# Total Offensive Alliances on Some Graphs

***Original Research Article***

## Abstract

Let $G = (V(G), E(G))$ be a nontrivial connected graph. A nonempty set of vertices $T \subseteq V(G)$ is defined as an offensive alliance in $G$ if, for every $v \in \partial(T)$, it holds that $|N[v] \cap T| \geq |N[v] \setminus T|$. Equivalently, this can be expressed as $deg_T(v) \geq deg_{V(G)\setminus T}(v) + 1$. The set $T$ is termed a total offensive alliance in $G$ if it is an offensive alliance and every vertex in $T$ has at least one neighbor within $T$. The minimum cardinality of a total offensive alliance set in $G$ is called the total offensive alliance number, denoted by $a_{to}(G)$. This paper presents a characterization of total offensive alliance sets and provides the corresponding minimum cardinality for various graph families, including path, cycle, complete, star, fan, and wheel graphs.

*Keywords: offensive alliance; total offensive alliance; total offensive alliance number.*

# 1 Introduction

In the real world, an alliance is an association or collection of entities formed for mutual benefit such that the union is stronger than the individual. Formally, it is affiliated as a formal agreement or treaty between two or more nations to collaborate for specific purposes [1] or joining of efforts and interests within families, states, parties, or individuals. This has motivated the study of Kristiansen et al. [2] which employed the concept of alliances in graphs and in the context of alliance of nations in war, the vertices of graph represent the nations and the edges correspond to possible relation between them. They defined three kinds of alliances, the defensive, offensive, and dual or powerful alliance. A defensive alliance in a graph $G$ is a set $S$ of vertices of $G$ with the property that every vertices in $S$ has atmost one more neighbor outside of $S$ than it has in $S$. Also, an offensive alliance in a graph $G$ is a set $S$ of vertices with the property that every vertex in the neighborhood of $S$ has at least one more neighbor in $S$ than it has outside of $S$ [3]. With these definitions, we say that vertices within a defensive alliance can be defended from possible attack by outside vertices and vertices in the neighborhood of an offensive alliance is vulnerable to possible attack by vertices in an offensive alliance [4]. Dual or powerful alliance, on the other hand, is both defensive and offensive. Variations of these defensive alliances can also be found in [5], [6] and variations of offensive alliances can be found in [7].

Throughout the years, researchers generate significant development on these three kinds of alliances. [3] Moreover, studying alliances in its broad sense continues to offer distinguished contributions specifically its variety of applications in business and social network modeling, bioinformatics, distributed computing, web communities, security and defense, biological networks, data clustering, etc. that are discussed in studies [1] and [2]. That said, the alliances in graphs remain an interesting study over the years.

In this paper, we focus on offensive alliances in graphs. We introduce the concept of total offensive alliance, an extension of the offensive alliance, with additional condition that may potentially be more useful in certain applications. Here, we present the total offensive alliances to some graph families particularly path, cycle, complete, star, fan, and wheel graphs. We examine the properties and nature of total offensive alliances within these graph structures to determine their characterizations. Also, we identify the corresponding total offensive alliance number of each graphs. The same method of finding characterizations can also be examined in [9], [10], [11].

## 2 Preliminary Notes

Some definitions of the concepts covered in this study are included below. You may refer on the remaining terms and definitions in [1], [2], [4], [3], [8].

**Definition 2.1.** The **join of graphs** $G$ and $H$ is a graph formed by the disjoint union, denoted by $G \cup H$, of $G$ and $H$ connecting every vertex of $G$ to every vertex of $H$. For $n \geq 2$, the **fan graph** $F_n$ of order $n+1$ is a graph join $P_n \cup G_T$ where $P_n$ denotes the path graph of order $n$ and $G_T$ denotes the trivial graph. Every vertex in $P_n$ is connected to the vertex in $G_T$ which we refer to as the universal vertex. For $n \geq 3$, the **wheel graph** $W_n$ of order $n+1$ is a graph join $C_n \cup G_T$ where $C_n$ denotes the cycle graph of order $n$ and $G_T$ denotes the trivial graph. Every vertex in $C_n$ is connected to the universal vertex in $G_T$.

**Definition 2.2.** Let $G = (V(G), E(G))$ be a nontrivial graph and let $u, v \in V(G)$. The subgraph of a graph $G$ induced by a nonempty set $T$ of vertices of $G$, is the **induced subgraph** with vertex set, $T$, denoted by $G[T]$, such that whenever $u$ and $v$ are vertices of $T$ and $uv$ is an edge of $G$, then $uv$ is an edge of $G[T]$ as well.

**Definition 2.3.** [8] Let $G$ be a simple graph and let $T \subseteq V(G)$. Then the **boundary** set of $T \subseteq V(G)$, denoted by $\partial(T)$, is the set of all vertices of $G$ which are adjacent to $T$, but not in $T$, i.e., $N(T) \setminus T$.

**Definition 2.4.** [1] Given a nontrivial connected graph $G$, a nonempty set of vertices $T \subseteq V(G)$ is an **offensive alliance** in $G$ if for every $v \in \partial(T)$, we have $|N[v] \cap T| \geq |N[v] \setminus T|$ or equivalently, $deg_T(v) \geq deg_{V(G) \setminus T}(v) + 1$.

**Example 2.1.** *Consider the graph in $Figure$ 1. Take $T = \{v_1, v_3\} \subseteq V(G)$. Then vertices $v_2, v_4 \in \partial(T)$. Now, for $v_2 \in \partial(T)$, $|N[v_2] \cap T| = 2 > |N[v_2] \setminus T| = 1$. Similarly for $v_4 \in \partial(T)$. Hence, $T$ is an offensive alliance of $G$.*

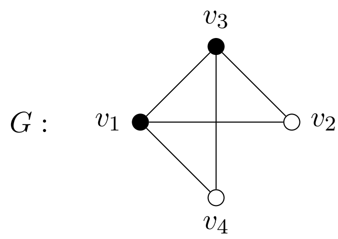


Figure 1: *A graph $G$ and its offensive alliance*

**Definition 2.5.** A nonempty set of vertices $T \subseteq V(G)$ is called a **total offensive alliance** in $G$ if $T$ is an offensive alliance in $G$ and every vertex in $T$ has at least one neighbor in $T$. Moreover, the minimum cardinality of a total offensive alliance in $G$ is called the **total offensive alliance number** of $G$, denoted by $a_{to}(G)$.

**Example 2.2.** *Consider the graph $H$ in Figure 2. Here, the set of vertices $T = \{v_2, v_4, v_5, v_6\} \subseteq V(H)$ is an offensive alliance set in $H$. Observe that $\partial(T) = \{v_1, v_7\}$. For $v_1 \in \partial(T)$, $|N[v_1] \cap T| = 2 \geq |N[v_1] \smallsetminus T| = 2$ and for $v_7 \in \partial(T)$, $|N[v_7] \cap T| = 2 \geq |N[v_7] \smallsetminus T| = 1$. However, $T$ is not a total offensive alliance set in $H$. Notice that the induced subgraph of $T$ in $H$ has an isolated vertex, as shown in graph $H_1$.*

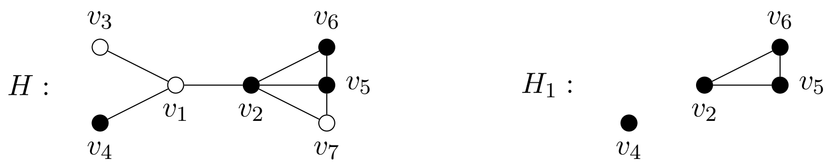


Figure 2: *A graph $H$ and its induced subgraph $H_1$ of an offensive alliance $T$*

**Example 2.3.** *Let $G$ be a graph as shown in Figure 3. Here, $T = \{v_2, v_3\} \subseteq V(G)$ is a total offensive alliance in $G$. In fact, $T$ is the minimum total offensive alliance in $G$.*

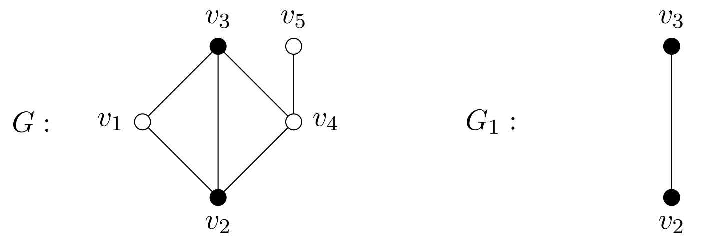


Figure 3: *A graph $G$ and its total offensive alliance*

*Remark* 2.1. If $T = V(G)$, then $T$ is not a total offensive alliance in $G$.

# 3 Main Results

In this section, the characteristics of a total offensive alliance for paths, cycles, complete graphs, star graphs, fan, and wheel graphs are provided. Moreover, the total offensive alliance number for each graph is also established. The term $TOA$ is used to represent total offensive alliance.

**Theorem 3.1.** *Let $G$ be a nontrivial connected graph with $\triangle(G) = 2$. If $\emptyset \neq T \subseteq V(G)$, then $T$ is a $TOA$ in $G$ if and only if no two vertices in $\partial(T)$ are neighbors in $\partial(T)$ and $G[T]$ has no isolated vertex.*

*Proof.* Let $\emptyset \neq T \subseteq V(G)$ be a $TOA$ in $G$. Suppose that there exists two vertices, say $u, v \in \partial(T)$, with $u \in N(v)$ and $v \in N(u)$ or $G[T]$ has an isolated vertex. Since $u, v \in \partial(T)$ are neighbors, $|N[u] \setminus T| = 2$. And since $T$ is a $TOA$ in $G$, $|N[u] \cap T| \geq 2$. However, $|N[u] \cap T| = 1$, a contradiction to the assumption that $T$ is a $TOA$ in $G$. Thus, $u, v \in \partial(T)$ in $G$ are not neighbors in $\partial(T)$. On the other hand, if $G[T]$ has an isolated vertex, then there exists a vertex $w \in T$ such that $w \notin N[x]$ for some $x \in T$, a contradiction. Thus, $G[T]$ has no isolated vertex. Therefore, no two vertices in $\partial(T)$ are neighbors in $\partial(T)$ and $G[T]$ has no isolated vertex.

Conversely, suppose that no two vertices in $\partial(T)$ are neighbors in $\partial(T)$ and $G[T]$ has no isolated vertex. Then for every $v \in \partial(T), |N[u] \cap T| = 1 \geq |N[u] \setminus T| = 1$. Thus, $T$ is an offensive alliance in $G$. Also, since $G[T]$ has no isolated vertex, clearly, $T$ is a $TOA$ in $G$. □

**Corollary 3.2.** *For a path graph $P_n$ of order $n \geq 3$,*

$$a_{to}(P_n) = \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 \text{(mod 3)} \\ \frac{2n-2}{3}, & \text{if } n \equiv 1 \text{(mod 3)} \\ \frac{2n-1}{3}, & \text{if } n \equiv 2 \text{(mod 3)} \end{cases}$$

.

*Proof.* Let $P_n = \{v_1, v_2, ..., v_n\}, n \geq 3$, and $\emptyset \neq T \subseteq V(P_n)$ be a $TOA$ in $P_n$. Consider the following cases:

Case 1: $n \equiv 0$ (mod 3)

Choose $T = \{v_2, v_3, ..., v_{3k+2}, v_{3k+3}, ..., v_{n-1}, v_n\}$, where $k = \frac{n-3}{3}$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{2n}{3}$. Now, $\partial(T) = \{v_1, v_4, ..., v_{n-2}\}$. Clearly, no two vertices of $\partial(T)$ are neighbors in $\partial(T)$. Also, $P_n[T]$ has no isolated vertex since for every $v \in T$, $\deg_T(v) = 1$. By Theorem 3.1, $T$ is a $TOA$ in $P_n$. It remains to show that $T$ is the minimum $TOA$ in $P_n$. Suppose $T$ is not the minimum $TOA$ in $P_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(P_n)$ such that $|T_0| < |T| = \frac{2n}{3}$. Without loss of generality, suppose $|T_0| = \frac{2n}{3} - 1$. Then there exists $v \in T_0$ such that $\deg_{T_0}(v) = 0$ or there exists $w \in \partial(T_0)$ such that $\deg_{\partial(T_0)}(w) = 1$. If $\deg_{T_0}(v) = 0$, then $P_n[T_0]$ has an isolated vertex, a contradiction to our assumption about $T_0$. Hence, $T_0$ is not a $TOA$ in $P_n$. If $\deg_{\partial(T_0)}(w) = 1$, then $|N[w] \cap T_0| = 1 \ngeq |N[w] \setminus T_0| = 2$, a contradiction. Thus, $T_0$ is not an offensive alliance in $P_n$. Therefore, $a_{to}(P_n) = |T| = \frac{2n}{3}$.

Case 2: $n \equiv 1$ (mod 3)

Choose $T = \{v_2, v_3, ..., v_{3k+2}, v_{3k+3}, ..., v_{n-2}, v_{n-1}\}$, where $k = \frac{n-1}{3} - 1$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{2n-2}{3}$. Now, $\partial(T) = \{v_1, v_4, ..., v_n\}$. Clearly, no two vertices of $\partial(T)$ are neighbors in $\partial(T)$. Also, $P_n[T]$ has no isolated vertex since for every $v \in T$, $\deg_T(v) = 1$. By Theorem 3.1, $T$ is a $TOA$ in $P_n$. It remains to show that $T$ is the minimum $TOA$ in $P_n$. Suppose $T$ is not the minimum $TOA$ in $P_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(P_n)$ such that $|T_0| < |T| = \frac{2n-2}{3}$. Without loss of generality, suppose $|T_0| = \frac{2n-2}{3} - 1$. Then there exists $v \in T_0$ such that $\deg_{T_0}(v) = 0$ or there exists $w \in \partial(T_0)$ such that $\deg_{\partial(T_0)}(w) = 1$. If $\deg_{T_0}(v) = 0$, then $P_n[T_0]$ has an isolated vertex, a contradiction to our assumption about $T_0$. Hence, $T_0$ is not a $TOA$ in $P_n$. If $\deg_{\partial(T_0)}(w) = 1$, then $|N[w] \cap T_0| = 1 \ngeq |N[w] \setminus T_0| = 2$, a contradiction. Thus, $T_0$ is not an offensive alliance in $P_n$. Therefore, $a_{to}(P_n) = |T| = \frac{2n-2}{3}$.

Case 3: $n \equiv 2$ (mod 3)

Choose $T = \{v_2, v_3, ..., v_{3k+2}, v_{3k+3}, ..., v_{n-3}, v_{n-2}, v_{n-1}\}$, where $k = \frac{n-2}{3} - 1$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{2n-1}{3}$. Now, $\partial(T) = \{v_1, v_4, ..., v_n\}$. Clearly, no two vertices of $\partial(T)$ are neighbors in $\partial(T)$. Also, $P_n[T]$ has no isolated vertex since for every $v \in T$, either $\deg_T(v) = 1$ or $\deg_T(v) = 2$. By Theorem 3.1, $T$ is a $TOA$ in $P_n$. It remains to show that $T$ is the minimum $TOA$ in $P_n$. Suppose $T$ is not the minimum $TOA$ in $P_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(P_n)$ such that $|T_0| < |T| = \frac{2n-1}{3}$. Without loss of generality, suppose $|T_0| = \frac{2n-1}{3} - 1$. Then there exists $v \in T_0$ such that $\deg_{T_0}(v) = 0$ or there exists $w \in \partial(T_0)$ such that $\deg_{\partial(T_0)}(w) = 1$. If $\deg_{T_0}(v) = 0$, then $P_n[T_0]$ has an isolated vertex, a contradiction to our assumption about $T_0$. Hence, $T_0$ is not a $TOA$ in $P_n$. If $\deg_{\partial(T_0)}(w) = 1$, then $|N[w] \cap T_0| = 1 \ngeq |N[w] \setminus T_0| = 2$, a contradiction. Thus, $T_0$ is not an offensive alliance in $P_n$. Therefore, $a_{to}(P_n) = |T| = \frac{2n-1}{3}$. □

**Corollary 3.3.** *For a cycle graph $C_n$ of order $n \geq 3$,*

$$a_{to}(C_n) = \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 \text{(mod 3)} \\ \frac{2n+1}{3}, & \text{if } n \equiv 1 \text{(mod 3)} \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 \text{(mod 3)} \end{cases}$$

.

*Proof.* Let $C_n = \{v_1, v_2, ..., v_n, v_1\}, n \geq 3$, and $\emptyset \neq T \subseteq V(C_n)$ be a $TOA$ in $C_n$. Consider the following cases:

Case 1: $n \equiv 0$ (mod 3)

Choose $T = \{v_1, v_2, ..., v_{3k+1}, v_{3k+2}, ..., v_{n-2}, v_{n-1}\}$, where $k = \frac{n-3}{3}$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{2n}{3}$. Now, $\partial(T) = \{v_3, v_6, ..., v_n\}$. Clearly, no two vertices of $\partial(T)$ are neighbors in $\partial(T)$. Also, $C_n[T]$ has no isolated vertex since for every $v \in T$, $\deg_T(v) = 1$. By Theorem 3.1, $T$ is a $TOA$ in $C_n$. It remains to show that $T$ is the minimum $TOA$ in $C_n$. Suppose $T$ is not the minimum $TOA$ in $C_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(C_n)$ such that $|T_0| < |T| = \frac{2n}{3}$. Without loss of generality, suppose $|T_0| = \frac{2n}{3} - 1$. Then there exists $v \in T_0$ such that $\deg_{T_0}(v) = 0$ or there exists $w \in \partial(T_0)$ such that $\deg_{\partial(T_0)}(w) = 1$. If $\deg_{T_0}(v) = 0$, then $C_n[T_0]$ has an isolated vertex, a contradiction to our assumption about $T_0$. Hence, $T_0$ is not a $TOA$ in $C_n$. If $\deg_{\partial(T_0)}(w) = 1$, then $|N[w] \cap T_0| = 1 \ngeq |N[w] \setminus T_0| = 2$, a contradiction. Thus, $T_0$ is not an offensive alliance in $C_n$. Therefore, $a_{to}(C_n) = |T| = \frac{2n}{3}$.

Case 2: $n \equiv 1$ (mod 3)

Choose $T = \{v_1, v_2, ..., v_{3k+1}, v_{3k+2}, ..., v_{n-3}, v_{n-2}, v_{n-1}\}$, where $k = \frac{n-1}{3} - 1$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{2n+1}{3}$. Now, $\partial(T) = \{v_3, v_6, ..., v_n\}$. Clearly, no two vertices of $\partial(T)$ are neighbors in $\partial(T)$. Also, $C_n[T]$ has no isolated vertex since for every $v \in T$, either $\deg_T(v) = 1$ or $\deg_T(v) = 2$. By Theorem 3.1, $T$ is a $TOA$ in $C_n$. It remains to show that $T$ is the minimum $TOA$ in $C_n$. Suppose $T$ is not the minimum $TOA$ in $C_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(C_n)$ such that $|T_0| < |T| = \frac{2n+1}{3}$. Without loss of generality, suppose $|T_0| = \frac{2n+1}{3} - 1$. Then there exists $v \in T_0$ such that $\deg_{T_0}(v) = 0$ or there exists $w \in \partial(T_0)$ such that $\deg_{\partial(T_0)}(w) = 1$. If $\deg_{T_0}(v) = 0$, then $C_n[T_0]$ has an isolated vertex, a contradiction to our assumption about $T_0$. Hence, $T_0$ is not a $TOA$ in $C_n$. If $\deg_{\partial(T_0)}(w) = 1$, then $|N[w] \cap T_0| = 1 \ngeq |N[w] \setminus T_0| = 2$, a contradiction. Thus, $T_0$ is not an offensive alliance in $C_n$. Therefore, $a_{to}(C_n) = |T| = \frac{2n+1}{3}$.

Case 3: $n \equiv 2$ (mod 3)

Choose $T = \{v_1, v_2, ..., v_{3k+1}, v_{3k+2}, ..., v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}\}$, where $k = \frac{n-2}{3} - 1$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{2n+2}{3}$. Now, $\partial(T) = \{v_1, v_4, ..., v_n\}$. Clearly, no two vertices of $\partial(T)$ are neighbors in $\partial(T)$. Also, $C_n[T]$ has no isolated vertex since for every $v \in T$, either $\deg_T(v) = 1$ or $\deg_T(v) = 2$. By Theorem 3.1, $T$ is a $TOA$ in $C_n$. It remains to show that $T$ is the minimum $TOA$ in $C_n$. Suppose $T$ is not the minimum $TOA$ in $C_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(C_n)$ such that $|T_0| < |T| = \frac{2n+2}{3}$. Without loss of generality, suppose $|T_0| = \frac{2n+2}{3} - 1$. Then there exists $v \in T_0$ such that $\deg_{T_0}(v) = 0$ or there exists $w \in \partial(T_0)$ such that $\deg_{\partial(T_0)}(w) = 1$. If $\deg_{T_0}(v) = 0$, then $C_n[T_0]$ has an isolated vertex, a contradiction to our assumption about $T_0$. Hence, $T_0$ is not a $TOA$ in $C_n$. If $\deg_{\partial(T_0)}(w) = 1$, then $|N[w] \cap T_0| = 1 \ngeq |N[w] \setminus T_0| = 2$, a contradiction. Thus, $T_0$ is not an offensive alliance in $C_n$. Therefore, $a_{to}(C_n) = |T| = \frac{2n+2}{3}$. □

**Theorem 3.4.** *Let $G$ be a complete graph $K_n$, $n \geq 3$, and $\emptyset \neq T \subseteq V(K_n)$. Then T is a TOA in $K_n$ if and only if $\lceil \frac{n}{2} \rceil \leq |T| \leq n - 1$.*

*Proof.* Let $\emptyset \neq T \subseteq V(K_n)$ be a $TOA$ in $K_n$. Clearly by Remark 2.1, $|T| \leq n - 1$. Now, we want to show that $\lceil \frac{n}{2} \rceil \leq |T|$. Suppose on contrary that $|T| < \lceil \frac{n}{2} \rceil$. Without loss of generality, suppose $|T| = \lceil \frac{n}{2} \rceil - 1$. If $|T| < \lceil \frac{n}{2} \rceil$, let $v \in \partial(T)$, then $|N[v] \cap T| \leq \lceil \frac{n}{2} \rceil$ - 1 and $|N[v] \setminus T| \leq \lceil \frac{n}{2} \rceil$. Thus, $|N[v] \cap T| \ngeq |N[v] \setminus T|$, a contradiction to our assumption that $T$ is a $TOA$ in $K_n$. Therefore, $\lceil \frac{n}{2} \rceil \leq |T|$.

Conversely, suppose $\lceil \frac{n}{2} \rceil \leq |T| \leq n - 1$. It suffices to show that if $|T| = \lceil \frac{n}{2} \rceil$ or $|T| = n - 1$, then $T$ is a $TOA$ in $K_n$. If $|T| = \lceil \frac{n}{2} \rceil$, then for every $v \in \partial(T)$, $|N[v] \cap T| = \lceil \frac{n}{2} \rceil = |N[v] \setminus T| = \lceil \frac{n}{2} \rceil$ when $n$ is even or $|N[v] \cap T| = \lceil \frac{n}{2} \rceil \geq |N[v] \setminus T| = \lceil \frac{n}{2} \rceil - 1$ when $n$ is odd. Also, if $|T| = n - 1$, then for every $v \in \partial(T)$, it is clear that $\deg_T(v) \geq \deg_{V(K_n) \setminus T}(v) + 1$. Thus, $T$ is an offensive alliance in $K_n$.

To this point, since every vertex in $K_n$ is connected, for $\lceil \frac{n}{2} \rceil \leq |T| \leq n-1$, every vertex in $T$ has at least one neighbor in $T$. Therefore, $T$ is a $TOA$ in $K_n$. □

**Corollary 3.5.** *If $G = K_n$, $n \geq 4$, then $a_{to}(K_n) = \lceil \frac{n}{2} \rceil$.*

*Proof.* Let $\emptyset \neq T \subseteq V(K_n)$ be the minimum $TOA$ in $K_n$. Then by Theorem 3.4, $|T| \geq \lceil \frac{n}{2} \rceil$. Thus, $|T| = \lceil \frac{n}{2} \rceil$. Therefore, $a_{to}(K_n) = |T| = \lceil \frac{n}{2} \rceil$. □

**Theorem 3.6.** *Let $G$ be a star graph $K_{1,n}$ of order $n+1$, $n \geq 2$, and $\emptyset \neq T \subseteq V(K_{1,n})$ such that $T = T_1 \cup T_2$, $T_1 = K_1$, $T_2 \subseteq V(\overline{K_n})$. Then $T$ is a $TOA$ in $K_{1,n}$ if and only if $1 \leq |T_2| \leq n-1$.*

*Proof.* Let $\emptyset \neq T \subseteq V(K_{1,n})$ such that $T = T_1 \cup T_2, T_1 = K_1$, $T_2 \subseteq V(\overline{K_n})$ be $TOA$ in $K_{1,n}$. Clearly by Remark 2.1, $|T_2| \leq n-1$. Now we want to show that 1 $\leq |T_2|$. Suppose on contrary, $|T_2| < 1$. Obviously, $K_{1,n}[T]$ has an isolated vertex, i.e., the central vertex $u \in T_1 \subseteq T$. A contradiction since $T$ is a $TOA$ in $K_{1,n}$. Hence, $1 \leq |T_2| \leq n-1$.

Conversely, suppose 1 $\leq |T_2| \leq n-1$. Then 1 $\leq |T_2|$ and $|T_2| \leq n-1$. For every vertex $v \in \partial(T)$ in $T_2$, $|N[v] \cap T| = 1 \geq |N[v] \setminus T| = 1$ since every leaf vertex in $T_2 \subseteq V(\overline{K_n})$ is connected to the central vertex in $T_1 = K_1$. Hence, $T$ is an offensive alliance in $K_{1,n}$. Also, since 1 $\leq |T_2|$, then the central vertex in $T$ has at least one neighbor in $T$. Therefore, $T$ is a $TOA$ in $K_{1,n}$. □

**Corollary 3.7.** *If $G = K_{1,n}$, $n \geq 2$, then $a_{to}(K_{1,n}) = 2$.*

*Proof.* Let $\emptyset \neq T \subseteq V(K_{1,n})$ such that $T = T_1 \cup T_2$, $T_1 = K_1$, $T_2 \subseteq V(\overline{K_n})$ be the minimum $TOA$ in $K_{1,n}$. By Theorem 3.6, 1 $\leq |T_2|$ and $T$ contains the central vertex $u \in T_1$. Therefore, $a_{to}(K_{1,n}) = 2$. □

For the next theorems, we consider two scenarios for fan and wheel graphs. For the first scenario, we examine a total offensive alliance $T$ such that $T \subseteq V(P_n) \subseteq V(F_n)$ for fan graphs and $T \subseteq V(C_n) \subseteq V(W_n)$ for wheel graphs. Here, $T$ must only contain vertices in $V(P_n)$ and $V(C_n)$ respectively. For the second scenario, we take $T \subseteq V(F_n)$ such that $T = T_1 \cup T_2$, $T_1 \subseteq V(P_n)$, $T_2 \subseteq G_T$ for fan graphs and $T \subseteq V(W_n)$ such that $T = T_1 \cup T_2$, $T_1 \subseteq V(C_n)$, $T_2 \subseteq G_T$ for wheel graphs. Here, $T$ must contain the universal vertex in $G_T$ and vertices in $V(P_n)$ and $V(C_n)$ respectively.

**Theorem 3.8.** *Let $G = F_n$ of order $n+1$, $n \geq 3$, and $\emptyset \neq T \subseteq V(P_n) \subseteq V(F_n)$. Then $T$ is a $TOA$ in $F_n$ if and only if one of the following holds:*

***(i)*** $T = V(P_n)$

***(ii)*** $\{v_1, v_2, v_{n-1}, v_n\} \subseteq T$ *and* $F_n[V(P_n) \setminus T]$ *is an empty graph in* $P_n$ *provided that* $F_n[T \setminus \{v_1, v_2, v_{n-1}, v_n\}]$ *in* $T$ *has no isolated vertex.*

*Proof.* Let $\emptyset \neq T \subseteq V(P_n) \subseteq V(F_n)$ be a $TOA$ in $F_n$. Clearly, $T = V(P_n)$. Now, suppose $\{v_1, v_2, v_{n-1}, v_n\} \nsubseteq T$ or $F_n[V(P_n) \setminus T]$ is an empty graph in $P_n$ provided that $F_n[T \setminus \{v_1, v_2, v_{n-1}, v_n\}]$ in $T$ has an isolated vertex. If $\{v_1, v_2, v_{n-1}, v_n\} \nsubseteq T$, then for every end-vertex $v_1 \in V(P_n) \setminus T$, $|N[v_1] \setminus T| = 3$. Same as with end-vertex $v_n \in V(P_n) \setminus T$. Since $T$ is a $TOA$ in $F_n$, $|N[v_1] \cap T| \geq 3$ but $|N[v_1] \cap T| = 0$, a contradiction. Hence, (i) holds. If $F_n[V(P_n) \setminus T]$ is an empty graph in $P_n$ provided that $F_n[T \setminus \{v_1, v_2, v_{n-1}, v_n\}]$ in $T$ has an isolated vertex, then clearly, there exists $v_i \in T \setminus \{v_1, v_2, v_{n-1}, v_n\}$ for $i = 1, 2, ..., n$ such that $v_i$ has no neighbor in $T$, again, a contradiction. Hence, (ii) holds.

For the converse, suppose (i) holds. Then for the universal vertex $u \in \partial(T)$ in $G_T$, $|N[u] \cap T| = n \geq |N[u] \setminus T| = 1$. Hence, $T$ is an offensive alliance in $F_n$. Also, since $T = V(P_n)$, then clearly, $T$ is a $TOA$ in $F_n$. Now, suppose (ii) holds. Since $F_n[V(P_n) \setminus T]$ is an empty graph in $P_n$, then for every vertex $v \in V(P_n) \setminus T$, $|N[v] \cap T| = 2$ and $|N[v] \setminus T| = 2$, which is itself and the universal vertex $u \in \partial(T)$ in $G_T$. Thus, $T$ is an offensive alliance in $F_n$. Also, since $\{v_1, v_2, v_{n-1}, v_n\} \subseteq T$ and $F_n[T \setminus \{v_1, v_2, v_{n-1}, v_n\}]$ in $T$ has no isolated vertex, clearly, $T$ is a $TOA$ in $F_n$. □

**Theorem 3.9.** *Let $G = F_n$ of order $n + 1$, $n \geq 3$, and $\emptyset \neq T \subseteq V(F_n)$ such that $T = T_1 \cup T_2$, $T_1 \subseteq V(P_n), T_2 \subseteq G_T$. Then $T$ is a $TOA$ in $F_n$ if and only if the following hold:*

***(i)*** *for every end-vertex $v \in \partial(T)$ of $V(P_n)$, $deg_T(v) = 2$; and*

***(ii)*** *every vertex in $\partial(T)$ that is not an end-vertex of $V(P_n)$ has at most one neighbor in $\partial(T)$.*

*Proof.* Let $\emptyset \neq T \subseteq V(F_n)$ be a $TOA$ in $F_n$. Suppose there exists an end-vertex $v \in \partial(T)$ of $V(P_n)$ such that $\deg_T(v) \neq 2$ or there exists a vertex $w \in \partial(T)$ that is not an end-vertex of $V(P_n)$ such that it has two neighbors in $\partial(T)$. If there exists an end-vertex $v \in \partial(T)$ of $V(P_n)$ such that $\deg_T(v) \neq 2$, then clearly, $\deg_T(v) = 1$, since it has at least one neighbor in $T$, which is immediately the universal vertex in $T_2 \subseteq G_T$. Thus, $|N[v] \cap T| = 1$ but $|N[v] \setminus T| = 2$, which is itself and its adjacent vertex in $V(P_n)$. This is a contradiction since $T$ is a $TOA$ in $F_n$. On the other hand, if there exists a vertex $w \in \partial(T)$ that is not an end-vertex of $V(P_n)$ such that it has two neighbors in $\partial(T)$, then $|N[v] \setminus T| = 3$, but $|N[v] \cap T| = 1$, which is the universal vertex in $T_2 \subseteq G_T$ only, a contradiction. Thus, both (i) and (ii) hold.

Conversely, suppose (i) and (ii) hold. Then for every end-vertex $v \in \partial(T)$ of $V(P_n)$ such that $\deg_T(v) = 2$, $|N[v] \cap T| = 2 \geq |N[v] \setminus T| = 1$. Also, for every vertex $w \in \partial(T)$ that is not an end-vertex of $V(P_n)$ with $|N(w)| = 3$ in $V(P_n)$, $|N[w] \cap T| = 3 \geq |N[w] \setminus T| = 1$ if $w \in \partial(T)$ has no neighbor in $\partial(T)$ or $|N[w] \cap T| = 2 \geq |N[w] \setminus T| = 2$ if $w \in \partial(T)$ has one neighbor in $\partial(T)$. Hence, $T$ is an offensive alliance in $F_n$. Moreover, since the universal vertex, say $u \in T_2 \subseteq G_T$ is in $T$, every vertex in $T$ has a neighbor in $T$. Therefore, $T$ is a $TOA$ in $F_n$. □

**Corollary 3.10.** *For a fan graph $F_n$ of order $n + 1$ where $n \geq 3$,*

$$a_{to}(F_n) = \begin{cases} \frac{n+3}{3}, & \text{if } n \equiv 0 (\text{mod } 3) \\ \frac{n+5}{3}, & \text{if } n \equiv 1 (\text{mod } 3) \\ \frac{n+4}{3}, & \text{if } n \equiv 2 (\text{mod } 3) \end{cases}$$

.

*Proof.* Let $\emptyset \neq T \subseteq V(F_n)$ such that $T = \{v_1, v_2, ..., v_{n-1}, v_n\} \cup \{u\}$ where $\{v_1, v_2, ..., v_{n-1}, v_n\} \in T_1 \subseteq V(P_n)$ and $\{u\} \in T_2 \subseteq G_T$ for $n \geq 3$ be a $TOA$ in $F_n$. Consider the following cases:

Case 1: n ≡ 0 (mod 3)

Choose $T = \{v_2, v_5, ..., v_{3k-1}, ..., v_{n-1}\} \cup \{u\}$ where $k = \frac{n}{3}$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{n+3}{3}$. Now, $\partial(T) = \{v_1, v_3, v_4, v_6, v_7, ..., v_n\}$. Clearly, every end-vertices $v_1, v_n \in \partial(T)$ of $V(P_n)$, $\deg_T(v_1) = 2 = \deg_T(v_n)$, which is its adjacent vertex in $T \subseteq V(P_n)$ and the universal vertex $u$. Also, every vertex in $\partial(T)$ that is not an end-vertex of $V(P_n)$ has at most one neighbor in $\partial(T)$. By Theorem 3.9, $T$ is a $TOA$ in $F_n$. Now, we want to show that $T$ is the minimum $TOA$ in $F_n$. Suppose $T$ is not the minimum $TOA$ in $F_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(F_n)$ such that $|T_0| < |T| = \frac{n+3}{3}$. Without loss of generality, suppose $|T_0| = \frac{n+3}{3} - 1$. Consider the following cases:

***(i)*** $v_{n-1} \notin T_0$ for some $v_{n-1} \in V(P_n)$
If $v_{n-1} \notin T_0$ for some $v_{n-1} \in V(P_n)$, then there exists $v_{n-1} \in \partial(T_0)$ such that $\deg_{V(F_n)\setminus T_0}(v_{n-1}) = 2$, its adjacent vertices in $\partial(T_0) \subseteq V(P_n)$, or there exists an end-vertex $v_n \in \partial(T)$ of $V(P_n)$ such that $\deg_T(v_n) = 1$. If $v_{n-1} \in \partial(T_0)$ such that $\deg_{V(F_n)\setminus T_0}(v_{n-1}) = 2$, then $|N[v_{n-1}] \cap T_0| = 1$ but $|N[v_{n-1}] \setminus T_0| = 3$, a contradiction. Also, if end-vertex $v_n \in \partial(T)$ of $V(P_n)$ such that $\deg_T(v_n) = 1$, then $|N[v_n] \cap T_0| = 2$ but $|N[v_n] \setminus T_0| = 1$, a contradiction as well. Hence, $T_0$ is not an offensive alliance in $F_n$.
***(ii)*** $u \notin T_0$
If $u \notin T_0$, then $F_n[T_0]$ contains isolated vertices of $V(P_n)$, a contradiction to the assumption that $T_0$ is a $TOA$ in $F_n$. Hence, $T_0$ is not a $TOA$ in $F_n$.
Therefore, $a_{to}(F_n) = |T| = \frac{n+3}{3}$.

Case 2: n ≡ 1 (mod 3)

Choose $T = \{v_2, v_5, ..., v_{3k-1}, ..., v_{n-2}, v_n\} \cup \{u\}$ where $k = \frac{n-1}{3}$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{n+5}{3}$. Now, $\partial(T) = \{v_1, v_3, v_4, v_6, v_7, ..., v_{n-3}, v_{n-1}\}$. Clearly, for end-vertex $v_1 \in \partial(T)$ of $V(P_n)$, $\deg_T(v_1) = 2$, which is its adjacent vertex in $T \subseteq V(P_n)$ and the universal vertex $u$. Also, every vertex in $\partial(T)$ that is not an end-vertex of $V(P_n)$ has at most one neighbor in $\partial(T)$. By Theorem 3.9, $T$ is a $TOA$ in $F_n$. Now, we want to show that $T$ is the minimum $TOA$ in $F_n$. Suppose $T$ is not the minimum $TOA$ in $F_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(F_n)$ such that $|T_0| < |T| = \frac{n+5}{3}$. Without loss of generality, suppose $|T_0| = \frac{n+5}{3} - 1$. Consider the following cases:
***(i)*** $v \notin T_0$ for some $v \in V(P_n)$
If $v \notin T_0$ for some $v \in V(P_n)$, then there exists $v \in \partial(T_0)$ such that $\deg_{T_0}(v) = 1$, the universal vertex $u$, if $v \in \partial(T_0)$ is an end-vertex of $V(P_n)$ or $\deg_{V(F_n)\setminus T_0}(v) = 2$, its adjacent vertices in $\partial(T_0) \subseteq V(P_n)$, if $v \in \partial(T_0)$ is not. And so, $|N[v] \cap T_0| = 1$ but $|N[v] \setminus T_0| = 2$ if $v \in \partial(T_0)$ is an end-vertex of $V(P_n)$ or $|N[v] \cap T_0| = 1$ but $|N[v] \setminus T_0| = 3$ if $v$ is not. But both are contradictions. Hence, $T_0$ is not an offensive alliance in $F_n$..
***(ii)*** $u \notin T_0$
If $u \notin T_0$, then $F_n[T_0]$ contains isolated vertices of $V(P_n)$, a contradiction to the assumption that $T_0$ is a $TOA$ in $F_n$. Hence, $T_0$ is not a $TOA$ in $F_n$.
Therefore, $a_{to}(F_n) = |T| = \frac{n+5}{3}$.

Case 3: n ≡ 2 (mod 3)

Choose $T = \{v_2, v_5, ..., v_{3k-1}, ..., v_n\} \cup \{u\}$ where $k = \frac{n-2}{3}$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{n+4}{3}$. Now, $\partial(T) = \{v_1, v_3, v_4, v_6, v_7, ..., v_{n-2}, v_{n-1}\}$. Clearly, for an end-vertex $v_1 \in \partial(T)$ of $V(P_n)$, $\deg_T(v_1) = 2$, which is its adjacent vertex in $T \subseteq V(P_n)$ and the universal vertex $u$. Also, every vertex in $\partial(T)$ that is not an end-vertex of $V(P_n)$ has at most one neighbor in $\partial(T)$. By Theorem 3.9, $T$ is a $TOA$ in $F_n$. Now, we want to show that $T$ is the minimum $TOA$ in $F_n$. Suppose $T$ is not the minimum $TOA$ in $F_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(F_n)$ such that $|T_0| < |T| = \frac{n+4}{3}$. Without loss of generality, suppose $|T_0| = \frac{n+4}{3} - 1$. Consider the following cases:
***(i)*** $v \notin T_0$ for some $v \in V(P_n)$
If $v \notin T_0$ for some $v \in V(P_n)$, then there exists $v \in \partial(T_0)$ such that $\deg_{T_0}(v) = 1$, the universal vertex $u$, if $v \in \partial(T_0)$ is an end-vertex of $V(P_n)$ or $\deg_{V(F_n)\setminus T_0}(v) = 2$, its adjacent vertices in $\partial(T_0) \subseteq V(P_n)$, if $v \in \partial(T_0)$ is not. And so, $|N[v] \cap T_0| = 1$ but $|N[v] \setminus T_0| = 2$ if $v$ is an end-vertex of $V(P_n)$ or $|N[v] \cap T_0| = 1$ but $|N[v] \setminus T_0| = 3$ if $v$ is not. But both are contradictions. Hence, $T_0$ is not an offensive alliance in $F_n$.
***(ii)*** $u \notin T_0$
If $u \notin T_0$, then $F_n[T_0]$ contains isolated vertices of $V(P_n)$, a contradiction to the assumption that $T_0$ is a $TOA$ in $F_n$. Hence, $T_0$ is not a $TOA$ in $F_n$.
Therefore, $a_{to}(F_n) = |T| = \frac{n+4}{3}$. □

**Theorem 3.11.** *Let $G = W_n$ of order $n + 1$, $n \geq 3$, and $\emptyset \neq T \subseteq V(C_n) \subseteq V(W_n)$. Then $T$ is a $TOA$ in $W_n$ if and only if one of the following holds:*

***(i)*** $T = V(C_n)$

***(ii)*** $W_n[V(C_n) \setminus T]$ *is an empty graph in $C_n$ and $W_n[T]$ has no isolated vertex.*

*Proof.* Let $\emptyset \neq T \subseteq V(C_n) \subseteq V(W_n)$ be a $TOA$ in $W_n$. Clearly, $T = V(C_n)$. Now, suppose $W_n[V(C_n) \setminus T]$ is not an empty graph in $C_n$ or $W_n[T]$ has an isolated vertex. If $W_n[V(C_n) \setminus T]$ is an not empty graph in $C_n$, then there exist vertices $u, v \in V(C_n) \setminus T$ such that $u$ and $v$ are neighbors in $V(C_n) \setminus T$. And so, $|N[v] \cap T| = 1$ but $|N[v] \setminus T| = 3$, a contradiction to our assumption that $T$ is a $TOA$ in $W_n$. Thus, $W_n[V(C_n) \setminus T]$ is an empty graph in $C_n$. On the other hand, if $W_n[T]$ has an isolated vertex, then there exists a vertex $w \in T$ such that $w \notin N[x]$ for some $x \in T$, a contradiction. Thus, $W_n[T]$ has no isolated vertex. Therefore, $W_n[V(C_n) \setminus T]$ is an empty graph in $C_n$ and $W_n[T]$ has no isolated vertex.

Conversely, suppose (i) holds. Then for the universal vertex $u \in \partial(T)$ in $G_T$, $|N[v] \cap T| = n \geq |N[v] \setminus T| = 1$. Hence, $T$ is an offensive alliance in $W_n$. Also, since $T = V(C_n)$, clearly, every vertex in $T$ has at least one neighbor in $T$. Thus, $T$ is a $TOA$ in $W_n$. On one hand, suppose (ii) holds. Since $W_n[V(C_n) \setminus T]$ is an empty graph in $C_n$, then for vertex $v \in V(C_n) \setminus T$, $|N[v] \cap T| = 2 \geq |N[v] \setminus T| = 2$. Thus, $T$ is an offensive alliance in $W_n$. Also, $W_n[T]$ has no universal vertex, clearly, $T$ is a $TOA$ in $W_n$. □

**Theorem 3.12.** *Let $G = W_n$ of order $n + 1$, $n \geq 3$, and $\emptyset \neq T \subseteq V(W_n)$ such that $T = T_1 \cup T_2$, $T_1 \subseteq V(C_n), T_2 \subseteq G_T$. Then $T$ is a $TOA$ if and only if every vertex in $\partial(T)$ has at most one neighbor in $\partial(T)$.*

*Proof.* Let $\emptyset \neq T \subseteq V(W_n)$ be a $TOA$ in $W_n$. Suppose on contrary that every vertex in $\partial(T)$ has more than one neighbor in $\partial(T)$. Let $v \in \partial(T)$ such that $\deg_{V(W_n) \setminus T}(v) = 2$. Then $|N[v] \cap T| = 1$ since $T_1 = G_T$. But $|N[v] \setminus T| = 3$ which is itself and its adjacent vertices in $\partial(T) \subseteq V(C_n)$, a contradiction. Thus, every vertex in $\partial(T)$ has at most one neighbor in $\partial(T)$.

Now, conversely, suppose every vertex in $\partial(T)$ has at most one neighbor in $\partial(T)$. If vertex $v \in \partial(T)$ has no neighbor in $\partial(T)$, then clearly, $|N[v] \cap T| = 3 \geq |N[v] \setminus T| = 1$. If vertex $v \in \partial(T)$ has one neighbor in $\partial(T)$, then clearly, $|N[v] \cap T| = 2 \geq |N[v] \setminus T| = 2$. Thus, $T$ is an offensive alliance in $W_n$. Moreover, since the isolated vertex $u \in T_2 \subseteq G_T$ is in $T$, every vertex $v \in T_1 \subseteq V(C_n)$ in $T$ has at least one neighbor in $T$. Therefore, $T$ is a $TOA$ in $W_n$. □

**Corollary 3.13.** *For a wheel graph $W_n$ of order $n + 1$ where $n \geq 3$,*

$$a_{to}(W_n) = \begin{cases} \frac{n+3}{3}, & \text{if } n \equiv 0 \text{(mod 3)} \\ \frac{n+5}{3}, & \text{if } n \equiv 1 \text{(mod 3)} \\ \frac{n+4}{3}, & \text{if } n \equiv 2 \text{(mod 3)} \end{cases}$$

.

*Proof.* Let $\emptyset \neq T \subseteq V(W_n)$ such that $T = \{v_1, v_2, ..., v_{n-1}, v_n, v_1\} \cup \{u\}$ where $\{v_1, v_2, ..., v_{n-1}, v_n, v_1\} \in T_1 \subseteq V(C_n)$ and $\{u\} \in T_2 \subseteq G_T$ for $n \geq 3$ be a $TOA$ in $W_n$. Consider the following cases:

Case 1: n ≡ 0 (mod 3)

Choose $T = \{v_1, v_4, ..., v_{3k-2}, ..., v_{n-2}\} \cup \{u\}$ where $k = \frac{n}{3}$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{n+3}{3}$. Now, $\partial(T) = \{v_2, v_3, v_5, v_6, ..., v_{n-1}, v_n\}$. Clearly, every vertex in $\partial(T)$ has at most one neighbor in $\partial(T)$.

By Theorem 3.12, $T$ is a $TOA$ in $W_n$. Now, we want to show that $T$ is the minimum $TOA$ in $W_n$. Suppose $T$ is not the minimum $TOA$ in $W_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(W_n)$ such that $|T_0| < |T| = \frac{n+3}{3}$. Without loss of generality, suppose $|T_0| = \frac{n+3}{3} - 1$. Consider the following cases:

***(i)*** $v_{3k-2} \notin T_0$ for some $v_{3k-2} \in V(C_n)$

If $v_{3k-2} \notin T_0$ for some $v_{3k-2} \in V(C_n)$, then $v_{3k-2} \in \partial(T_0)$ such that $\deg_{V(W_n)\setminus T_0}(v_{3k-2}) = 2$. And so, $|N[v_{3k-2}] \cap T_0| = 1$ but $|N[v_{3k-2}] \setminus T_0| = 3$, a contradiction. Hence, $T_0$ is not an offensive alliance in $W_n$.

***(ii)*** $u \notin T_0$

If $u \notin T_0$, then $W_n[T_0]$ contains isolated vertices of $V(C_n)$, a contradiction to the assumption that $T_0$ is a $TOA$ in $W_n$. Hence, $T_0$ is not a $TOA$ in $W_n$.

Therefore, $a_{to}(W_n) = |T| = \frac{n+3}{3}$.

Case 2: n $\equiv$ 1 (mod 3)

Choose $T = \{v_1, v_4, ..., v_{3k-2}, ..., v_{n-3}, v_{n-1}\} \cup \{u\}$ where $k = \frac{n-1}{3}$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{n+5}{3}$. Now, $\partial(T) = \{v_2, v_3, v_5, v_6, ..., v_{n-2}, v_n\}$. Then, every vertex in $\partial(T)$ has at most one neighbor in $\partial(T)$. By Theorem 3.9, $T$ is a $TOA$ in $W_n$. Now, we want to show that $T$ is the minimum $TOA$ in $W_n$. Suppose $T$ is not the minimum $TOA$ in $W_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(W_n)$ such that $|T_0| < |T| = \frac{n+5}{3}$. Without loss of generality, suppose $|T_0| = \frac{n+5}{3} - 1$. Consider the following cases:

***(i)*** $v_{3k-2} \notin T_0$ for some $v_{3k-2} \in V(C_n)$

If $v_{3k-2} \notin T_0$ for some $v_{3k-2} \in V(C_n)$, then $v_{3k-2} \in \partial(T_0)$ such that $\deg_{V(W_n)\setminus T_0}(v_{3k-2}) = 2$. And so, $|N[v_{3k-2}] \cap T_0| = 1$ but $|N[v_{3k-2}] \setminus T_0| = 3$, a contradiction. Hence, $T_0$ is not an offensive alliance in $W_n$.

***(ii)*** $u \notin T_0$

If $u \notin T_0$, then $W_n[T_0]$ contains isolated vertices of $V(C_n)$, a contradiction to the assumption that $T_0$ is a $TOA$ in $W_n$. Hence, $T_0$ is not a $TOA$ in $W_n$.

Therefore, $a_{to}(W_n) = |T| = \frac{n+5}{3}$.

Case 3: n $\equiv$ 2 (mod 3)

Choose $T = \{v_1, v_4, ..., v_{3k-2}, ..., v_{n-1}\} \cup \{u\}$ where $k = \frac{n-2}{3}$, $k \in \mathbb{Z}^+$. Then $|T| = \frac{n+4}{3}$. Now, $\partial(T) = \{v_2, v_3, v_5, v_6, ..., v_{n-2}, v_n\}$. Then, every vertex in $\partial(T)$ has at most one neighbor in $\partial(T)$. By Theorem 3.9, $T$ is a $TOA$ in $W_n$. Now, we want to show that $T$ is the minimum $TOA$ in $W_n$. Suppose $T$ is not the minimum $TOA$ in $W_n$. Then there exists a $\emptyset \neq T_0 \subseteq V(W_n)$ such that $|T_0| < |T| = \frac{n+4}{3}$. Without loss of generality, suppose $|T_0| = \frac{n+4}{3} - 1$. Consider the following cases:

***(i)*** $v_{3k-2} \notin T_0$ for some $v_{3k-2} \in V(C_n)$

If $v_{3k-2} \notin T_0$ for some $v_{3k-2} \in V(C_n)$, then $v_{3k-2} \in \partial(T_0)$ such that $\deg_{V(W_n)\setminus T_0}(v_{3k-2}) = 2$. And so, $|N[v_{3k-2}] \cap T_0| = 1$ but $|N[v_{3k-2}] \setminus T_0| = 3$, a contradiction. Hence, $T_0$ is not an offensive alliance in $W_n$.

***(ii)*** $u \notin T_0$

If $u \notin T_0$, then $W_n[T_0]$ contains isolated vertices of $V(C_n)$, a contradiction to the assumption that $T_0$ is a $TOA$ in $W_n$. Hence, $T_0$ is not a $TOA$ in $W_n$.

Therefore, $a_{to}(W_n) = |T| = \frac{n+4}{3}$. □

# 4 CONCLUSIONS

In this article, total offensive alliances in path graphs, cycle graphs, complete graphs, star graphs, fan graphs, and wheel graphs are characterized. Moreover, the total offensive alliance number is also identified. As future line of research, we intend to investigate the total offensive alliances and total offensive alliance number for other graph families.

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