The Stability for the Two-dimensional Incompressible MHD System with Mixed Dissipation*

Abstract. The global well-posedness and stability on the two-dimensional (2D) incompressible magnetohydrodynamics (MHD) system are studied in this paper. More precisely, when the MHD system only has the dissipation of the second component of the magnetic field and mixed dissipation of velocity, this system is stable. Another main goal is to establish the large-time behavior of the solution for the linear system.

Keywords. MHD; stability; global well-posedness; large-time behavior

AMS Subject Classifications (2020). 35B20, 35B35

1 Introduction

The MHD system is coupled from Naiver-stokes and Maxwell's equations. Extensive physical experiments and numerical simulations have shown an important phenomenon that a background magnetic field can actually stabilize and dump electrically conducting fluids (see, e.g., [1, 2, 3, 4, 11]). The anisotropic incompressible MHD system in $\mathbb{R}^2$ have the following form

$$\begin{cases} \partial_t u_1 + u\cdot\nabla u_1 - B\cdot\nabla B_1 + \partial_1 p = \mu_{11}\partial_{11}u_1 + \mu_{12}\partial_{22}u_1, \\ \partial_t u_2 + u\cdot\nabla u_2 - B\cdot\nabla B_2 + \partial_2 p = \mu_{21}\partial_{11}u_2 + \mu_{22}\partial_{22}u_2, \\ \partial_t B_1 + u\cdot\nabla B_1 - B\cdot\nabla u_1 = \eta_{11}\partial_{11}B_1 + \eta_{12}\partial_{22}B_1, \\ \partial_t B_2 + u\cdot\nabla B_2 - B\cdot\nabla u_2 = \eta_{21}\partial_{11}B_2 + \eta_{22}\partial_{22}B_2, \\ \mathrm{div}u = \mathrm{div}B = 0, \\ (u,B)(x,0) = (u_0,B_0), \end{cases} \quad (x,t)\in\mathbb{R}^2\times\mathbb{R}^+, \tag{1.1}$$

where $u=(u_1,u_2)$, $B=(B_1,B_2)$ and $p(x,t)$ are unknown velocity, magnetic field, and pressure, respectively. The constants $\mu_{ij},\eta_{ij}\geq 0$ with $i,j=1,2$ are viscous and magnetic diffusion coefficients, respectively.

For simplicity, we denote

$$A_1 = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix}, A_2 = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}$$

as the matrices of viscous and magnetic diffusion coefficients. When all coefficients μ_{ij}, $\eta_{ij}(i,j=1,2)$ are positive, Duvet [9] established the global existence of classical solutions to the 2D MHD

*Supported by NNSFC (Grant No. 12101200) and China Postdoctoral Science Foundation (Grant No. 2022M721035)

equations for initial data $(u_0, b_0) \in H^s(\mathbb{R}^2)(s > 2)$. For the anisotropic incompressible MHD system, Cao-Wu [6] first proved the global existence of the classical solutions for the system (1.1) with cases

$$(A_1, A_2) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad (A_1, A_2) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then, Du-Zhou [7] established some similar results for the MHD system with maximized partial dissipation and magnetic diffusion in more general cases.

However, stability for the perturbations near the trivial steady state $(u, B) = (0, 0)$ and their large-time behavior remain open. In this paper, we will focus on the following MHD system (1.2) near the equilibrium state $(U^{(0)}, B^{(0)})$

$$\begin{cases} \partial_t u + u \cdot \nabla u - b \cdot \nabla b + \nabla p = (\partial_2^2 u_1, \partial_1^2 u_2)^\top + \partial_2 b, \\ \partial_t b + u \cdot \nabla b - b \cdot \nabla u = (0, \partial_1^2 b_2)^\top + \partial_2 u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ (u, b)|_{t=0} = (u_0, b_0). \end{cases} \tag{1.2}$$

where

$$U^{(0)} \equiv 0, \quad B^{(0)} \equiv e_2 := (0, 1).$$

Recently, Lin-Ji-Wu-Yan [14] obtained the existence and large-time behavior for the global solutions of $U^{(0)} \equiv 0, \quad B^{(0)} \equiv (1, 0)$ with

$$\textbf{case } 1. \quad (A_1|A_2) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

For $U^{(0)} \equiv 0, \quad B^{(0)} \equiv (0, 1)$, Li-Wu-Xu [15] obtained the stability of system (1.1) with

$$\textbf{case } 2. \quad (A_1|A_2) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Guo-Jia-Dong [12] got the stability of (1.1) with

$$\textbf{case } 3. \quad (A_1|A_2) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Lin-Chen–Bai-Zhang [16] proved the stability of (1.1) with

$$\textbf{case } 4. \quad (A_1|A_2) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Our result is inspired by the work of [16]. If the magnetic field has less dissipation, the system is still stable. For the system (1.1) with equilibrium state $U^{(0)} \equiv 0, \quad B^{(0)} \equiv (0, 1)$, we consider the case

$$\textbf{case } 5. \quad (A_1|A_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Our main results can then be stated as follows.

Theorem 1.1. *Consider (1.2) with the initial data* $(u_0, b_0) \in H^2(\mathbb{R}^2)$ *satisfies* $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. *Then there exists a positive constant* $\varepsilon>0$*, such that if*

$$\|(u_0, b_0)\|_{H^2} \le \varepsilon,$$

then the system (1.2) has a unique global solution for any $t>0$*, satisfying*

$$\|(u,b)(t)\|_{H^2}^2 + 2\int_0^t \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 u_2\|_{H^2}^2 + \|\partial_1 b_2\|_{H^2}^2 + \|\partial_2 b_1\|_{H^1}^2 d\tau \le C\varepsilon^2, \tag{1.3}$$

where $C>0$ *is a generic positive constant independent of* ε *and* t*. In addition,* (u,b) *obeys the following large-time behavior:*

$$\|(\nabla u, \nabla b)(t)\|_{L^2} \to 0, \qquad \|(\nabla^2 u, \nabla^2 b)(t)\|_{L^2} \to 0, \qquad as \quad t\to\infty. \tag{1.4}$$

We observe that (1.3) of Theorem 1.1 rigorously assesses that any small initial perturbation leads to a unique global solution of (1.2) and remains consistently small in H^2. Since the local existence result can be shown via the standard method, we only need to establish a global prior estimate of the solutions. To use the bootstrapping argument, we introduce an energy functional specifically to achieve our desired estimates. Let

$$\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t),$$

where

$$\begin{aligned}\mathcal{E}_1(t) &= \sup_{0\le\tau\le t} \|(u,b)(\tau)\|_{H^2}^2 + 2\int_0^t (\|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 u_2\|_{H^2}^2 + \|\partial_1 b_2\|_{H^2}^2) d\tau,\\ \mathcal{E}_2(t) &= \int_0^t \|\partial_2 b_1(\tau)\|_{H^1}^2 d\tau.\end{aligned}$$

The main goal of the proof is to establish the following estimate:

$$\mathcal{E}(t) \le C\mathcal{E}(0) + C\mathcal{E}^{3/2}(t). \tag{1.5}$$

The proof of (1.5) is not obvious and requires significant effort. We need to establish the following three inequalities respectively, and there exists a generic positive constant C,

$$\mathcal{E}_1(t) \le C\mathcal{E}_1(0) + C\mathcal{E}_1^{3/2}(t) + C\mathcal{E}_2^{3/2}(t), \tag{1.6}$$

$$\mathcal{E}_2(t) \le C\mathcal{E}_1(0) + C\mathcal{E}_1(t) + C\mathcal{E}_1^{3/2}(t) + C\mathcal{E}_2^{3/2}(t). \tag{1.7}$$

For any $t>0$, adding (1.7) to (1.6) by the appropriate constant, then we can yield the estimate of (1.5). The bootstrapping argument implies that if

$$\mathcal{E}(0) = \|(u_0, b_0)\|_{H^2}^2 \le \varepsilon^2,$$

for suitable $\varepsilon>0$, then $\mathcal{E}(t)$ remains uniformly bounded for $0<t<\infty$,

$$\mathcal{E}(t) \le C\varepsilon^2,$$

for some pure constant $C>0$. The more details are provided in Section 3.

Another main goal of this paper is to establish the large-time behavior for the following linear system

$$\begin{cases}\partial_t u_1 = \partial_2^2 u_1 + \partial_2 b_1 + (-\Delta)^{-1}\partial_1^2\partial_2^2 u_1 + (-\Delta)^{-1}\partial_2\partial_1^3 u_2,\\ \partial_t u_2 = \partial_1^2 u_2 + \partial_2 b_2 + (-\Delta)^{-1}\partial_1\partial_2^3 u_1 + (-\Delta)^{-1}\partial_1^2\partial_2^2 u_2,\\ \partial_t b_1 = \partial_2 u_1,\\ \partial_t b_2 = \partial_1^2 b_2 + \partial_2 u_2,\\ \nabla\cdot u = 0, \qquad \nabla\cdot b = 0.\end{cases} \tag{1.8}$$

The decay estimates can then be stated as follows.

Theorem 1.2. *Let $(u_0, b_0) \in \dot{H}^s(\mathbb{R}^2)$ satisfies $\nabla\cdot u_0 = \nabla\cdot b_0$, where $s \geq 0$ and (u, b) is the solution of (1.8). Let $(\nabla u_0, \nabla b_0) \in \dot{H}^s(\mathbb{R}^2)$. Then (u, b) satisfies,*

$$\|\nabla u(t)\|_{\dot{H}^s} + \|\nabla b(t)\|_{\dot{H}^s} \leq C(1+t)^{-1/2}. \tag{1.9}$$

Moreover, if $(\partial_1^2 u_{20}, \partial_2^2 u_{10}, \partial_1^2 b_{20}) \in \dot{H}^s(\mathbb{R}^2)$. Then we have

$$\|\partial_t u(t)\|_{\dot{H}^s} + \|\partial_t b(t)\|_{\dot{H}^s} \leq C(1+t)^{-1/2}. \tag{1.10}$$

The rest of this paper is divided into three sections. Section 2 presents several tool lemmas to be used in the proof of Theorem 1.1 and Theorem 1.2. The proofs of inequalities (1.6) and (1.7), and Theorem 1.1 are completed in Section 3. The last section demonstrates Theorem 1.2.

2 Preliminaries

In this section, we provide several lemmas that will be very important in subsequent proofs. Lemma 2.1 and Lemma 2.2 can help us treat the difficulty caused by the absence of dissipation. Lemma 2.3 and Lemma 2.4 contribute to obtaining some results on the large-time behavior.

Lemma 2.1. *(see [5]) Assume $f, g, h, \partial_1 g$ and $\partial_2 h$ all in $L^2(\mathbb{R}^2)$, it holds that*

$$\int |fgh|dx \leq C\|f\|_{L^2}\|g\|_{L^2}^{1/2}\|\partial_1 g\|_{L^2}^{1/2}\|h\|_{L^2}^{1/2}\|\partial_2 h\|_{L^2}^{1/2}.$$

Lemma 2.2. *(see [10]) The following estimates hold when the right-hand sides are all bounded in $\mathbb{R}^2$.*

$$\|f\|_{L^\infty} \leq C\|f\|_{L^2}^{1/4}\|\partial_1 f\|_{L^2}^{1/4}\|\partial_2 f\|_{L^2}^{1/4}\|\partial_1\partial_2 f\|_{L^2}^{1/4},$$

which implies that

$$\|f\|_{L^\infty} \leq C\|f\|_{H^1}^{1/2}\|\partial_1 f\|_{H^1}^{1/2},$$
$$\|f\|_{L^\infty} \leq C\|f\|_{H^1}^{1/2}\|\partial_2 f\|_{H^1}^{1/2}.$$

Lemma 2.3. *(see [13]) Let $f = f(t)$ with $t \in [0,\infty)$ be nonnegative continuous function. Assume that*

$$\int_0^\infty f(t)dt < \infty.$$

Suppose that for any $\rho > 0$, there is $\delta > 0$ such that, for any $0 \leq s < t$ with $t - s \leq \delta$,

$$\text{either} \quad f(t) \leq f(s) \quad \text{or} \quad f(t) > f(s) \quad \text{and} \quad f(t) - f(s) \leq \rho.$$

Then

$$f(t) \to 0, \quad \text{as} \quad t \to \infty.$$

Lemma 2.4. *(see [8]) For given Constants $C_0>0$ and $C_1>0$, assume that $f = f(t)$ is a non-negative continuous function satisfying*

$$\int_0^\infty f(\tau)d\tau \le C_0, \quad \text{and} f(t) \le C_1 f(s), \quad \text{for any} \quad 0 \le s<t.$$

Then there exists a positive constant $C_2 := \max\{2C_1 f(0), 4C_0C_1\}$ such that for any $t>0$,

$$f(t) \le C_2(1+t)^{-1}.$$

3 The global well-posedness

The main purpose of this section is to prove Theorem 1.1. In the following, we establish the validity of (1.6) and (1.7) respectively.

3.1 Proof of (1.6)

Proof. First, we take the L^2-inner product of (1.2) with (u, b) to obtain

$$\frac{1}{2}\frac{d}{dt}\|(u,b)\|_{L^2}^2 + \|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2 + \|\partial_1 b_2\|_{L^2}^2 = 0. \tag{3.1}$$

Next, to estimate the $\dot{H}^1$-norm, applying ∇ to (1.2) and dotting them with $(\nabla u, \nabla b)$ in L^2, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(\nabla u,\nabla b)\|_{L^2}^2 + \|\partial_1 \nabla u_2\|_{L^2}^2 + \|\partial_2 \nabla u_1\|_{L^2}^2 + \|\partial_1 \nabla b_2\|_{L^2}^2 \\ &= -\sum_{i=1}^{2}\int \partial_i u\cdot\nabla b\cdot\partial_i b dx + \sum_{i=1}^{2}\int \partial_i b\cdot\nabla u\cdot\partial_i b dx \\ &:= I_1 + I_2, \end{aligned} \tag{3.2}$$

where we used the significant fact that

$$\sum_{i=1}^{2}\int \partial_i(u\cdot\nabla u)\cdot\partial_i u dx = 0, \qquad \sum_{i=1}^{2}\int \partial_i b\cdot\nabla b\cdot\partial_i u dx = 0.$$

By integration by parts and $\nabla\cdot b = 0$,

$$\sum_{i=1}^{2}\int b\cdot\partial_i\nabla b\cdot\partial_i u dx + \sum_{i=1}^{2}\int b\cdot\partial_i\nabla u\cdot\partial_i b dx = 0.$$

Now, we bound I_1. By Hölder's inequality, we have

$$\begin{aligned} I_1 &= -\int \partial_1 u\cdot\nabla b\cdot\partial_1 b dx - \int \partial_2 u\cdot\nabla b\cdot\partial_2 b dx \\ &\le C\|\nabla u\|_{L^4}\|\nabla b\|_{L^2}\|\nabla b\|_{L^4} \\ &\le C\|b\|_{H^2}\|\nabla b\|_{L^2}\|\nabla u\|_{H^1} \\ &\le C\|b\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b_2\|_{H^1}^2 + \|\partial_2 b_1\|_{H^1}^2), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \|\nabla u\|_{H^2} &= \|\partial_1 u\|_{H^2} + \|\partial_2 u\|_{H^2} \\ &= \|\partial_1 u_1\|_{H^2} + \|\partial_1 u_2\|_{H^2} + \|\partial_2 u_1\|_{H^2} + \|\partial_2 u_2\|_{H^2} \\ &\le 2\|\partial_1 u_2\|_{H^2} + 2\|\partial_2 u_1\|_{H^2}, \end{aligned}$$

similarly,

$$\|\nabla b\|_{H^1} \leq 2\|\partial_1 b_2\|_{H^1} + 2\|\partial_2 b_1\|_{H^1}.$$

For the term I_2,

$$\begin{aligned} I_2 &= \int \partial_1 b \cdot \nabla u \cdot \partial_1 b dx + \int \partial_2 b \cdot \nabla u \cdot \partial_2 b dx \\ &\leq C\|\nabla u\|_{L^2}\|\nabla b\|_{L^4}^2 \\ &\leq C\|u\|_{H^2}(\|\partial_1 b_2\|_{H^1}^2 + \|\partial_2 b_1\|_{H^1}^2). \end{aligned} \tag{3.4}$$

Collecting the estimates in (3.3) and (3.4), we yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(\nabla u, \nabla b)\|_{L^2}^2 + \|\partial_1 \nabla u_2\|_{L^2}^2 + \|\partial_2 \nabla u_1\|_{L^2}^2 + \|\partial_1 \nabla b_2\|_{L^2}^2 \\ &\quad \leq C\|(u,b)\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b_2\|_{H^1}^2 + \|\partial_2 b_1\|_{H^1}^2). \end{aligned} \tag{3.5}$$

To estimate the $\dot{H}^2$-norm of (u,b), applying $\partial_i^2 (i=1,2)$ to (1.2) and dotting them with $(\partial_i^2 u, \partial_i^2 b)$ in L^2, one can obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{i=1}^2 \|(\partial_i^2 u, \partial_i^2 b)\|_{L^2}^2 + \sum_{i=1}^2 \|\partial_1 \partial_i^2 u_2\|_{L^2}^2 + \sum_{i=1}^2 \|\partial_2 \partial_i^2 u_1\|_{L^2}^2 + \sum_{i=1}^2 \|\partial_1 \partial_i^2 b_2\|_{L^2}^2 \\ &\quad = -\sum_{i=1}^2 \int \partial_i^2(u\cdot\nabla u)\cdot \partial_i^2 u dx + \sum_{i=1}^2 \int \partial_i^2 (b\cdot\nabla b)\cdot\partial_i^2 u dx \\ &\quad\quad - \sum_{i=1}^2 \int \partial_i^2(u\cdot\nabla b)\cdot\partial_i^2 b dx + \sum_{i=1}^2 \int \partial_i^2(b\cdot\nabla u)\cdot\partial_i^2 b dx \\ &\quad := H_1 + H_2 + H_3 + H_4. \end{aligned}$$

Due to Newton-Leibniz formula and the fact of $\nabla\cdot u = 0$, it follows

$$\begin{aligned} H_1 &= -\sum_{i=1}^2 \int \partial_i^2 u\cdot\nabla u\cdot\partial_i^2 u dx - 2\sum_{i=1}^2 \int \partial_i u\cdot\partial_i\nabla u\cdot\partial_i^2 u dx \\ &\leq C\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^4}^2 \\ &\leq C\|u\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2). \end{aligned} \tag{3.6}$$

Then, we estimate H_2 and H_4 together, by $\nabla\cdot b = 0$, one has

$$\begin{aligned} H_2 + H_4 &= \sum_{i=1}^2 \int \partial_i^2 b\cdot\nabla b\cdot\partial_i^2 u dx + 2\sum_{i=1}^2 \int \partial_i b\cdot\partial_i\nabla b\cdot\partial_i^2 u dx \\ &\quad + \sum_{i=1}^2 \int \partial_i^2 b\cdot\nabla u\cdot\partial_i^2 b dx + 2\sum_{i=1}^2 \int \partial_i b\cdot\partial_i\nabla u\cdot\partial_i^2 b dx \\ &:= H_{21} + H_{22} + H_{41} + H_{42}, \end{aligned}$$

where

$$\sum_{i=1}^2 \int b\cdot\partial_i^2\nabla b\cdot\partial_i^2 u dx + \sum_{i=1}^2 \int b\cdot\partial_i^2\nabla u\cdot\partial_i^2 b dx = 0. \tag{3.7}$$

By Hölder's inequality, we yield

$$\begin{aligned}H_{21}+H_{42}&=\int\partial_1^2b\cdot\nabla b\cdot\partial_1^2udx+\int\partial_2^2b\cdot\nabla b\cdot\partial_2^2udx\\&\quad+2\int\partial_1b\cdot\partial_1\nabla u\cdot\partial_1^2bdx+2\int\partial_2b\cdot\partial_2\nabla u\cdot\partial_2^2bdx\\&\le C\|\nabla^2b\|_{L^2}\|\nabla b\|_{L^4}\|\nabla^2u\|_{L^4}\\&\le C\|b\|_{H^2}(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2+\|\partial_1b_2\|_{H^2}^2+\|\partial_2b_1\|_{H^1}^2).\end{aligned}\tag{3.8}$$

Similarly,

$$\begin{aligned}H_{22}+H_{41}&=2\int\partial_1b\cdot\partial_1\nabla b\cdot\partial_1^2udx+2\int\partial_2b\cdot\partial_2\nabla b\cdot\partial_2^2udx\\&\quad+\int\partial_1^2b\cdot\nabla u\cdot\partial_1^2bdx+\int\partial_2^2b\cdot\nabla u\cdot\partial_2^2bdx\\&\le C\|\nabla b\|_{L^4}\|\nabla^2b\|_{L^2}\|\nabla^2u\|_{L^4}+C\|\nabla u\|_{L^\infty}\|\nabla^2b\|_{L^2}^2\\&\le C\|b\|_{H^2}(\|\nabla u\|_{H^2}^2+\|\nabla b\|_{H^1}^2)\\&\le C\|b\|_{H^2}(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2+\|\partial_1b_2\|_{H^2}^2+\|\partial_2b_1\|_{H^1}^2).\end{aligned}\tag{3.9}$$

Combining the estimates in (3.7)–(3.9), one immediately gets

$$H_2+H_4\le C\|b\|_{H^2}(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2+\|\partial_1b_2\|_{H^2}^2+\|\partial_2b_1\|_{H^1}^2).\tag{3.10}$$

Then, we estimate the term of H_3, by $\nabla\cdot b=0$, we get

$$\begin{aligned}H_3&=-\sum_{i=1}^2\int\partial_i^2u\cdot\nabla b\cdot\partial_i^2bdx-2\sum_{i=1}^2\partial_iu\cdot\partial_i\nabla b\cdot\partial_i^2bdx\\&\le C\|\nabla^2u\|_{L^4}\|\nabla b\|_{L^4}\|\nabla^2b\|_{L^2}+C\|\nabla u\|_{L^\infty}\|\nabla^2b\|_{L^2}^2\\&\le C\|b\|_{H^2}\|\nabla b\|_{H^1}\|\nabla u\|_{H^2}\\&\le C\|b\|_{H^2}(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2+\|\partial_1b_2\|_{H^2}^2+\|\partial_2b_1\|_{H^1}^2).\end{aligned}\tag{3.11}$$

Putting (3.6), (3.10) and (3.11) together, we yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^2\|(\partial_i^2u,\partial_i^2b)\|_{L^2}^2+\sum_{i=1}^2\|\partial_1\partial_i^2u_2\|_{L^2}^2+\sum_{i=1}^2\|\partial_i^2\partial_2\partial_i^2u_1\|_{L^2}^2+\sum_{i=1}^2\|\partial_1\partial_i^2b_2\|_{L^2}^2\\&\quad\le C\|(u,b)\|_{H^2}(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2+\|\partial_1b_2\|_{H^2}^2+\|\partial_2b_1\|_{H^1}^2).\end{aligned}\tag{3.12}$$

Combining (3.1), (3.5) with (3.12) and integrating it over $[0,t]$ yields

$$\begin{aligned}&\|(u,b)(t)\|_{H^2}^2+2\int_0^t(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2+\|\partial_1b_2\|_{H^2}^2)d\tau\\&\quad\le C\|(u_0,b_0)\|_{H^2}^2+C\sup_{0\le\tau\le t}\|(u,b)\|_{H^2}\int_0^t(\|\partial_1u_2\|_{H^2}^2+\|\partial_2u_1\|_{H^2}^2+\|\partial_1b_2\|_{H^2}^2\\&\qquad+\|\partial_2b_1\|_{H^1}^2)d\tau\\&\quad\le C\mathcal{E}_1(0)+C\mathcal{E}_1^{3/2}(t)+C\mathcal{E}_2^{3/2}(t).\end{aligned}$$

The proof of (1.6) is therefore complete. □

3.2 Proof of (1.7)

The dissipation of $\mathcal{E}_2(t)$ is generated by the background magnetic field. In order to establish the bound of $\mathcal{E}_2(t)$, we need the following special structure of equation $(1.2)_1$:

$$\partial_2 b_1 = \partial_t u_1 + u\cdot\nabla u_1 - b\cdot\nabla b_1 + \partial_1 p - \partial_2^2 u_1.$$

Proof. First, multiplying $(1.2)_1$ by $\partial_2 b_1$ and integrating over $\mathbb{R}^2$, it follows

$$\begin{aligned}\|\partial_2 b_1\|_{L^2}^2 &= \int \partial_2 b_1 \partial_t u_1 dx + \int \partial_2 b_1 (u\cdot\nabla u_1 - b\cdot\nabla b_1)dx + \int \partial_2 b_1 \partial_1 p dx - \int \partial_2 b_1 \partial_2^2 u_1 dx\\ &:= M_1 + M_2 + M_3 + M_4.\end{aligned}$$

By integration by parts and applying the magnetic equation in $(1.2)_2$,

$$\begin{aligned}M_1 &= \frac{d}{dt}\int \partial_2 b_1 u_1 dx - \int u_1 \partial_2(\partial_2 u_1 + b\cdot\nabla u_1 - u\cdot\nabla b_1)dx\\ &= \frac{d}{dt}\int \partial_2 b_1 u_1 dx + \int \partial_2 u_1(\partial_2 u_1 + b\cdot\nabla u_1 - u\cdot\nabla b_1)dx\\ &:= M_{11} + M_{12}.\end{aligned}$$

It is easily conclude that

$$\begin{aligned}M_{12} &= \int \partial_2 u_1(\partial_2 u_1 + b\cdot\nabla u_1 - u\cdot\nabla b_1)dx\\ &\le \|\partial_2 u_1\|_{L^2}^2 + C\|\partial_2 u_1\|_{L^2}(\|b\|_{L^\infty}\|\nabla u_1\|_{L^2} + \|u\|_{L^\infty}\|\nabla b_1\|_{L^2})\\ &\le \|\partial_2 u_1\|_{L^2}^2 + C\|(u,b)\|_{H^2}\|\partial_2 u_1\|_{L^2}(\|\nabla u_1\|_{L^2} + \|\nabla b_1\|_{L^2})\\ &\le \|\partial_2 u_1\|_{L^2}^2 + C\|(u,b)\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b_2\|_{H^2}^2 + \|\partial_2 b_1\|_{H^1}^2).\end{aligned}$$

Therefore,

$$\begin{aligned}M_1 \le{}& \frac{d}{dt}\int \partial_2 b_1 u_1 dx + \|\partial_2 u\|_{L^2}^2\\ &+ C\|(u,b)\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b_2\|_{H^2}^2 + \|\partial_2 b_1\|_{H^1}^2).\end{aligned} \tag{3.13}$$

By Hölder's inequality and Young's inequality, one has

$$\begin{aligned}M_2 &= \int \partial_2 b_1(u\cdot\nabla u_1 - b\cdot\nabla b_1)dx\\ &\le C\|\partial_2 b_1\|_{L^2}(\|u\|_{L^\infty} + \|b\|_{L^\infty})(\|\nabla u_1\|_{L^2} + \|\nabla b_1\|_{L^2})\\ &\le C\|(u,b)\|_{H^2}\|\partial_2 b_1\|_{L^2}(\|\nabla u_1\|_{L^2} + \|\nabla b_1\|_{L^2})\\ &\le C\|(u,b)\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b_2\|_{H^2}^2 + \|\partial_2 b_1\|_{H^1}^2).\end{aligned} \tag{3.14}$$

To bound M_4, by Young's inequality, we get

$$M_4 = -\int \partial_2 b_1 \partial_2^2 u_1 dx \le \frac{1}{4}\|\partial_2 b_1\|_{L^2}^2 + C\|\partial_2 u_1\|_{H^1}^2. \tag{3.15}$$

Now, we need to estimate M_3, applying $\nabla\cdot$ to $(1.2)_1$, one can obtain

$$p = (-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u) - (-\Delta)^{-1}\nabla\cdot(b\cdot\nabla b) - (-\Delta)^{-1}\partial_1\partial_2^2 u_1 - (-\Delta)^{-1}\partial_2\partial_1^2 u_2. \tag{3.16}$$

Due to Hölder's inequality,

$$M_3 = \int \partial_2 b_1 \partial_1 p dx \le C\|\partial_2 b_1\|_{L^2}\|\partial_1 p\|_{L^2}, \tag{3.17}$$

where

$$\begin{aligned}\|\partial_1 p\|_{L^2} \le & C\|(-\Delta)^{-1}\nabla\cdot\partial_1(u\cdot\nabla u)\|_{L^2} + C\|(-\Delta)^{-1}\nabla\cdot\partial_1(b\cdot\nabla b)\|_{L^2}\\ &+ C\|(-\Delta)^{-1}\partial_1^2\partial_2^2 u_1\|_{L^2} + C\|(-\Delta)^{-1}\partial_2\partial_1^3 u_2\|_{L^2}\\ &:= M_{31} + M_{32} + M_{33} + M_{34}.\end{aligned}$$

Using the fact of Riesz operator $\partial_i(-\Delta)^{-1/2}$ with $i = 1, 2$ is bounded in $L^r, 0<r<\infty$, one find

$$\begin{aligned}M_{31} + M_{32} &= C\|(-\Delta)^{-1}\nabla\cdot\partial_1(u\cdot\nabla u)\|_{L^2} + C\|(-\Delta)^{-1}\nabla\cdot\partial_1(b\cdot\nabla b)\|_{L^2}\\ &\le C\|u\cdot\nabla u\|_{L^2} + C\|b\cdot\nabla b\|_{L^2}\\ &\le C\|(u,b)\|_{H^2}(\|\nabla u\|_{H^1} + \|\nabla b\|_{H^1})\\ &\le C\|(u,b)\|_{H^2}(\|\partial_1 u_2\|_{H^1} + \|\partial_2 u_1\|_{H^1} + \|\partial_1 b_2\|_{H^1} + \|\partial_2 b_1\|_{H^1}).\end{aligned} \tag{3.18}$$

Obviously,

$$\begin{aligned}M_{33} + M_{34} &= C\|(-\Delta)^{-1}\partial_1^2\partial_2^2 u_1\|_{L^2} + C\|(-\Delta)^{-1}\partial_2\partial_1^3 u_2\|_{L^2}\\ &\le C\|\partial_2 u_1\|_{H^1} + C\|\partial_1 u_2\|_{H^1}.\end{aligned} \tag{3.19}$$

Combining the estimates in (3.18) and (3.19) together, we have

$$\begin{aligned}\|\partial_1 p\|_{L^2} \le & C\|\partial_2 u_1\|_{H^1} + C\|\partial_1 u_2\|_{H^1}\\ &+ C\|(u,b)\|_{H^2}(\|\partial_1 u_2\|_{H^1} + \|\partial_2 u_1\|_{H^1} + \|\partial_1 b_2\|_{H^1} + \|\partial_2 b_1\|_{H^1}).\end{aligned} \tag{3.20}$$

Putting (3.20) into (3.17), one can get

$$\begin{aligned}M_3 \le & \frac{1}{4}\|\partial_2 b_1\|_{L^2}^2 + C\|\partial_2 u_1\|_{H^1}^2 + C\|\partial_1 u_2\|_{H^1}^2\\ &+ C\|(u,b)\|_{H^2}(\|\partial_1 u_2\|_{H^1}^2 + \|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 b_2\|_{H^1}^2 + \|\partial_2 b_1\|_{H^1}^2).\end{aligned} \tag{3.21}$$

Combining the estimates (3.13), (3.14), (3.15) and (3.21) respectively, it follows

$$\begin{aligned}\|\partial_2 b_1\|_{L^2}^2 \le & 2\frac{d}{dt}\int \partial_2 b_1 u_1 dx + C\|\partial_2 u_1\|_{H^1}^2 + C\|\partial_1 u_2\|_{H^1}^2\\ &+ C\|(u,b)\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b_2\|_{H^2}^2 + \|\partial_2 b_1\|_{H^1}^2).\end{aligned} \tag{3.22}$$

Next, applying $\partial_i (i = 1, 2)$ to $(1.2)_1$ and dotting it with $\partial_2\partial_i b_1$ in L^2, we can deduce that

$$\begin{aligned}\sum_{i=1}^{2}\|\partial_2\partial_i b_1\|_{L^2}^2 &= \sum_{i=1}^{2}\int \partial_2\partial_i b_1 \partial_i(\partial_t u_1 + u\cdot\nabla u_1 - b\cdot\nabla b_1 + \partial_1 p - \partial_2^2 u_1)dx\\ &= \frac{d}{dt}\sum_{i=1}^{2}\int \partial_2\partial_i b_1\partial_i u_1 dx - \sum_{i=1}^{2}\int \partial_2\partial_i\partial_t b_1\partial_i u_1 dx\\ &\quad + \sum_{i=1}^{2}\int \partial_2\partial_i b_1 \partial_i(u\cdot\nabla u_1 - b\cdot\nabla b_1 + \partial_1 p - \partial_2^2 u_1)dx\\ &:= Q_0 + Q_1 + Q_2 + Q_3 + Q_4 + Q_5.\end{aligned} \tag{3.23}$$

Applying the struture of equation $(1.2)_2$ and integration by parts,

$$\begin{aligned} Q_1 &= \sum_{i=1}^{2} \int \partial_2 \partial_i u_1 \partial_i (\partial_2 u_1 + b \cdot \nabla u_1 - u \cdot \nabla b_1) dx \\ &:= Q_{11} + Q_{12} + Q_{13}, \end{aligned}$$

where

$$\begin{aligned} Q_{12} &= \sum_{i=1}^{2} \int \partial_2 \partial_i u_1 (\partial_i b \cdot \nabla u_1 + b \cdot \partial_i \nabla u_1) dx \\ &\leq C \|\partial_2 \nabla u\|_{L^4} (\|\nabla b\|_{L^2} \|\nabla u_1\|_{L^4} + \|b\|_{L^4} \|\nabla^2 u_1\|_{L^2}) \\ &\leq C \|b\|_{H^2} \|\nabla u\|_{H^2}^2 \\ &\leq C \|b\|_{H^2} (\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2). \end{aligned}$$

Similarly,

$$\begin{aligned} Q_{13} &= -\sum_{i=1}^{2} \int \partial_2 \partial_i u_1 (\partial_i u \cdot \nabla b_1 + u \cdot \partial_i \nabla b_1) dx \\ &\leq \|\partial_2 \nabla u_1\|_{L^4} (\|\nabla u\|_{L^2} \|\nabla b_1\|_{L^4} + \|u\|_{L^4} \|\nabla^2 b_1\|_{L^2}) \\ &\leq C \|u\|_{H^2} \|\partial_2 \nabla u_1\|_{H^1} \|\nabla b\|_{H^1} \\ &\leq C \|u\|_{H^2} (\|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b_2\|_{H^1}^2 + \|\partial_2 b_1\|_{H^1}^2). \end{aligned}$$

Combining the estimates for Q_{12} and Q_{13} respectively, we get

$$Q_1 \leq \|\partial_2 \nabla u_1\|_{L^2}^2 + C \|(u,b)\|_{H^2} (\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b_2\|_{H^1}^2 + \|\partial_2 b_1\|_{H^1}^2). \tag{3.24}$$

Using Hölder's inequality and Young's inequality to get

$$\begin{aligned} Q_2 &= \sum_{i=1}^{2} \int \partial_2 \partial_i b_1 (\partial_i u \cdot \nabla u_1 + u \cdot \partial_i \nabla u_1) dx \\ &\leq C \|\partial_2 \nabla b_1\|_{L^2} (\|\nabla u\|_{L^4} \|\nabla u_1\|_{L^4} + \|u\|_{L^\infty} \|\nabla^2 u_1\|_{L^2}) \\ &\leq C \|u\|_{H^2} \|\partial_2 \nabla b_1\|_{L^2} \|\nabla u_1\|_{H^1} \\ &\leq C \|u\|_{H^2} (\|\partial_1 u_2\|_{H^1}^2 + \|\partial_2 u_1\|_{H^1}^2 + \|\partial_2 b_1\|_{H^1}^2). \end{aligned} \tag{3.25}$$

Similar as Q_2,

$$\begin{aligned} Q_3 &= -\sum_{i=1}^{2} \int \partial_2 \partial_i b_1 (\partial_i b \cdot \nabla b_1 + b \cdot \partial_i \nabla b_1) dx \\ &\leq C \|\partial_2 \nabla b_1\|_{L^2} (\|\nabla b\|_{L^4}^2 + \|b\|_{L^\infty} \|\nabla^2 b_1\|_{L^2}) \\ &\leq C \|b\|_{H^2} (\|\partial_1 b_2\|_{H^1}^2 + \|\partial_2 b_1\|_{H^1}^2). \end{aligned} \tag{3.26}$$

By Young's inequality, one find

$$Q_5 = -\sum_{i=1}^{2} \int \partial_2 \partial_i b_1 \partial_2^2 \partial_i u_1 dx \leq \frac{1}{4} \|\partial_2 \nabla b_1\|_{L^2}^2 + C \|\partial_2 u_1\|_{H^2}^2. \tag{3.27}$$

Now, we need to establish the estimate of Q_4.

$$Q_4 = \sum_{i=1}^{2} \int \partial_2 \partial_i b_1 \partial_1 \partial_i p dx \leq C \sum_{i=1}^{2} \|\partial_2 \partial_i b_1\|_{L^2} \|\partial_1 \partial_i p\|_{L^2}.$$

Thanks to (3.16),

$$\begin{aligned}\|\partial_1\partial_i p\|_{L^2} \le C\|(-\Delta)^{-1}\nabla\cdot\partial_1\partial_i(u\cdot\nabla u)\|_{L^2} + C\|(-\Delta)^{-1}\nabla\cdot\partial_1\partial_i(b\cdot\nabla b)\|_{L^2}\\ + C\|(-\Delta)^{-1}\partial_1^2\partial_2^2\partial_i u_1\|_{L^2} + C\|(-\Delta)^{-1}\partial_2\partial_1^3\partial_i u_2\|_{L^2}.\end{aligned}$$

Due to the boundness of Riesz operator $\partial_i(-\Delta)^{-1/2}$ with $i=1,2$ in L^2, then

$$\begin{aligned}\|(-\Delta)^{-1}\nabla\cdot\partial_1\partial_i(u\cdot\nabla u)\|_{L^2} &\le C\|(\nabla u\cdot\nabla u + u\cdot\nabla^2 u)\|_{L^2}\\ &\le C\|\nabla u\|_{L^4}^2 + C\|u\|_{L^\infty}\|\nabla^2 u\|_{L^2}\\ &\le C\|u\|_{H^2}\|\nabla u\|_{H^1}\\ &\le C\|u\|_{H^2}(\|\partial_1 u_2\|_{H^1} + \|\partial_2 u_1\|_{H^1}).\end{aligned}$$

In a similar manner,

$$\begin{aligned}\|(-\Delta)^{-1}\nabla\cdot\partial_1\partial_i(b\cdot\nabla b)\|_{L^2} &\le C\|b\|_{H^2}\|\nabla b\|_{H^1}\\ &\le C\|b\|_{H^2}(\|\partial_1 b_2\|_{H^1} + \|\partial_2 b_1\|_{H^1}).\end{aligned}$$

Obviously,

$$\|(-\Delta)^{-1}\partial_1^2\partial_2^2\partial_i u_1\|_{L^2} + \|(-\Delta)^{-1}\partial_2\partial_1^3\partial_i u_2\|_{L^2} \le C\|\partial_2 u_1\|_{H^2} + C\|\partial_1 u_2\|_{H^2}.$$

Therefore,

$$\begin{aligned}Q_4 \le \frac{1}{4}\|\partial_2\nabla b_1\|_{L^2}^2 + C\|\partial_2 u_1\|_{H^2}^2 + C\|\partial_1 u_2\|_{H^2}^2\\ + C\|(u,b)\|_{H^2}(\|\partial_1 u_2\|_{H^1}^2 + \|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 b_2\|_{H^1}^2 + \|\partial_2 b_1\|_{H^1}^2).\end{aligned} \tag{3.28}$$

Putting (3.24)–(3.28) into (3.23), we can decude

$$\begin{aligned}\|\partial_2\nabla b_1\|_{L^2}^2 \le 2\frac{d}{dt}\sum_{i=1}^{2}\int \partial_2\partial_i b_1\partial_i u_1 dx + C\|\partial_1 u_2\|_{H^2}^2 + C\|\partial_2 u_1\|_{H^2}^2\\ + C\|(u,b)\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2 + \|\partial_1 b_2\|_{H^2}^2 + \|\partial_2 b_1\|_{H^1}^2).\end{aligned} \tag{3.29}$$

Adding (3.22) and (3.29) together, one derives

$$\begin{aligned}\|\partial_2 b_1\|_{H^1}^2 \le 2\frac{d}{dt}\int \partial_2 b_1 u_1 dx + 2\frac{d}{dt}\sum_{i=1}^{2}\int \partial_2\partial_i b_1\partial_i u_1 dx + C\|\partial_1 u_2\|_{H^2}^2\\ + C\|\partial_2 u_1\|_{H^2}^2 + C\|(u,b)\|_{H^2}(\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{L^2}^2 + \|\partial_1 b_2\|_{H^2}^2 + \|\partial_2 b_1\|_{H^1}^2).\end{aligned}$$

Integrating it over $[0,t]$, it follows

$$\begin{aligned}\int_0^t \|\partial_2 b_1\|_{H^1}^2 d\tau &\le C\|(u_0,b_0)\|_{H^2}^2 + C\|(u,b)\|_{H^2}^2 + C\int_0^t (\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{H^2}^2)d\tau\\ &\quad + C\sup_{0\le\tau\le t}\|(u,b)\|_{H^2}\int_0^t (\|\partial_1 u_2\|_{H^2}^2 + \|\partial_2 u_1\|_{L^2}^2 + \|\partial_1 b_2\|_{H^2}^2 + \|\partial_2 b_1\|_{H^1}^2)d\tau\\ &\le C\mathcal{E}_1(0) + C\mathcal{E}_1(t) + C\mathcal{E}_1^{3/2}(t) + C\mathcal{E}_2^{3/2}(t),\end{aligned}$$

which implies (1.7). □

3.3 Proof of Theorem 1.1

This subsection completes the proof of Theorem 1.1, which can be achieved by the bootstrapping argument [17]. As we know, the local well-posedness of (1.2) in $\mathbb{R}^2$ can be established via a standard procedure. We only need to establish the global bounds and then apply the bootstrapping argument to obtain the desired stability result in Theorem 1.1. The key components are the following energy inequalities established previously in subsections 3.1 and 3.2.

$$\mathcal{E}_1(t) \leq C\mathcal{E}_1(0) + C\mathcal{E}_1^{3/2}(t) + C\mathcal{E}_2^{3/2}(t), \tag{3.30}$$

$$\mathcal{E}_2(t) \leq C\mathcal{E}_1(0) + C\mathcal{E}_1(t) + C\mathcal{E}_1^{3/2}(t) + C\mathcal{E}_2^{3/2}(t). \tag{3.31}$$

Proof. For any $t>0$, adding (3.31) to (3.30) by the appropriate constant yields,

$$\mathcal{E}(t) \leq C_0\mathcal{E}(0) + C_0\mathcal{E}^{3/2}(t). \tag{3.32}$$

where $\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t)$, and $C_0>0$ is a pure constant. We take

$$\|(u_0, b_0)\|_{H^2}^2 \leq \frac{1}{16C_0^3}.$$

The bootstrapping argument starts with the ansatz that

$$\mathcal{E}(t) \leq \frac{1}{4C_0^2}.$$

It follows from (3.32) that

$$\mathcal{E}(t) \leq C_0\mathcal{E}(0) + C_0\mathcal{E}^{1/2}(t)\mathcal{E}(t) \leq C_0\mathcal{E}(0) + C_0\frac{1}{2C_0}\mathcal{E}(t) = C_0\mathcal{E}(0) + \frac{1}{2}\mathcal{E}(t),$$

then,

$$\mathcal{E}(t) \leq 2C_0\mathcal{E}(0).$$

The bootstrapping argument then implies that, for any $t \geq 0$,

$$\mathcal{E}(t) \leq \frac{1}{8C_0^2}.$$

This completes the proof of (1.3).

Now, we establish the decay results in (1.4). First of all, we establish the large-time behavior of $(\nabla u, \nabla b)$, based on $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$, we rewrite (3.2) as,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla u, \nabla b)\|_{L^2}^2 + \|\partial_1 \nabla u_2\|_{L^2}^2 + \|\partial_2 \nabla u_1\|_{L^2}^2 + \|\partial_1 \nabla b_2\|_{L^2}^2 \\
&\quad = -\int \nabla u \cdot \nabla b \cdot \nabla b dx + \int \nabla b \cdot \nabla u \cdot \nabla b dx \\
&\quad \leq C\|\nabla u\|_{L^2}\|\nabla b\|_{L^4}^2 \leq C\|u\|_{H^2}\|b\|_{H^2}^2.
\end{aligned}$$

For any $0 \leq s \leq t<\infty$, integrating it in time, by the upper bound in (1.3), that is

$$\|(\nabla u, \nabla b)(t)\|_{L^2}^2 - \|(\nabla u, \nabla b)(s)\|_{L^2}^2 \leq C\varepsilon^3(t-s).$$

Due to

$$\|(\nabla u, \nabla b)\|_{L^2} \leq C(\|\partial_1 u_2\|_{L^2} + \|\partial_2 u_1\|_{L^2}) + C(\|\partial_1 b_2\|_{L^2} + \|\partial_2 b_1\|_{L^2}).$$

Invoking (1.3), we then infer

$$\int_0^\infty \|(\nabla u, \nabla b)\|_{L^2}^2 dt \leq \infty,$$

then using Lemma 2.3 to obtain

$$\|(\nabla u, \nabla b)(t)\|_{L^2}^2 \to 0, \qquad as \quad t \to \infty.$$

Next, we need to establish the large-time behavior of $(\nabla^2 u, \nabla^2 b)$. Applying ∇^2 to (1.2) and dotting them with $(\nabla^2 u, \nabla^2 b)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla^2 u, \nabla^2 b)\|_{L^2}^2 + \|\partial_1 \nabla^2 u_2\|_{L^2}^2 + \|\partial_2 \nabla^2 u_1\|_{L^2}^2 + \|\partial_1 \nabla^2 b_2\|_{L^2}^2 \\
&\quad = -\int \nabla^2(u\cdot\nabla u)\cdot\nabla^2 u dx + \int \nabla^2(b\cdot\nabla b)\cdot\nabla^2 u dx \\
&\quad\quad - \int \nabla^2(u\cdot\nabla b)\cdot\nabla^2 b dx + \int \nabla^2(b\cdot\nabla u)\cdot\nabla^2 b dx \\
&\quad := F_1 + F_2 + F_3 + F_4.
\end{aligned} \tag{3.33}$$

By $\nabla\cdot u = 0$ and Lemma 2.1, we conclude

$$\begin{aligned}
F_1 &= -\int (\nabla^2 u\cdot\nabla u\cdot\nabla^2 u + 2\nabla u\cdot\nabla^2 u\cdot\nabla^2 u)dx \\
&= -\int \nabla^2 u\cdot\nabla u_1 \nabla^2 u_1 dx - \int \nabla^2 u\cdot\nabla u_2\nabla^2 u_2 dx \\
&\quad - 2\int \nabla u\cdot\nabla^2 u_1\nabla^2 u_1 dx - 2\int \nabla u\cdot\nabla^2 u_2\nabla^2 u_2 dx \\
&\leq C\|\nabla^2 u\|_{L^2}\|\nabla u_1\|_{L^2}^{1/2}\|\partial_1\nabla u_1\|_{L^2}^{1/2}\|\nabla^2 u_1\|_{L^2}^{1/2}\|\partial_2\nabla^2 u_1\|_{L^2}^{1/2} \\
&\quad + C\|\nabla^2 u\|_{L^2}\|\nabla u_2\|_{L^2}^{1/2}\|\partial_2\nabla u_2\|_{L^2}^{1/2}\|\nabla^2 u_2\|_{L^2}^{1/2}\|\partial_1\nabla^2 u_2\|_{L^2}^{1/2} \\
&\quad + C\|\nabla^2 u_1\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\partial_1\nabla u\|_{L^2}^{1/2}\|\nabla^2 u_1\|_{L^2}^{1/2}\|\partial_2\nabla^2 u_1\|_{L^2}^{1/2} \\
&\quad + C\|\nabla^2 u_2\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\partial_2\nabla u\|_{L^2}^{1/2}\|\nabla^2 u_2\|_{L^2}^{1/2}\|\partial_1\nabla^2 u_2\|_{L^2}^{1/2} \\
&\leq C\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^2(\|\partial_1\nabla^2 u_2\|_{L^2}^{1/2} + \|\partial_2\nabla^2 u_1\|_{L^2}^{1/2}) \\
&\leq C(\overline{\delta})\|u\|_{H^2}^{10/3} + \overline{\delta}\|\partial_1\nabla^2 u_2\|_{L^2}^2 + \overline{\delta}\|\partial_2\nabla^2 u_1\|_{L^2}^2.
\end{aligned} \tag{3.34}$$

Similarly, by $\nabla\cdot b = 0$, it follows

$$\begin{aligned}
F_3 &= -\int (\nabla^2 u\cdot\nabla b\cdot\nabla^2 b + 2\nabla u\cdot\nabla^2 b\cdot\nabla^2 b)dx \\
&= -\int \nabla^2 u_1\partial_1 b_1\nabla^2 b_1 dx - \int \nabla^2 u_2\partial_2 b_2\nabla^2 b_1 dx - \int \nabla^2 u\cdot\nabla b_2\nabla^2 b_2 dx \\
&\leq C\|\nabla^2 b_1\|\nabla^2 u_1\|_{L^2}^{1/2}\|\partial_2\nabla^2 u_1\|_{L^2}^{1/2}\|\partial_1 b_1\|_{L^2}^{1/2}\|\partial_1^2 b_1\|_{L^2}^{1/2} \\
&\quad + C\|\nabla^2 b_1\|_{L^2}\|\nabla^2 u_2\|_{L^2}^{1/2}\|\partial_1\nabla^2 u_2\|_{L^2}^{1/2}\|\partial_2 b_2\|_{L^2}^{1/2}\|\partial_2^2 b\|_{L^2}^{1/2} \\
&\quad + C\|\nabla^2 u\|_{L^2}\|\nabla b_2\|_{L^2}^{1/2}\|\partial_2\nabla b_2\|_{L^2}^{1/2}\|\nabla^2 b_2\|_{L^2}^{1/2}\|\partial_1\nabla^2 b_2\|_{L^2}^{1/2} \\
&\leq C\|b\|_{H^2}^2\|u\|_{H^2}^{1/2}(\|\partial_1\nabla^2 u_2\|_{L^2}^{1/2} + \|\partial_2\nabla^2 u_1\|_{L^2}^{1/2}) + C\|u\|_{H^2}\|b\|_{H^2}^{3/2}\|\partial_1\nabla^2 b_2\|_{L^2}^{1/2} \\
&\leq C(\overline{\delta})(\|u\|_{H^2}^{2/3}\|b\|_{H^2}^{8/3} + \|u\|_{H^2}^{4/3}\|b\|_{H^2}^2) + \overline{\delta}(\|\partial_1\nabla^2 u_2\|_{L^2}^2 + \|\partial_2\nabla^2 u_1\|_{L^2}^2 + \|\partial_1\nabla^2 b_2\|_{L^2}^2).
\end{aligned} \tag{3.35}$$

By integration by parts and $\nabla \cdot b = 0$, it is easily seen that

$$
\begin{aligned}
F_2 + F_4 &= \int \nabla^2 b \cdot \nabla b \cdot \nabla^2 u dx + 2\int \nabla b \cdot \nabla^2 b \cdot \nabla^2 u dx \\
&\quad + \int \nabla^2 b \cdot \nabla u \cdot \nabla^2 b dx + 2\int \nabla b \cdot \nabla^2 u \cdot \nabla^2 b dx \\
&:= F_{21} + F_{22} + F_{41} + F_{44}.
\end{aligned}
$$

Similar to F_3, by Hölder's inequality and Young's inequality,

$$
\begin{aligned}
F_{21} + F_{22} &= \int \nabla^2 b \cdot \nabla b_1 \nabla^2 u_1 dx + \int \nabla^2 b \cdot \nabla b_2 \nabla^2 u_2 dx \\
&\quad + 2\int \nabla b \cdot \nabla^2 b_1 \nabla^2 u_1 dx + 2\int \nabla b \cdot \nabla^2 b_2 \nabla^2 u_2 dx \\
&\leq C\|u\|_{H^2}^{1/2}\|b\|_{H^2}^2(\|\partial_2\nabla^1 u_2\|_{L^2}^{1/2} + \|\partial_2\nabla^2 u_1\|_{L^2}^{1/2}) \\
&\leq C(\overline{\delta})\|u\|_{H^2}^{2/3}\|b\|_{H^2}^{8/3} + \overline{\delta}(\|\partial_2\nabla^1 u_2\|_{L^2}^2 + \|\partial_2\nabla^2 u_1\|_{L^2}^2).
\end{aligned}
$$

By Lemma 2.2,

$$
\begin{aligned}
F_{41} + F_{42} &= \int \nabla^2 b_1 \partial_1 u_1 \nabla^2 b_1 dx + \int \nabla^2 b_2 \partial_2 u_1 \nabla^2 b_1 dx + \int \nabla^2 b \cdot \nabla u_2 \nabla^2 b_2 dx \\
&\quad + 2\int \nabla b \cdot \nabla^2 u_1 \nabla^2 b_1 dx + 2\int \nabla b \cdot \nabla^2 u_2 \nabla^2 b_2 dx \\
&\leq C\|\partial_2 u_2\|_{L^2}^{1/4}\|\partial_1\partial_2 u_2\|_{L^2}^{1/4}\|\partial_2^2 u_2\|_{L^2}^{1/4}\|\partial_2^2\partial_1 u_2\|_{L^2}^{1/4}\|\nabla^2 b_1\|_{L^2}^2 \\
&\quad + C\|\nabla^2 b\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\partial_1\nabla u\|_{L^2}^{1/2}\|\nabla^2 b_2\|_{L^2}^{1/2}\|\partial_1\nabla^2 b_2\|_{L^2}^{1/2} \\
&\quad + C\|\nabla^2 b\|_{L^2}\|\nabla b\|_{L^2}^{1/2}\|\nabla^2 b\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2}(\|\partial_1\nabla^2 u_2\|_{L^2}^{1/2} + \|\partial_2\nabla^2 u_1\|_{L^2}^{1/2}) \\
&\leq C\|u\|_{H^2}^{3/4}\|b\|_{H^2}^2\|\partial_1\nabla^2 u_2\|_{L^2}^{1/4} + C\|u\|_{H^2}\|b\|_{H^2}^{3/2}\|\partial_1\nabla^2 b_2\|_{L^2}^{1/2} \\
&\quad + C\|u\|_{H^2}^{1/2}\|b\|_{H^2}^2(\|\partial_1\nabla^2 u_2\|_{L^2}^{1/2} + \|\partial_2\nabla^2 u_1\|_{L^2}^{1/2}) \\
&\leq C(\overline{\delta})(\|u\|_{H^2}^{6/7}\|b\|_{H^2}^{16/7} + \|u\|_{H^2}^{4/3}\|b\|_{H^2}^2 + \|u\|_{H^2}^{2/3}\|b\|_{H^2}^{8/3}) + \overline{\delta}\|\partial_1\nabla^2 u_2\|_{L^2}^2 \\
&\quad + \overline{\delta}\|\partial_2\nabla^2 u_1\|_{L^2}^2 + \overline{\delta}\|\partial_1\nabla^2 b_2\|_{L^2}^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
F_2 + F_4 &\leq C(\overline{\delta})(\|u\|_{H^2}^{6/7}\|b\|_{H^2}^{16/7} + \|u\|_{H^2}^{4/3}\|b\|_{H^2}^2 + \|u\|_{H^2}^{2/3}\|b\|_{H^2}^{8/3}) + \overline{\delta}\|\partial_1\nabla^2 u_2\|_{L^2}^2 \\
&\quad + \overline{\delta}\|\partial_2\nabla^2 u_1\|_{L^2}^2 + \overline{\delta}\|\partial_1\nabla^2 b_2\|_{L^2}^2.
\end{aligned}
\tag{3.36}
$$

Choosing $\overline{\delta} = \frac{1}{6}$ and inserting the estimates in (3.34), (3.35) and (3.36) into (3.33) leads to

$$
\begin{aligned}
&\frac{d}{dt}\|(\nabla^2 u, \nabla^2 b)\|_{L^2}^2 + \|\partial_1\nabla^2 u_2\|_{L^2}^2 + \|\partial_2\nabla^2 u_1\|_{L^2}^2 + \|\partial_1\nabla^2 b_2\|_{L^2}^2 \\
&\qquad \leq C(\|u\|_{H^2}^{10/3} + \|u\|_{H^2}^{2/3}\|b\|_{H^2}^{8/3} + \|u\|_{H^2}^{4/3}\|b\|_{H^2}^2 + \|u\|_{H^2}^{6/7}\|b\|_{H^2}^{16/7}).
\end{aligned}
\tag{3.37}
$$

Foe any $0 \leq s<t$, integrating (3.37) in time, by the upper bound in (1.3), that is

$$
\|(\nabla^2 u, \nabla^2 b)(t)\|_{L^2}^2 - \|(\nabla^2 u, \nabla^2 b)(s)\|_{L^2}^2 \leq C(\varepsilon^{10/3} + \varepsilon^{22/7})(t-s).
$$

Due to

$$
\|(\nabla^2 u, \nabla^2 b)\|_{L^2}^2 \leq C(\|\partial_1\nabla u_2\|_{L^2} + \|\partial_2\nabla u_1\|_{L^2}) + C(\|\partial_1\nabla b_2\|_{L^2} + \|\partial_2\nabla b_1\|_{L^2}),
$$

and (1.3) implies that

$$\int_0^\infty \|(\nabla^2 u, \nabla^2 b)\|_{L^2}^2 dt<\infty.$$

Then, applying Lemma 2.3, we derive

$$\|(\nabla^2 u, \nabla^2 b)\|_{L^2}^2 \to 0, \qquad as \quad t\to\infty.$$

Therefore, we finish the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

This section aims to establish the proof of Theorem 1.2. It main relies on Lemma 2.4.

Proof. We first prove (1.9) in Theorem 1.2. Taking the H^1-inner product of (1.8), we have

$$\|(u,b)(t)\|_{H^1}^2 + 2\int_0^t (\|\partial_1 u_2\|_{H^1}^2 + \|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 b_2\|_{H^1}^2) ds = \|(u_0,b_0)\|_{H^1}^2. \tag{4.1}$$

By integration by parts and Hölder's inequality,

$$\|\partial_1 u_1\|_{L^2}^2 = -\int \partial_1 u_2 \partial_2 u_1 dx \le \frac{1}{2}(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2),$$

then we yield

$$\|\nabla u\|_{L^2}^2 \le 2(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_2 u_1\|_{L^2}^2).$$

Thanks to (1.3), we can conclude that

$$\int_0^\infty \|\nabla u(t)\|_{L^2}^2 dt \le C, \qquad \int_0^\infty \|\partial_1 b_2(t)\|_{L^2}^2 dt \le C.$$

In order to obtain the integrability for $\|\nabla b\|_{L^2}^2$ on the time. We need to first establish the estimation of $\|\partial_2 b_1\|_{L^2}^2$. Dotting $(1.8)_1$ by $\partial_2 b_1$ and integrate over $\mathbb{R}^2$, we obtain

$$\begin{aligned}
\|\partial_2 b_1\|_{L^2}^2 &= \int \partial_t u_1 \partial_2 b_1 dx - \int \partial_2^2 u_1 \partial_2 b_1 dx \\
&\quad + \int (-\Delta)^{-1}\partial_1^2\partial_2^2 u_1 \partial_2 b_1 dx \int (-\Delta)^{-1}\partial_1^3\partial_2 u_2 \partial_2 b_1 dx \\
&= \frac{d}{dt}\int u_1\partial_2 b_1 dx - \int u_1 \partial_2\partial_t b_1 dx - \int \partial_2^2 u_1 \partial_2 b_1 dx \\
&\quad + \int (-\Delta)^{-1}\partial_1^2\partial_2^2 u_1 \partial_2 b_1 dx \int (-\Delta)^{-1}\partial_1^3\partial_2 u_2 \partial_2 b_1 dx \\
&:= K_0 + K_1 + K_2 + K_3 + K_4.
\end{aligned}$$

By integration by parts, Hölder's inequality and $(1.8)_3$,

$$K_1 = -\int u_1 \partial_2 \partial_t b_1 dx = -\int u_1 \partial_2^2 u_1 dx = \|\partial_2 u_1\|_{L^2}^2.$$

Due to the Riesz operator $\partial_i(-\Delta)^{-1/2} (i=1,2)$ is bounded in L^2, we have

$$\begin{aligned}
K_2 + K_3 + K_4 &\le C\|\partial_2 b_1\|_{L^2}(\|\partial_2^2 u_1\|_{L^2} + \|\partial_1^2 u_2\|_{L^2}) \\
&\le \frac{1}{2}\|\partial_2 b_1\|_{L^2}^2 + C\|\partial_2^2 u_1\|_{L^2}^2 + C\|\partial_1^2 u_2\|_{L^2}^2.
\end{aligned}$$

Therefore,

$$\|\partial_2 b_1\|_{L^2}^2 \le 2\frac{d}{dt}\int u_1\partial_2 b_1 dx + C\|\partial_2 u_1\|_{H^1}^2 + C\|\partial_1 u_2\|_{H^1}^2.$$

Integrating it over $[0,t]$, it follows

$$\int_0^t \|\partial_2 b_1\|_{L^2}^2 d\tau \le C\|(u_0,b_0)\|_{H^1}^2 + C\|(u,b)\|_{H^1}^2 + C\int_0^t (\|\partial_2 u_1\|_{H^1}^2 + \|\partial_1 u_2\|_{H^1}^2) d\tau. \tag{4.2}$$

Adding (4.2) to (4.1) by the appropriate constant, we get

$$\begin{aligned}&\|(u,b)\|_{H^1}^2 + C\int_0^t (\|\partial_1 u_2\|_{H^1}^2 + \|\partial_2 u_1\|_{H^1}^2) d\tau + \int_0^t (\|\partial_1 b_2\|_{H^1}^2 + \|\partial_2 b_1\|_{L^2}^2) d\tau \\ &\quad \le C\|(u_0,b_0)\|_{H^1}^2.\end{aligned}$$

It is easily seen that

$$\int_0^\infty (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) dt \le C.$$

Applying ∇ to (1.8) and dotting them with $(\nabla u, \nabla b)$, one can obtain

$$\frac{d}{dt}\|(\nabla u,\nabla b)(t)\|_{L^2}^2 + 2(\|\partial_1\nabla u_2(t)\|_{L^2}^2 + \|\partial_2\nabla u_1(t)\|_{L^2}^2) + 2\|\partial_1\nabla b_2\|_{L^2}^2 = 0.$$

Therefore, by Lemma 2.4, we know that

$$\|(\nabla u,\nabla b)\|_{L^2}^2 \le C(1+t)^{-1/2}.$$

Next, we prove the decay rate in (1.10). Applying ∂_t to (1.8) and multiplying them with $(\partial_t u, \partial_t b)$, we obtain

$$\frac{d}{dt}(\|\partial_t u(t)\|_{L^2}^2 + \|\partial_t b(t)\|_{L^2}^2) \le 0.$$

Taking the L^2-norm of the velocity and magnetic field equations in (1.8), and integrating them on $[0,t]$ for any $t \ge 0$, we can infer

$$\int_0^t \|\partial_t u(\tau)\|_{L^2}^2 d\tau \le \int_0^t \|\partial_1^2 u_2(\tau)\|_{L^2}^2 d\tau + \int \|\partial_2^2 u_1(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\partial_2 b(\tau)\|_{L^2}^2 d\tau,$$

and

$$\int_0^t \|\partial_t b(\tau)\|_{L^2}^2 d\tau \le \int_0^t \|\partial_1^2 b_2(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\partial_2 u(\tau)\|_{L^2}^2 d\tau.$$

Through (1.3) and Lemma 2.4, we derive

$$\|\partial_t u\|_{L^2}^2 + \|\partial_t b\|_{L^2}^2 \le C(1+t)^{-1/2}.$$

Similarly, applying ∇^s to (1.8),

$$\begin{cases}\partial_t\widetilde{U_1} = \partial_2^2\widetilde{U_1} + \partial_2\widetilde{B_1} + (-\Delta)^{-1}\partial_1^2\partial_2^2\widetilde{U_1} + (-\Delta)^{-1}\partial_2\partial_1^3\widetilde{U_2},\\ \partial_t\widetilde{U_2} = \partial_1^2\widetilde{U_2} + \partial_2\widetilde{B_2} + (-\Delta)^{-1}\partial_1\partial_2^3\widetilde{U_1} + (-\Delta)^{-1}\partial_1^2\partial_2^2\widetilde{U_2},\\ \partial_t\widetilde{B_1} = \partial_2\widetilde{U_1},\\ \partial_t\widetilde{B_2} = \partial_1^2\widetilde{B_2} + \partial_2\widetilde{U_2},\\ \nabla\cdot\widetilde{U} = 0, \qquad \nabla\cdot\widetilde{B} = 0,\end{cases}$$

where $\widetilde{U} = \nabla^s u, \widetilde{B} = \nabla^s b$. Repeating the above process, then we can yield (1.9) and (1.10). This completes the proof of Theorem 1.2. □

4.1 The stucture of the solutions

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