Rational Generating Functions of Numerical Sequences

Abstract

If division is performed on a rational (non-integer) function, an infinite series is obtained that is relatively easy to find. But the inverse problem can also be solved and, given an infinite numerical sequence, the rational function that can generate it can be found. In this article, different cases are studied in which this generating function can be found in a more or less simple way.

Key words: Generating function; Recurrence relation; Convolution; k–Fibonacci numbers; Extended Fibonacci numbers.

MSC2000: 11B39; 11B30

1 Introduction

In this section we will review some concepts that we will need in the preparation of this article.

1.1 Generating function of a numerical sequence

$f(x)$ is said to be the generating function of the numerical sequence $\{u_n\}$ if, by developing f(x) in power series, we obtain $f(x) = \sum_{n=0} u_n\, x^n$.

The series expansion of a function can be obtained by means of the Taylor formula $f(x) = \sum_{n=0} \frac{f^{(n}(a)}{n!}(x-a)^n$ and where $f^{(n}(a)$ is the value at $x = a$ of the derivative of order n of the function $f(x)$. It is usually simpler if $a = 0$ (Maclaurin series).

This is a process that is generally complicated, but is greatly simplified if $f(x)$ is a rational function $f(x) = \frac{N(x)}{D(x)}$. In this case, it is enough to divide $N(x)$ by $D(x)$, placing both polynomials in increasing powers. Moreover, all the coeficients are integer and the constant term of the denominator must be 1, otherwise, the generated sequence would not be integer.

For this reason, the rational generating functions to which this article is dedicated are specifically studied.

Example 1 *For example, given the function* $f(x) = \frac{1+x}{1-x-x^2}$, *dividing* $1+x$ *by* $1-x-x^2$ *gives the quotient* $1+2\,x+3\,x^2+5\,x^3+8\,x^4+\cdots$. *The coefficients of* x^n *are the Fibonacci numbers* F_n *through* F_2 *so* $f(x) = \sum_{n=0} F_{n+2}x^n$.

1.2 k–Fibonacci and k–Lucas numbers

In [Falcon and Plaza(2006)], the k–Fibonacci numbers $F_{k,n}$ are defined by mean of the recurrence relation $F_{k,n} = k\,F_{k,n-1} + F_{k,n-2}$ with the initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$. These numbers can be calculated by the Binet Identity $F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$ with $\sigma_{1,2} = \frac{k \pm \sqrt{k^2+4}}{2}$. Moreover, also the negative numbers are defined as $F_{k,-n} = (-1)^{n+1} F_{k,n}$. Finally, the generating function of the k–Fibonacci sequence is $f(k,x) = \dfrac{x}{1 - k\,x - x^2}$

The k–Lucas numbers are defined by the same recurrence relation but with the initial conditions $L_{k,0} = 2$, $L_{k,1} = k$. In this case, Binet Identity takes the form $L_{k,n} = \sigma_1^n + \sigma_2^n$ and both numbers are related as $L_{k,n} = F_{k,n-1} + F_{k,n+1}$. From Binet Identity it is easy to prove that $L_{k,-n} = (-1)^n L_{k,n}$. Its generating function is $l(k,x) = \dfrac{2 - k\,x}{1 - k\,x - x^2}$

1.3 Extended Fibonacci numbers

Falcon defines the extended (k,t)–Fibonacci numbers $T(k,t,n)$ in [Falcon(2024)] by mean of the recurrence relation $T(k,t,n) = k\,T(k,t,n-1) + T(k,t,n-2) + t$ with the iniitial conditions , $T(k,t,0) = 1,\ T(k,t,1) = 1$. The sequence generated is

$\{T(k,t,n)\} = \{1, 1, k + (t+1), k^2 + (t+1)k + (t+1), k^3 + (t+1)k^2 + (t+2)k + (2t+1), \ldots\}$

The extended (k,t)–Fibonacci numbers are related to the k–Fibonacci numbers by the formula $T(k,t,n) = \dfrac{(k+t)(F_{k,n} + F_{l,n-1}) - t}{k}$

If $k = 1$, this sequence takes the form

$\{1, 1, 2+t, 3+2t, 5+4t, 8+7t, 13+12t, 21+20t, \ldots\}$ which can be considered a generalization of Leonardo sequence and it is indicated as $\Big\{Le_n(t)\Big\}$.

Later, if $t = 1$, the classical Leonardo sequence appears:

$\{Le_n\} = \{1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177 \ldots\}$

2 Sum, product and derivative of generating functions

In this section we will study the nuimerical sequences generated by sum or product of generating functions.

- **Sum.** If u_n is a linear combination of other sequences, its generating function is the same linear combination of the generating functions of the respective numerical sequences.

  Exercise 1 *Find the generating function of the extended Fibonacci numbers.*

  In Introduction we have seen that $T(k,t,n) = \dfrac{(k+t)(F_{k,n}+F_{l,n-1})-t}{k}$. On the other hand, the generating function of $\{F_{k,n}\}$ is $f_1(k,x) = \dfrac{x}{1-k\,x-x^2}$ so that of $\{F_{k,n-1}\}$ is $f_2(k,x) = \dfrac{1-k\,x}{1-k\,x-x^2}$; and that $\{1\}$ is $f_3(x) = \dfrac{1}{1-x}$. Consequently, the generating function of the extended (k,t)–Fibonacci numbers is

$$\begin{aligned} g(k,t,x) &= \frac{1}{k}\left((k+t)\left(\frac{x}{1-k\,x-x^2}+\frac{1-k\,x}{1-k\,x-x^2})\right)-t\,\frac{1}{1-x}\right) \\ &= \frac{1-k\,x-(1-t-k)x^2}{(1-k\,x-x^2)(1-x)} \end{aligned}$$

- **Product.** The convolution of the sequences $A = \{a_n\}$ and $B = \{b_n\}$ is defined as the sequence $A * B = \left\{\sum_{j=0}^{n} a_j\, b_{n-j}\right\}$ [Umar and Yushau(2007)]. In fact, the convolution coincides with the sequence of coefficients of the product of two infinite polynomials $\left\{\sum_{n=0} a_n\, x^n\right\}$ and $\left\{\sum_{n=0} b_n\, x^n\right\}$.

  Convolution verifies commutative, associative, and distributive properties:

  $A * B = B * A$, $A * (B * C) = (A * B) * C$, $A * (B + C) = A * B + A * C$.

  The unit element is the sequence $\{1, 0, 0, 0, \ldots\}$, but the inverse sequence, if exists, is not integer.

Theorem 1 *If $f(x)$ and $g(x)$ are the respective generating functions of the sequences $A = \{a_n\}$ and $B = \{b_n\}$, then $f(x) \cdot g(x)$ is the generating function of the convolution sequence $A * B$*

Proof 1 *Proof is easy because $A(x) = \sum_{j\geq 0} a_j x^j$ and $B(x) = \sum_{j\geq 0} b_j x^j$, then $A(x) \cdot B(x) = \sum_{j\geq 0} \left(\sum_{i=0}^{j} a_i b_{j-i} \right) x^j$, and the coefficients of $A(x) \cdot B(x)$ are the elements of the convolution $A * B$.*

Example 2 *It is well known that $f(x) = \dfrac{1}{1 - a\,x}$ is the generating function of the sequence $\{a^n\} = \{1, a, a^2, a^3, \ldots\}$. Then*

$$\begin{aligned}
\frac{1}{1-x} &\mapsto \{1\} = \{1,1,1,1,1,\ldots\} = \left\{ \binom{n}{0} \right\} \\
\frac{1}{(1-x)^2} &\mapsto \{1\} * \{1\} = \{1\}^{(2} = \{1,2,3,4,5,\ldots\} = \left\{ \binom{n}{1} \right\} : \\
&\qquad \textit{Linear numbers} \\
\frac{1}{(1-x)^3} &\mapsto \{1\} * \{1\}^{(2} = \{1\}^{(3} = \{1,3,6,10,15,\ldots\} = \left\{ \binom{n+1}{2} \right\} : \\
&\qquad \textit{Triangular numbers} \\
\frac{1}{(1-x)^4} &\mapsto \{1\} * \{1\}^{(3} = \{1\}^{(4} = \{1,4,10,20,35,\ldots\} = \left\{ \binom{n+2}{3} \right\} : \\
&\qquad \textit{Tetrahedral numbers} \\
&\ldots \\
\frac{1}{(1-x)^r} &\mapsto \{1\}^{(r} = \left\{ \binom{n+r-2}{r-1} \right\}
\end{aligned}$$

- **Derivative.**

Theorem 2 (Derivative of the generating function) *If $g(x)$ is the generating function of the sequence u_n, then $x \cdot g'(x)$ is the generating function of the sequence $\{n\, u_n\}$*

Proof 2 *If $g(x) = \sum_{n=0} u_n\, x^n$, then $g'(x) = \sum_{n=1} n\, u_n\, x^{n-1}$ and so $f(x) = x \cdot g'(x) = \sum_{n=1} n \cdot u_n\, x^n$*

Exercise 2 *Find the generating function of the sequence of squares $\{n^2\}$*

It is well known that the generating function of the constant sequence $\{1\}$ is

$g(x) = \dfrac{1}{1-x}$. Then, the generating function of $\{n\}$ is $x \cdot g'(x) = \dfrac{x}{(1-x)^2}$. And so, the generating function of the sequence of squares $\{n^2\}$ is $f(x) = x(x \cdot g'(x))' = \dfrac{x(1+x)}{(1-x)^3}$

This is a very interesting example because, by repeating the process we can find the generating function of the power sequences $\{n^r\}$.
And so, the generating function of $\{n^3\}$ is $f_3(x) = \dfrac{(1+4x+x^2)x}{(1-x)^4}$, and that of $\{n^4\}$ it is
$f_4(x) = \dfrac{(1+11x+11x^2+x^3)x}{(1-x)^5}$, etc.

In short, if $f(x)$ is the generating function of the sequence $\{u_n\}$, then $(x \cdot D)^r\, f(x)$ is the generating function of the sum sequence $\left\{\sum_{n=0} n^r\, u_n\right\}$.

Even though the integral of the generating function can be found to find a new sequence, this trick is of no interest because the resulting numerical sequence is not an integer.

3 Generating function and recurrence relation

Many times the recurrence relation verified by the terms of a numerical sequence is known. In this case, the process to follow is indicated in the following theorem.

In order to simplify the calculations, let us assume that the recurrence relation relates four consecutive terms of a sequence $\{u_n\}$.

Theorem 3 *If the sequence $\{u_n\}$ verifies the recurrence relation $u_n = a\,u_{n-1} + b\,u_{n-2} + c\,u_{n-3}$, its generating function $g(x)$ is*

$$g(x) = \frac{u_0 + (u_1 - a\,u_0)x + (u_2 - a\,u_1 - b\,u_0)x^2}{1 - a\,x - b\,x^2 - c\,x^3} \qquad (1)$$

Proof 3 (First form) *Let $g(x)$ be the generating function of the sequence $\{u_n\}$. Then $g(x) = \sum_{n=0} u_n\, x^n$ and*

$$\begin{aligned}
g(x) &= u_0 + u_1\, x + u_2 x^2 + u_3\, x^3 + u_4\, x^4 + \cdots \\
a\, x\, g(x) &= \quad a\, u_0\, x + a\, u_1\, x^2 + a\, u_2 x^3 + a\, u_3\, x^4 + \cdots \\
b\, x^2\, g(x) &= \qquad b\, u_0\, x^2 + b\, u_1\, x^3 + b\, u_2 x^4 + \cdots \\
c\, x^3\, g(x) &= \qquad\qquad c\, u_0\, x^3 + c\, u_1\, x^4 + \cdots \\
& (1 - a\, x - b\, x^2 - c\, x_3) g(x) \\
& = u_0 + (u_1 - a\, u_0) x + (u_2 - a\, u_1 - b\, u_0) x^2 \quad (*) \\
g(x) &= \frac{u_0 + (u_1 - a\, u_0) x + (u_2 - a\, u_1 - b\, u_0) x^2}{1 - a\, x - b\, x^2 - c\, x^3}
\end{aligned}$$

(*) The remaining addends are all null due to the recurrence relation.

Exercise 3 *The sequence of the squares of the k–Fibonacci numbers $\left\{F_{k,n}^2\right\}$ verifies the recurrence relation $F_{k,n}^2 = (k^2+1) F_{k,n-2}^2 + (k^2+1) F_{k,n-2}^2 - F_{k,n-3}^2$. Find its f.g.*

Then, if $g(k, x)$ is its generating function , as $a = k^2 + 1$, $b = k^2 + 1$, $c = -1$ and $u_0 = F_{k,0}^2 = 0$, $u_1 = F_{k,1}^2 = 1$ and $u_2 = F_{k,2}^2 = k^2$, it is

$$\begin{aligned}
g(k, x) &= \frac{u_0 + (u_1 - a\, u_0) x + (u_2 - b\, u_1 - c\, u_0) x^2}{1 - a\, x - b\, x^2 - c\, x^3} \\
&= \frac{x - x^2}{1 - (k^2+1) x - (k^2+1) x^2 + x^3}
\end{aligned}$$

In a second way, we will use the generating function itself to find its mathematical expression.

Proof 4 (Second form) *If $g(x) = \sum_{n=0} u_n\, x^n$.*

$$
\begin{aligned}
g(x) &= u_0 + u_1\,x + u_2x^2 + u_3\,x^3 + u_4\,x^4 + \cdots \\
&= u_0 + u_1\,x + u_2\,x^2 + (a\,u_2 + b\,u_1 + c\,u_0)x^3 \\
&\qquad\qquad + (a\,u_3 + b\,u_2 + c\,u_1)x^4 + (a\,u_4 + b\,u_3 + c\,u_2)x^5 + \cdots \\
&= u_0 + u_1\,x + u_2\,x^2 + a(u_2\,x^3 + u_3\,x^4 + u_4\,x^5 + \cdots) \\
&\qquad\qquad + b(u_1\,x^3 + u_2\,x^4 + u_3\,x^5 + \cdots) \\
&\qquad\qquad + c(u_0\,x^3 + u_1\,x^4 + u_2\,x^5 + \cdots) \\
&= u_0 + u_1\,x + u_2\,x^2 + a\,x(u_2\,x^2 + u_3\,x^3 + u_4\,x^4 + \cdots) \\
&\qquad\qquad + b\,x^2(u_1\,x + u_2\,x^2 + u_3\,x^3 + \cdots) \\
&\qquad\qquad + c\,x^3(u_0 + u_1\,x + u_2\,x^2 + \cdots) \\
&= u_0 + u_1\,x + u_2\,x^2 + a\,x(g(x) - u_0 - u_1\,x) + b\,x^2(g(x) - u_0) + c\,x^3\,g(x) \\
&= u_0 + u_1\,x + u_2\,x^2 + (a\,x + b\,x_2 + c\,x^3)g(x) - a\,x(u_0 + u_1\,x) - b\,x^2\,u_0 \\
&\quad g(x)(1 - a\,x - b\,x^2 - c\,x^3) = u_0 + u_1\,x + u_2\,x^2 - a\,x(u_0 + u_1\,x) - b\,x^2\,u_0 \\
g(x) &= \frac{u_0 + u_1\,x + u_2\,x^2 - a\,x(u_0 + u_1\,x) - b\,x^2\,u_0}{1 - a\,x - b\,x^2 - c\,x^3}
\end{aligned}
$$

Exercise 4 *In [Falcon(2024)] it is proven that the extended* (k, t)*–Fibonacci numbers* $T(k, t, n)$ *verify the recurrence relation* $T(k, t, n) = (k + 1)\,T(k, t, n - 1) + (1 - k)\,T(k, t, n - 2) - T(k, t, n - 3)$. *Find its g.f.*

In this case $a = k + 1$, $b = 1 - k$, and $c = -1$, so, applying Equation (1), the generating function of this sequence is [Falcon(2024)]

$$f(k, t, x) = \frac{1 - k\,x + (k - 1 + t)\,x^2}{1 - (k + 1)\,x + (k - 1)\,x^2 + x^3}$$

Remark. In general, to find the generating function knowing the recurrence relation, it is usually more practical to follow the entire theoretical process with the data provided by the statement of the problem.

Exercise 5 *If* $u_n = 2k\,u_{n-1} + (2 - k^2)\,u_{n-2} - 2k\,u_{n-3} - u_{n-4}$ *and initial conditions are* $u_0 = 0,\ u_1 = 0,\ u_2 = 1,\ u_3 = k$*, find the generating function of the sequence* $\{u_n\}$.

We follow the first method. If $f(k, x)$ is the generating function

$$\begin{aligned}
f(k,x) &= u_0 + u_1\,x + u_2\,x^2 + u_3\,x^3 + u_4\,x^4 + \cdots \\
2k\,x\,f(k,x) &= 2k\,u_o\,x + 2k\,u_1\,x^2 + 2k\,u_2\,x^3 + 2k\,u_3x^4 + \cdots \\
(2-k^2)\,x^2\,f(k,x) &= (2-k^2)\,u_0\,x^2 + (2-k^2)\,u_1\,x^3 + (2-k^2)\,u_2\,x^4 + \cdots \\
-2k\,x^3\,f(k,x) &= \qquad\qquad -2k\,u_0\,x^3 - 2k\,u_1\,x^4 - \cdots \\
-x^4\,f(k,x) &= \qquad\qquad\qquad -u_0\,x^4 + \cdots
\end{aligned}$$

Adding these equations according to the recurrence relation and taking into account the initial conditions,

$$\begin{aligned}
& (1 - 2k\,x - (2-k^2)\,x^2 + 2k\,x^3 + x^4)\,f(k,x) \\
&= u_0 + (u_1 - 2k\,u_0)x + (u_2 - 2k\,u_1 - (2-k^2)\,u_0)x^2 \\
&\quad + (u_3 - 2k\,u_2 - (2-k^2)\,u_1 + 2k\,u_0)x^3 = x^2 \\
f(k,x) &= \frac{x^2}{1 - 2k\,x - (2-k^2)\,x^2 + 2k\,x^3 + x^4}
\end{aligned}$$

Remark This is the generating function of the derivative of the k–Fibonacci sequence.

3.1 Recurrence relation

In the previous section we have seen how to find the generating function of a numerical sequence if we know the recurrence relation verified by its terms. But there is also the reciprocal problem: given the generating function of a sequence, find the recurrence relation of its elements. This is the problem that will be solved in this section.

The process to follow is the following. Let $f(x) = \dfrac{N(x)}{D(x)}$ be the generating function of the sequence $\{u_n\}$. The denominator $D(x)$ is a polynomial of degree r. Cancelling the denominator and setting $x^p = u_{n-p}$ gives the recurrence relation sought, which will take the form $u_n = \sum_{j=1}^{r} a_{n-j}\,u_{n-j}$.
n initial conditions are needed which can be found either by following the

formula found previously, or by choosing the first n coefficients of the quotient of $N(x)$ by $D(x)$ in increasing powers.

Exercise 6 *If* $f(x) = \dfrac{1-x+x^2}{1+3x+x^2-2x^3}$ *is the generating function of the sequence* $\{u_n\}$*, find its recurrence relation.*

In this case, $D(x) = 1+3x+x^2-2x^3 = 0 \to u_n + 3\,u_{n-1} + u_{n-2} - 2\,u_{n-3} = 0$, so the recurrence relation is $u_n = -3\,u_{n-1} - u_{n-2} + 2\,u_{n-3}$
Dividing $N(x)$ by $D(x)$ gives $1 - 4x + 12x^2 - 30x^3 + \cdots$ so the three initial conditions are $u_0 = 1$, $u_1 = -4$ and $u_3 = 12$.

The initial conditions can also be found by solving the system $u_0 + (u_1 - a\,u_0)x + (u_2 - a\,u_1 - b\,u_0)x^2 = 1 - x + x^2$, with $a = -3$, $b = -1$.

3.2 Sequence of partial sums

Let $U = \{u_0, u_1, u_2, \ldots\}$ be the sequence generated by the function $f(x)$. The convolution of this sequence and the constant sequence $1^{(1} = \{1, 1, 1, 1, \ldots\}$ is $U * 1^1 = \{u_0, u_0 + u_1, u_0 + u_1 + u_2, \ldots, \sum_{j=0}^{n} u_j, \ldots\}$ that is the sequence of partial sums. Taking into account that $\dfrac{1}{1-x}$ is the generating function of the sequence $1^{(1}$, then $\dfrac{1}{1-x}\,f(x) = \dfrac{f(x)}{1-x}$ is the generating function of the sequence of partial sums $\left\{\sum_{j=0}^{n} u_n\right\}$.

Exercise 7 *Find the generating function of the sequence of partial sums of the 3–Fibonacci sequence.*

The generating function of the sequence $F_3 = \{0, 1, 3, 10, 33, 109, 360, \ldots\}$ is $f(x) = \dfrac{x}{1-3x-x^2}$ so, the generating function of the partial sums of 3–

Fibonaccci sequence is $\dfrac{x}{(1-x)(1-3x-x^2)} = \dfrac{x}{1-4x+2x^2+x^3}$.

4 Alternated sequences

In this section we will study the sequences that change sign alternatively in each term.

Theorem 4 *If $g(x)$ is the generating function of the sequence $\{u_n\}$, then $g(-x)$ is the generating function of the alternated sequence $\{(-1)^n\, u_n\}$*

Proof 5 *If $g(x)$ is the generating function of the sequence $\{u_n\}$, then*
$g(x) = \sum\limits_{n=0} u_n\, x^n = u_0 + u_1\, x + u_2\, x^2 + u_3\, x^3 + \cdots$ *so*
$g(-x) = u_0 - u_1\, x + u_2\, x^2 - u_3\, x^3 + \cdots = \sum\limits_{n=0} (-1)^n\, u_n\, x^n$

Exercise 8 *Given the generating function of $\left\{\sum\limits_{n=0} F_{k,n}^2\right\}$ (Exercise 3), find the generating function of the alternated sequence $\left\{\sum\limits_{j=0} (-1)^n F_{k,n}^2\right\}$*

Its generating function is $f(k,x) = g(k,-x) = \dfrac{-x-x^2}{1+(k^2+1)x-(k^2+1)x^2-x^3}$

Corollary 1 *If $f(x)$ is the generating function of the sequence $\{u_n\}$, then $\dfrac{f(x)+f(-x)}{2}$ is the generating function of the terms of even order, the remaining ones being zero. And $\dfrac{f(x)-f(-x)}{2}$ generates only the odd terms, the even ones being zero.*

Proof 6

$$\begin{aligned}
f(x) &= u_0 + u_1\, x + u_2\, x^2 + u_3\, x^3 + u_4\, x^4 + \cdots \\
f(-x) &= u_0 - u_1\, x + u_2\, x^2 - u_3\, x^3 + u_4\, x^4 + \cdots \\
f(x) + f(-x) &= 2(u_0 + u_2\, x^2 + u_4\, x^4 + \cdots) \\
f(x) - f(-x) &= 2(u_1\, x + u_3\, x^3 + u_5\, x^5 + \cdots)
\end{aligned}$$

Example 3 $f(k,x) = \dfrac{x}{1-k\,x-x^2}$ *is the generating function of the* k*–Fibonacci numbers. Then* $\dfrac{f(k,x)+f(k,-x)}{2} = \dfrac{k\,x^2}{(1-k\,x-x^2)(1+k\,x-x^2)}$ *is the generating function of the sequence* $\{0,0,k,0,2k+k^3,0,3k+4k^3+k^5,0,4k+10k^3+6k^5+k^7,0,\ldots\}$

5 Generating functions of displaced sequences

In this section we will study the generating function of different sequences obtained from $\{u_n\}$.

Theorem 5 (Sequence with $r-1$ null terms) *If* $f(x)$ *is the generating function of the sequence* $\{u_n\}$*, then* $f(x^r)$ *is the generating function of the same sequence but with the successive terms separated with* $r-1$ *null terms.*

Proof 7

$$\begin{aligned} f(x) &= u_0 + u_1\,x + u_2\,x^2 + u_3\,x^3 + \cdots \\ f(x^r) &= u_0 + u_1\,x^r + u_2\,x^{2\,r} + u_3\,x^{3\,r} + \cdots \end{aligned}$$

and therefore the coefficients of x^n are zero if n is not a multiple of r.

Example 4 $f(x) = \dfrac{2-x}{1-x-x^2}$ *is the generating function of the classical Lucas numbers* $\{2,1,3,4,7,11,18,29,\ldots\}$*. Then* $f(x^3) = \dfrac{2-x^3}{1-x^3-x^6}$ *is the generating function ot the sequence* $\{2,0,0,1,0,0,3,0,0,4,0,0,7,0,0,11,\ldots\}$

5.1 On the sequence starting at u_{n+r}

Theorem 6 (Sequence $\{u_{n+1}\}$) *If* $f(x)$ *is the generating function of the sequence* $\{u_n\} = \{u_0,u_1,u_2,\ldots\}$*, then* $g(x) = \dfrac{f(x)-u_0}{x}$ *is the generating func-*

tion of the sequence $\{u_{n+1}\} = \{u_1, u_2, u_3, \ldots\}$

Proof 8 *If* $g(x)$ *is the generating function of* $\{u_{n+1}\}$*, then*
$$g(x) = \sum_{n=0} u_{n+1}x^n = \frac{1}{x}\sum_{n=0} u_{n+1}x^{n+1} = \frac{1}{x}\sum_{m=1} u_m x^m = \frac{1}{x}(f(x) - u_0)$$

being $u_0 = f(0)$.

Exercise 9 *If* $f(x) = \dfrac{1+5x}{1+3x-2x^2}$ *is the generating function of a sequence* $\{u_n\} = \{u_0, u_1, u_2, \ldots\}$*, find the generating function of* $\{u_{n+1}\} = \{u_1, u_2, u_3, \ldots\}$

As $u_0 = f(0) = 1$, the generating function of $\{u_{n+1}\}$ is

$$g(x) = \frac{f(x) - f(0)}{x} = \frac{1}{x}\left(\frac{1+5x}{1+3x-2x^2} - 1\right) = \frac{2+2x}{1+3x-2x^2}$$
It is $\{u_n\} = \{1, 2, -4, 16, -56, 200, \ldots$ while $\{u_n{+}1\} = \{2, -4, 16, -56, 200, -712 \ldots$

This method can be applied iteratively and so, the generating function of the sequence $\{u_{n+2}\} = \{u_2, u_3, u_4, \ldots\}$ is $g(x) = \dfrac{f(x) - (u_1 + u_0\, x)}{x^2}$. To find $\{u_0, u_1\}$, it is enough to divide $1 + 5x$ between $1 + 3x - 2x^2$.

Corollary 2 *The generating function of the sequence* $\{u_{n+r}\}$ *is*
$$g(x) = \frac{1}{x^r}\left(f(x) - \sum_{j=0}^{r-1} u_j\, x^j\right)$$

This method is important not only in itself but can be applied to solve more complex problems, as shown in the following exercise.

Exercise 10 *Find the generating function of the sequence of sums* $\left\{\sum_{n=o} F_{k,n+3}\right\}$

The generating function of the sequence of the k–Fibonacci numbers is $f(k,x) = \dfrac{x}{1 - k\,x - x^2}$. For the sequence whose first element is $F_{k,n+3}$, according to this last formula, is

$$g(k,x) = \frac{1}{x^3}\left(f(k,x) - (F_{k,0} + F_{k,1}x + F_{k,2}x^2)\right)$$
$$= \frac{1}{x^3}\left(\frac{x}{1-k\,x-x^2} - (x + k\,x^2)\right) = \frac{1+k^2+k\,x}{1-k\,x-x^2}$$

And taking into account Theorem 2, the generating function of the sum sequence it is
$$h(k,x) = \frac{1}{1-x}g(k,x) = \frac{1+k^2+k\,x}{(1-x)(1-k\,x-x^2)}$$

In the displaced sequence $\{u_{n+r}\}$, it may happen that r is a negative integer. In this case, the process to follow is the same as the one followed previously as indicated below.

Theorem 7 *If $f(x)$ is the generating function of the sequence $\{u_n\}$ and u_{-1} is known, $h(x) = u_{-1} + x\,f(x)$ is the generating function of the sequence $\{u_{n-1}\}$*

In this case

$$h(x) = \sum_{n=0} u_{n-1}x^n = u_{-1} + u_0\,x + u_1\,x^2 + u_2\,x^3 + \cdots$$
$$= u_{-1} + x(u_0 + u_1\,x + u_2\,x^2 + u_3\,x^3 + \dots) = u_{-1} + x\sum_{n=0} u_n\,x^n$$
$$= u_{-1} + x\,f(x)$$

Corollary 3 *By reiterating the above process, we can find that the generating function of the sequence u_{n-r} is $h(x,r) = \sum_{r=0}^{n-1} u_{r-n}\,x^{r-1} + x^r\,f(x)$, where the u_{r-n} are known.,*

Exercise 11 *Find the generating function of the sequence of displaced $k-$Lucas numbers $L_{k,n-2}$*

From the above formula and taking into account that $L_{k,-2} = (-1)^2\,L_{k,2} = k^2+2$ and $L_{k,-1} = (-1)^1 L_{k,1} = -k$,

$$
\begin{aligned}
h(x) &= \sum_{r=0}^{n-1} L_{k,r-n}\, x^{r-1} + x^r\, l(k,x) = L_{k,-2} + L_{k,-1}\, x + x^2\, \frac{2 - k\, x}{1 - k\, x - x^2} \\
&= (k^2+2) - k\, x + \frac{x^2(2 - k\, x)}{1 - k\, x - x^2} = \frac{(k^2+2) - 3k\, x - k^2\, x^2}{1 - k\, x - x^2}
\end{aligned}
$$

Of course, the sequence generated begins in $L_{k,-2}$ and is

$\{2 + k^2, -k, 2, k, 2 + k^2, 3k + k^3, 2 + 4k^2 + k^4, 5k + 5k^3 + k^5, \ldots\}$

Conclusions

In this article we have studied the generating functions of numerical sequences of different types, not only those of the general form $\{u_n\}$. We have also studied sequences that do not start at $n = 0$ but at any term u_{n+p}, where p is a positive or negative integer.

We have also studied the generating functions of alternating sequences and those whose first r terms are zero or which have an infinite number of intercalated zeros. We have solved the reciprocal problem of finding the recurrence relation of a sequence whose generating function is known.

Another application has been to calculate the form of the generating function of the sequence of partial sums $\left\{\sum_{j=0}^{n} u_j\right\}$.

Disclamer (Artificial Intellligence)

Author hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text generators have been used during writing of this manuscript.

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