is invertible. Since the operators on the right-hand side of (2.1) commute, every $L - \alpha_i$ is Fredholm. Since L is algebraically (\wp, λ, ϱ) -paranormal, L has SVEP [4, Corollary 2.10]. It follows from [1, Theorem 2.6] that $i(L - \alpha_i) \leq 0$ for each $i = 1, 2, \dots, n$. Therefore $\lambda \notin f(\omega(L))$ and hence $f(\omega(L)) = \omega(f(L))$.

Now recall ([LeLe, Lemma]) that if L is isoloid then

$$f(\sigma(L) \setminus \pi_{00}(L)) = \sigma(f(L)) \setminus \pi_{00}(f(L)) \text{ for every } f \in H(\sigma(L)).$$

Since L is isoloid (by Lemma 2.3) and Weyl's theorem holds for L ,

$$\sigma(f(L)) \setminus \pi_{00}(f(L)) = f(\sigma(L) \setminus \pi_{00}(L)) = f(\omega(L)) = \omega(f(L))$$

which implies that Weyl's theorem holds for $f(L)$. This completes the proof. □

Corollary 2.5. *Let L be algebraically (\wp, λ, ϱ) -paranormal operator. Then $\omega(f(L)) = f(\omega(L))$ for every $f \in H(\sigma(L))$.*

3 Weyl's Theorem for Algebraically (\wp, λ, ϱ) -*-Paranormal Operator

For each $\wp > 0$, $\lambda \geq 0$ and $\varrho > 0$, an operator L is (\wp, λ, ϱ) -*-paranormal if $\left\| |L|^\wp \cup |L|^\lambda \mathfrak{S} \right\|^{\frac{1}{\varrho}} \geq \left\| |L|^{\frac{\wp+\lambda}{\varrho}} \cup^* \mathfrak{S} \right\|$.

We say that L is algebraically (\wp, λ, ϱ) -*-paranormal if there exists a nonconstant complex polynomial p such that $p(L)$ is (\wp, λ, ϱ) -*-paranormal.

We now establish a result for algebraically (\wp, λ, ϱ) -*-paranormal operator.

Lemma 3.1. *Let L be a (\wp, λ, ϱ) -*-paranormal operator, $\lambda \in C$ and assume that $\sigma(L) = \{\lambda\}$. Then $L = \lambda$.*

Proof. We consider two cases:

Case I ($\lambda = 0$): Since L is (\wp, λ, ϱ) -*-paranormal, L is normaloid. Therefore $L = 0$.

Case II ($\lambda \neq 0$): Here L is invertible and since L is (\wp, λ, ϱ) -*-paranormal, we see that T^{-1} is also (\wp, λ, ϱ) -*-paranormal. Therefore L^{-1} is normaloid. On the other hand, $\sigma(L^{-1}) = \frac{1}{\lambda}$, so $\|L\| \|L^{-1}\| = |\lambda| \left| \frac{1}{\lambda} \right| = 1$. It follows from [Mla, Lemma 3] that L is convexoid, so $W(L) = \lambda$. Therefore $T = \lambda$. \square

Lemma 3.2. *Let L be an algebraically (\wp, λ, ϱ) -*-paranormal operator. Then L is nilpotent.*

Proof. Suppose that $p(L)$ is (\wp, λ, ϱ) -*-paranormal for some nonconstant polynomial p . Since $\sigma(p(L)) = p(\sigma(L))$, the operator $p(L) - p(0)$ is quasinilpotent. It follows from Lemma 3.1 that $cL^m(L - \lambda_1) \dots (L - \lambda_n) \equiv p(L) - p(0) = 0$ (where $m \geq 1$). Since $L - \lambda_i$ is invertible for every $\lambda_i \neq 0$, we must have $L^m = 0$. \square

Lemma 3.3. *Let L be an algebraically (\wp, λ, ϱ) -*-paranormal operator. Then L is isoloid.*

Proof. Let $\lambda \in \text{iso}\sigma(L)$ and let $P = \frac{1}{2\pi i} \int_{\partial D} (\mu - L)^{-1} d\mu$ be the associated Riesz idempotent, where D is a closed disk centered at λ which contains no other points of $\sigma(L)$. We can then represent L as the direct sum

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \text{ where } \sigma(L_1) = \{\lambda\} \text{ and } \sigma(L_2) = \sigma(L) \setminus \{\lambda\}$$

Since L is algebraically (\wp, λ, ϱ) -*-paranormal, $p(L)$ is (\wp, λ, ϱ) -*-paranormal for some nonconstant polynomial p . Since $\sigma(L_1) = \{\lambda\}$, we must have $\sigma(p(L_1)) = p(\sigma(L_1)) = \{p(\lambda)\}$. Therefore $p(L_1) - p(\lambda)$ is quasinilpotent. Since $p(L_1)$ is (\wp, λ, ϱ) -*-paranormal, it follows from Lemma 3.1 that $p(L_1) - p(\lambda) = 0$. Put $q(z) = p(z) - p(\lambda)$. Then $q(L_1) = 0$ and hence L_1 is algebraically (\wp, λ, ϱ) -*-paranormal. Since $L_1 - \lambda$ is quasinilpotent and algebraically (\wp, λ, ϱ) -*-paranormal, it follows from Lemma 3.2 that $L_1 - \lambda$ is nilpotent. Therefore $\lambda \in \pi_0(L_1)$ and hence $\lambda \in \pi_0(L)$. This shows that L is isoloid. \square

Theorem 3.4. *Let L be an algebraically (\wp, λ, ϱ) -*-paranormal operator. Then Weyl's theorem holds for $f(L)$ for every $f \in H(\sigma(L))$.*

Proof. We first show that Weyl's theorem holds for L . Suppose that $\lambda \in \sigma(L) \setminus \omega(L)$. Then $L - \lambda$ is Weyl and not invertible. We claim that $\lambda \in \partial\sigma(L)$. Assume to the contrary that λ is an interior point of $\sigma(L)$. Then there exists a neighborhood U of λ such that $\dim N(L - \mu) > 0$ for all $\mu \in U$. It follows from [Fin, Theorem 10] that L does not have SVEP. On the other hand, since $p(L)$ is (\wp, λ, ϱ) -*-paranormal for some nonconstant polynomial p , it follows from [4, Corollary 2.10] that $p(L)$ has SVEP. Hence by [13, Theorem 3.3.9], L has SVEP, a contradiction. Therefore $\lambda \in \partial\sigma(L) \setminus \omega(L)$, and it follows from the punctured neighborhood theorem that $\lambda \in \pi_{00}(L)$.

Conversely, suppose that $\lambda \in \pi_{00}(L)$, with associated Riesz idempotent $P = \frac{1}{2\pi i} \int_{\partial D} (\mu - L)^{-1} d\mu$, where D is a closed disk centered at λ which contains no other points of $\sigma(L)$. As before, we can then represent L as the direct sum

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \text{ where } \sigma(L_1) = \{\lambda\} \text{ and } \sigma(L_2) = \sigma(L) \setminus \{\lambda\}$$

We consider two cases:

Case I ($\lambda = 0$): Here L_1 is algebraically (\wp, λ, ϱ) -*-paranormal and quasinilpotent, so from Lemma 3.2 it follows that L_1 is nilpotent. We claim that $\dim R(P) < \infty$. For, if $N(L_1)$ were infinite dimensional, then $0 \notin \pi_{00}(L)$, a contradiction. Therefore L_1 is a finite dimensional operator, therefore Weyl. But since L_2 is invertible, we can conclude that L is Weyl. Thus $0 \in \sigma(L) \setminus \omega(L)$.

Case II ($\lambda \neq 0$): By the proof of Lemma 3.3, $L_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(L)$, $L_1 - \lambda$ is a finite dimensional operator, so $L_1 - \lambda$ is Weyl. Since $L_2 - \lambda$ is invertible, $L - \lambda$ is Weyl.

Thus Weyl's theorem holds for L . Next we claim that $f(\omega(L)) = \omega(f(L))$ for all $f \in H(\sigma(L))$. Let

$f \in H(\sigma(L))$. Since $\omega(f(L)) \subseteq f(\omega(L))$ with no other restriction on L , it suffices to show that $f(\omega(L)) \subseteq \omega(f(L))$. Suppose $\lambda \neq \omega(f(L))$. Then $f(L) - \lambda$ is Weyl and

$$f(L) - \lambda = c(L - \alpha_1)(L - \alpha_2)\dots(L - \alpha_n)g(L), \tag{3.1}$$

where $c, \alpha_1, \alpha_2, \dots, \alpha_n \in C$ and $g(L)$ is invertible. Since the operators on the right-hand side of (3.1) commute, every $L - \alpha_i$ is Fredholm. Since L is algebraically (\wp, λ, ϱ) -*-paranormal, L has SVEP [4, Corollary 2.10]. It follows from [1, Theorem 2.6] that $i(L - \alpha_i) \leq 0$ for each $i = 1, 2, \dots, n$. Therefore $\lambda \notin f(\omega(L))$ and hence $f(\omega(L)) = \omega(f(L))$.

Now recall ([LeLe, Lemma]) that if L is isoloid then

$$f(\sigma(L) \setminus \pi_{00}(L)) = \sigma(f(L)) \setminus \pi_{00}(f(L)) \text{ for every } f \in H(\sigma(L)).$$

Since L is isoloid (by Lemma 3.3) and Weyl's theorem holds for L ,

$$\sigma(f(L)) \setminus \pi_{00}(f(L)) = f(\sigma(L) \setminus \pi_{00}(L)) = f(\omega(L)) = \omega(f(L))$$

which implies that Weyl's theorem holds for $f(L)$. This completes the proof. □

Corollary 3.5. *Let L be algebraically (\wp, λ, ϱ) -*-paranormal operator. Then $\omega(f(L)) = f(\omega(L))$ for every $f \in H(\sigma(L))$.*

4 α -Browders Theorem for Algebraically (\wp, λ, ϱ) -Paranormal and Algebraically (\wp, λ, ϱ) -*-Paranormal Operators

In general, we cannot expect that Weyl's theorem holds for operators having only SVEP. Consider the following example: let $L \in B(l_2)$ be defined by

$$L(x_1, x_2, x_3, \dots) = \left(\frac{1}{2}x_2, \frac{1}{2}x_3, \dots \right)$$

Then L is quasinilpotent and so L has SVEP. But $\sigma(L) = \omega(L) = \{0\}$ and $\pi_{00}(L) = \{0\}$, hence Weyl's theorem does not hold for L . However, α -Browder's theorem holds for L , as Theorem 3.4 below shows. We first need the following auxiliary result, essentially due to C.K. Fong [Fon]; for completeness, we include a proof. Recall that $X \in B(H)$ is called a quasiaffinity if it has trivial kernel and dense range. $\text{Sin}B(H)$ is said to be a quasiaffine transform of L if there is a quasiaffinity X such that $XS = LX$. If both $S \prec L$ and $L \prec S$, then we say that S and L are quasisimilar.

Lemma 4.1. *Suppose L has SVEP and $S \prec L$. Then S has SVEP.*

Proof. Let $U \subseteq C$ be an open set and let $f : U \rightarrow H$ be an analytic function such that $(S - \lambda)f(\lambda) = 0$ for all $\lambda \in U$. Since $S \prec L$, there exists a quasiaffinity X such that $XS = LX$. So $X(S - \lambda) = (L - \lambda)X$ for all $\lambda \in U$. Since $(S - \lambda)f(\lambda) = 0$ for all $\lambda \in U$, $0 = X(S - \lambda)f(\lambda) = (L - \lambda)Xf(\lambda)$ for all $\lambda \in U$. But L has SVEP, so $Xf(\lambda) = 0$ for all $\lambda \in U$. Since X is one-to-one, $f(\lambda) = 0$ for all $\lambda \in U$. Therefore S has SVEP. □

For $L \in B(H)$, it is known that the inclusion $\sigma_{ea}(f(L)) \subseteq f(\sigma_{ea}(L))$ holds for every $f \in H(\sigma(L))$ with no restrictions on L ([16, Theorem 3.3]). The next theorem shows that for algebraically (\wp, λ, ϱ) -Paranormal and algebraically (\wp, λ, ϱ) -*-Paranormal operators the spectral mapping theorem holds for the essential approximate point spectrum.

Theorem 4.2. *Assume that L or L^* is algebraically (\wp, λ, ϱ) -paranormal. Then $\sigma_{ea}(f(L)) = f(\sigma_{ea}(L))$ for every $f \in H(\sigma(L))$.*

Proof. Let $f \in H(\sigma(L))$. It suffices to show that $f(\sigma_{ea}(L)) \subseteq \sigma_{ea}(f(L))$. Suppose that $\lambda \notin \sigma_{ea}(f(L))$. Then $f(L) - \lambda \in \Phi_+^-(H)$ and

$$f(L) - \lambda = c(L - \alpha_1)(L - \alpha_2)\dots(L - \alpha_n)g(L), \quad (4.1)$$

where $c, \alpha_1, \alpha_2, \dots, \alpha_n \in C$ and $g(L)$ is invertible. If L is algebraically (\wp, λ, ϱ) -paranormal, it follows from [1, Theorem 2.6] that $i(L - \alpha_i) \leq 0$ for each $i = 1, 2, \dots, n$. Therefore $\lambda \notin f(\sigma_{ea}(L))$ and hence $\sigma_{ea}(f(L)) = f(\sigma_{ea}(L))$.

Suppose now that L^* is algebraically (\wp, λ, ϱ) -paranormal. Then L^* has SVEP, and so by [1, Theorem 2.8] $i(L - \alpha_i) \geq 0$ for each $i = 1, 2, \dots, n$. Since

$$0 \leq \sum_{i=1}^n i(L - \alpha_i) = i(f(L) - \lambda) \leq 0$$

$L - \alpha_i$ is Weyl for each $i = 1, 2, \dots, n$. Hence $\lambda \notin f(\sigma_{ea}(L))$ and so $\sigma_{ea}(f(L)) = f(\sigma_{ea}(L))$. This completes the proof. \square

Theorem 4.3. *Assume that L or L^* is algebraically (\wp, λ, ϱ) -*-paranormal. Then $\sigma_{ea}(f(L)) = f(\sigma_{ea}(L))$ for every $f \in H(\sigma(L))$.*

Proof. Let $f \in H(\sigma(L))$. It suffices to show that $f(\sigma_{ea}(L)) \subseteq \sigma_{ea}(f(L))$. Suppose that $\lambda \notin \sigma_{ea}(f(L))$. Then $f(L) - \lambda \in \Phi_+^-(H)$ and

$$f(L) - \lambda = c(L - \alpha_1)(L - \alpha_2)\dots(L - \alpha_n)g(L), \quad (4.2)$$

where $c, \alpha_1, \alpha_2, \dots, \alpha_n \in C$ and $g(L)$ is invertible. If L is algebraically (\wp, λ, ϱ) -*-paranormal, it follows from [1, Theorem 2.6] that $i(L - \alpha_i) \leq 0$ for each $i = 1, 2, \dots, n$. Therefore $\lambda \notin f(\sigma_{ea}(L))$ and hence $\sigma_{ea}(f(L)) = f(\sigma_{ea}(L))$.

Suppose now that L^* is algebraically (\wp, λ, ϱ) -*-paranormal. Then L^* has SVEP, and so by [1, Theorem 2.8] $i(L - \alpha_i) \geq 0$ for each $i = 1, 2, \dots, n$. Since

$$0 \leq \sum_{i=1}^n i(L - \alpha_i) = i(f(L) - \lambda) \leq 0$$

$L - \alpha_i$ is Weyl for each $i = 1, 2, \dots, n$. Hence $\lambda \notin f(\sigma_{ea}(L))$ and so $\sigma_{ea}(f(L)) = f(\sigma_{ea}(L))$. This completes the proof. \square

Theorem 4.4. *Assume L has SVEP and let $S \prec L$. Then a -Browder's theorem holds for $f(S)$ for every $f \in H(\sigma(L))$.*

Proof. We first show that a -Browders theorem holds for S . It is well known that $\sigma_{ea}(S) \subseteq \sigma_{ea}(L)$. Conversely, suppose that $\lambda \in \sigma_a(S) \setminus \sigma_{ea}(S)$. Then $S - \lambda \in \Phi_+^-(H)$ and $S - \lambda$ is not bounded below. Since S has SVEP (by Lemma 3.1) and $S - \lambda \in \Phi_+^-(H)$, it follows from [1, Theorem 2.6] that $S - \lambda$ has finite ascent. Therefore, by [16, Theorem 2.1], $\lambda \in \sigma_a(S) \setminus \sigma_{ea}(S)$. Thus, a -Browders theorem holds for S .

It now follows from Theorem 3.2 and Theorem 3.3 that

$$\sigma_{ab}(f(S)) = f(\sigma_{ab}(S)) = f(\sigma_{ea}(S)) = \sigma_{ea}(f(S))$$

So a -Browders theorem holds for $f(S)$. \square

Corollary 4.5. *Let L be an algebraically (\wp, λ, ϱ) -paranormal operator and let $S \prec L$. Then a -Browder's theorem holds for $f(S)$ for every $f \in H(\sigma(L))$.*

Proof. Straightforward from Theorem 3.3 and the fact that algebraically (\wp, λ, ϱ) -paranormal operators have SVEP. \square

Corollary 4.6. *Let L be an algebraically (\wp, λ, ϱ) -*-paranormal operator and let $S \prec L$. Then α -Browder's theorem holds for $f(S)$ for every $f \in H(\sigma(L))$.*

Proof. Straightforward from Theorem 3.3 and the fact that algebraically (\wp, λ, ϱ) -*-paranormal operators have SVEP. \square

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