Original research article

IMPACT OF DIFFERENTIAL EQUATIONS ON TRANSFER FUNCTIONS, IMPULSE AND FREQUENCY RESPONSES IN FILTERS

Abstract

For some examples of filters, defined by the "input-output" correspondence, a correspondence that can be translated in many cases by linear differential equations (first or second order filters), we specify the notions of step response, impulse response, transfer function, frequency response. We also specify the qualities of the filters considered from the point of view of causality, stability and rationality. To avoid any risk of confusion, some usual definitions and criteria are repeated below.

Keywords: differential equations, step response, impulse response, transfer function, frequency response.

INTRODUCTION

I. FILTER CONCEPT

Consider a physical system that receives input signals $t \longmapsto e(t)$and delivers output signals:$t \longmapsto s(t)$.

To represent this correspondence between input signal and output signal, we can define a mathematical operator ϕacting on the admissible signals, such that for any input $e$and the $s$associated output verifies

$s = \phi(e)$. (1)

Assuming that the set $\mathcal{A}$of admissible signals is a vector space of functions or distributions also equipped with a notion of convergence for the sequences (convergence in the sense of functions or in the sense of distributions), this operator ϕcan present remarkable properties:

- linearity: the input signal $\alpha e_1 + \beta e_2$is transformed into an output signal $\alpha\phi(e_1) + \beta\phi(e_2)$.
- of continuity in the following sense: if, when $n$tends towards $+\infty$,an input signal of type $\sum_0^n e_k$admits a limit $e$at $\mathcal{A}$, then the output signal $\sum_0^n \phi(e_k)$also admits a limit $s$and $s = \phi(e)$.4

The same is true for saying that the sum of a converging series of admissible signals is transformed into the sum of the series of associated outputs.

- of temporal invariance: if $s = \phi(e)$then the translated signal

$e_t: t \longmapsto e(t-\tau)$turns into translated $s_t: t \longmapsto s(t-\tau)$, in other words$s_t = \phi(e_t)$

When a system has these three properties, it is said to be a "filter" or "linear filter."

II. DEFINITION

$i)$a priori $\mathbb{L}^1$function in $\varkappa$,or in $\mathbb{L}^2$, response of the stepped system$\varkappa$.

$ii)$Transfer function and frequency response: if, formally, we denote $E$and $S$the bilateral Laplace transforms of the input $e$and output ,$s$the "input-output" relationship in the filter translates to:

$$Q(z)S(z) = P(z)S(z) \text{(2)}$$

from where $$S(z) = \frac{P(z)}{Q(z)}E(z). \text{(3)}$$

The function H defined by :$H(z) = \frac{P(z)}{Q(z)}$, to which we possibly associate its domain of validity, is called the transfer function of the filter. Often; H is rational, $P\ et\ Q$are then polynomials. We will mainly examine the case where $\deg(P) \leq \deg(Q)$.When, on the axis of imaginaries, there is no pole of H, we can define the frequency response $G$by

$$G(\lambda) = H(2i\pi\lambda). \text{(4)}$$

$iii)$transformable a priori $\mathcal{F}$,function or distribution by h,which constitutes the response of the filter to the input. δ.It is also the derivative, in the sense of distributions, of the step response $\varkappa$and the Fourier transform $h$of the frequency response.

$iv)$Causality: we have seen that this property depends on the class of signals admissible at the input of the filter. In all cases, causality is expressed, when it $h$is known, by $\forall t > 0, h(t) = 0$;in other words by the causality of h.In the case where H is a rational fraction with $\deg(P) \leq \deg(Q)$,this amounts to saying that H is a unilateral transform whose domain of validity contains the axis of imaginaries, therefore that all the poles of H are of strictly negative real parts. When this property is verified, the bilateral Laplace transform of $h$merges with its unilateral transform; it is H whose summability domain is the half-plane to the right of the pole of the smallest real part.

$v)$Stability: In the case where H is a rational fraction, we use the condition on the poles, otherwise, in the general case, there remains the definition which imposes bounded outputs for bounded inputs.

$vi)$Dynamic filter: it is a causal and stable filter whose transfer function is also rational.

III. Applications

1. “Time Average” Filter

We consider a system which, for any functional input $t \longmapsto e(t)$, corresponds to the output $s$defined by:

$$s(t) = \frac{1}{A}\int_{t-A}^{t} e(\theta)d\theta \quad avec\ A > 0 \qquad (1.1)$$

a. Let us show that this system has the properties of linearity and time invariance.

From the linearity of the integral over each of the intervals $[t - A, t]$, we show the proof that the system is linear. Consider a $e_a: t \longmapsto e(t-a)$. The corresponding output being provisionally denoted y,we have:

$$y(t) = \frac{1}{A}\int_{t-A}^{t} e(\theta - a)d\theta \quad (1.2)$$

A simple change of variable provides, $s$as output, the e,desired result:

$$y(t) = \frac{1}{A}\int_{t-a-A}^{t-a} e(u)du = s(t-a), \quad (1.3)$$

Let a be the translated index of the output. e.We conclude that the system is invariant. The system is therefore a linear filter. We do not examine continuity issues for the filters studied here.

b. Let us determine the step response of this filter. Let us deduce its impulse response $h$and then the transfer function H. Let us show that h can be extended, by means of a power series, to the entire complex field and deduce the frequency response .$G(\lambda)$Is this filter causal?

Let's start by using a graphical representation:

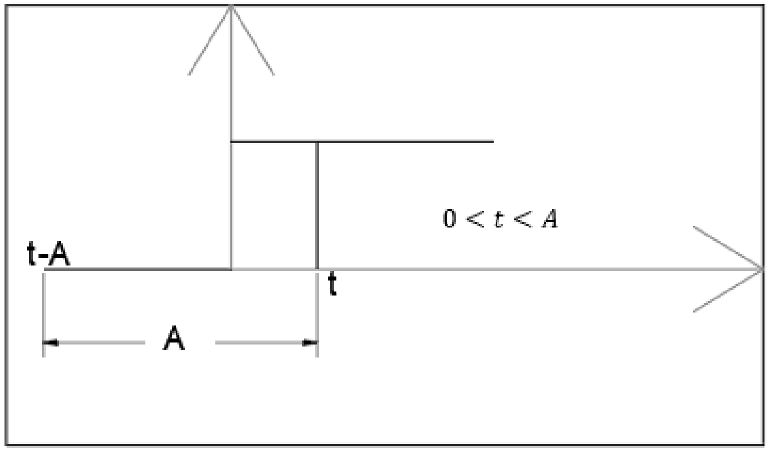

Figure 1. Graphical representation of the time averaging filter

This involves calculating the area of a rectangle. We find:

$$\begin{cases} si\ t < 0, u(t) = 0 \\ si\ 0 \leq t \leq A, u(t) = \dfrac{t}{A} \\ si\ t > A, u(t) = 1 \end{cases}$$

This response being causal, the filter studied is causal. The impulse response is the derivative, in the sense of distributions, of this step response. The function being continuous, it suffices to calculate in the sense of functions this derivative which is easily expressed using the step:

$$h(t) = \frac{1}{A}\left(\varkappa(t) - \varkappa(t-A)\right) \quad (1.4)$$

The Laplace transform of this causal function $h$provides us

$$H(p) = \frac{1}{AP}\left(1 - e^{-PA}\right) \quad (1.5)$$

This formula is not valid a priori at point 0, but, using the entire series development of the exponential which is valid in the entire complex plane, we can extend by the series:

$$H(p) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}(pA)^{n-1}}{n!}, \quad (1.6)$$

valid everywhere. No difficulty in replacing $p$by $2i\pi\lambda$to obtain the frequency response from which:

$$G(\lambda) = \frac{1-e^{-2i\pi\lambda A}}{2i\pi\lambda A} \quad (1.7)$$

It should be noted that the direct calculation of the Fourier transform is not simple since it would involve the unit step transformation. We would find:

$$G(\lambda) = \frac{1}{A}(1 - \exp(-2i\pi A\lambda)\mathcal{F}(\varkappa)(\lambda) = \frac{1 - \exp(-2i\pi A\lambda)}{A}\left(\frac{\delta}{2} + \frac{1}{2i\pi}V_p\left(\frac{1}{\lambda}\right)\right) \quad (1.8)$$

c. Let us check that this filter is well characterized by the relation:

$$s = e * h \quad (1.9)$$

Let's carry out e. the convolution $e * h$, defined by the classical formula , since these are functions:

$$(e * h)(t) = \frac{1}{A}\int_{-\infty}^{+\infty} e(t-u)\big(\varkappa(u) - \varkappa(u-A)\big)du \quad (1.10)$$

$$ou \quad (e * h)(t) = \frac{1}{A}\int_{0}^{A} e(t-u)du = \frac{1}{A}\int_{t-A}^{t} e(\theta)d\theta \quad (1.11)$$

which gives the output $s$associated with$e$.

d. Is this filter stable (in the strict sense, in the broad sense)? Is it a dynamic filter?

Since the function H is not rational, we cannot use the criterion on the poles. However, the definition of stability applies since $h$being with bounded support is summable. The function $h$being bounded, this results in stability in the broad sense. Since the function H is not rational, the filter is not dynamic.

2. Filter "translator"

Let an "input-output" system be such that it $e$is linked by s:

$$s(t) = e(t-a) \quad (2.1)$$

where$a$is a fixed number > 0. Let's deal with the previous questions for this filter.

a. It is simple to demonstrate the properties of linearity, continuity (while retaining the same notion of convergence) and invariance with respect to an integral and the domain of validity.
b. We take the bilateral transforms of two members of the "input-output" relation, we obtain

$$S(p) = \exp(-pa)E(p) \quad (2.2)$$

, therefore $H(p) = \exp(-pa)$the frequency response is therefore

$$\lambda \mapsto \exp(-2i\pi a\lambda). \quad (2.3)$$

The impulse response is therefore the Dirac distribution: $\delta_a$which is causal only if$a > 0$.

c. We note that the filter is stable in the broad sense since it $h$contains only a Dirac distribution. Stability in the strict sense is obvious because if an input is bounded, the output which is a translated output is also bounded. Finally, the filter is not dynamic since it is not stable in the strict sense.

3. Filter "R – C"

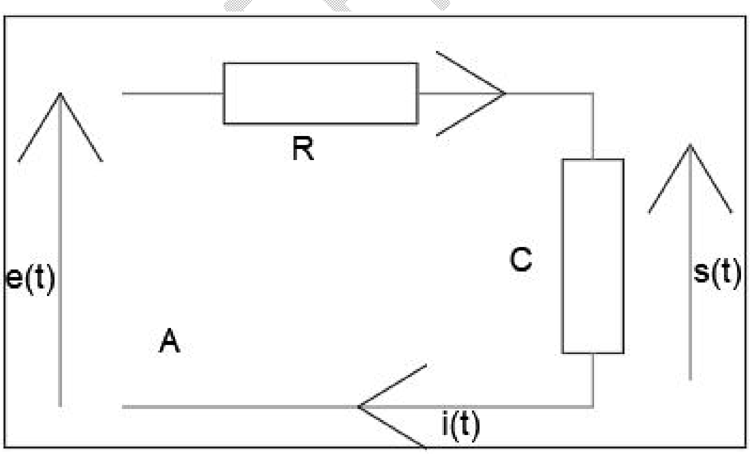

Figure 2 Graphical representation of the “R – C” time averaging filter

From this diagram, we are led to the "input-output" relationship which is expressed by the relationships:

$$\begin{cases} e(t) = Ri(t) + s(t) \\ i(t) = C\dfrac{ds}{dt}(t) \end{cases} \quad (3.1)$$

a. Let us find the output $s$as a function of the $e$assumed causality, by expressing it by an integral with one of the limits zero and an arbitrary constant. Let us calculate this constant. Let us show that the expression of $s$as an integral is the translation of a convolution product of by a $e$causal function that we will explain. Let us deduce the impulse response and the frequency response.

From this classical scheme we obtain the following differential equation link$e$And$s$.

$$RC\, s' + s = e \qquad (3.2)$$

Let us ask ourselves classically:

$$s(t) = v \exp\left(-\frac{t}{RC}\right), (3.3)$$

The equation deviates

$$v'(t) = \frac{1}{RC} \exp\left(\frac{t}{RC}\right) e(t), (3.4)$$

from where:

$$v(t) = K + \frac{1}{RC} \int_0^t \exp\left(\frac{\theta}{RC}\right) e(\theta) d\theta, (3.5)$$

which finally gives

$$s(t) = \exp\left(-\frac{t}{RC}\right)\left(K + \frac{1}{RC} \int_0^t \exp\left(\frac{\theta}{RC}\right)\right) e(\theta) d\theta. (3.6)$$

The initial condition which is given imposes, $s(\theta) = 0$and by continuity, from where $K = 0$. We can, by sliding the exponential under the integral, make a convolution appear:

$$s(t) = \frac{1}{RC} \int_0^t \exp\left(-\frac{t-\theta}{RC}\right) e(\theta) d\theta (3.7)$$

$$= \frac{1}{RC} \int_{-\infty}^{+\infty} e(\theta) \varkappa(t-\theta) \exp\left(-\frac{t-\theta}{RC}\right) d\theta (3.8)$$

We deduce that the impulse response $h$is defined by the causal function:

$$h(t) = \frac{1}{RC} \varkappa(t) \exp\left(-\frac{t}{RC}\right) (3.9)$$

The frequency response is then

$$G(\lambda) = \frac{1}{1+2i\pi\lambda RC} \quad (3.10)$$

by direct calculation

b. Let us resume the calculation of the transfer function by directly transforming the equations of the system. Let us then determine its impulse response.

With our assumptions, we transform the two equations of the circuit by; the capital letters denote the Fourier transforms, we have:

$$E = RI + S \quad et\ I = 2i\pi\lambda CS, \quad (3.11)$$

from where

$$E(\lambda) = (RC(2i\pi\lambda) + 1)S(\lambda). \quad (3.12)$$

The advantage of this method is to obtain the frequency response immediately.

$$G(\lambda) = \frac{1}{1+2i\pi\lambda RC} \quad (3.13)$$

It remains to be shown that this function is indeed a Fourier transform, but it is true, at least in the sense of distributions. It is in fact a continuous and bounded function. We use the transformation $\bar{\mathcal{F}}$, we then obtain the impulse response above.

By transforming the "input-output" link equation using the bilateral transformation, we obtain:

$$(RCp + 1)S(p) = E(p) \quad (3.14)$$

from where

$$H(p) = \frac{1}{RCp+1} \quad (3.15)$$

The only pole of this fraction has the affix $-\frac{1}{RC}$, therefore a strictly negative real number.

We deduce that the filter is causal. The Laplace transform H is then the unilateral transform and we find the impulse response.

$$h(t) = \varkappa(t)\exp\left(-\frac{t}{RC}\right) \quad (3.16)$$

c. Let's check if the filter is stable and dynamic

In fact, the filter is stable in the broad and strict sense and it is a dynamic filter.

4. Filtered $''R-L-C''$

This filter translates the “input-output” relationship into the circuit below:

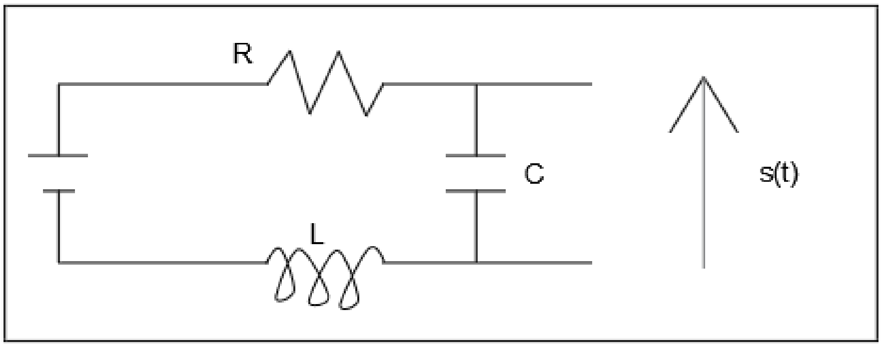

Figure 3The filter translates the “input-output” relationship into the circuit

This relationship being governed by a second order differential equation:

$$LCs'' + RC\,s' + s = e \quad (4.1)$$

From this differential equation we say that the filter is second order.

a. Using the direct method, let us calculate the transfer function and the frequency response. Is the filter causal?
   Using the $c$previous one (3°), the same method leads to the transfer function H.

We find

$$H(p) = \frac{1}{LCp^2+RC\,p+1} \quad (4.2)$$

Due to the sign of the coefficients, the roots of the denominator are either strictly negative real or complex with strictly negative real parts. We deduce that the filter is causal. The frequency response is defined by:

$$G(\lambda) = \frac{E}{S} = \frac{1}{-4\pi^2\lambda^2 LC+2i\pi\lambda RC+1} \quad (4.3)$$

b. Let us determine, in all cases, the impulse response. Without doing all the calculations, we will give the different forms of this impulse response by passing either by an inverse Fourier transform, or by an inverse Laplace transform.

The denominator is factorized into a product of two factors, and the impulse response is easily obtained using a dictionary of Fourier images. The residue theorem can also be used to calculate Fourier transforms of rational fractions. For two distinct roots, we obtain a function of the type:

$$h(t) = \varkappa(t)(Aexp\,(z_1 t) + b\exp(z_2 t);(4.4)$$

in the case of a double root, a function of the type:

$$h(t) = At\,\varkappa(t)\exp\left(-\frac{R}{2L}\right)(4.5)$$

c. Let's study stability in the strict sense and in the broad sense. Is it a dynamic filter?

Based on our assumptions and results above, the filter is stable and it is a dynamic filter.

5. Study of another second order filter

Given that $a > 0$,we consider the system in which $e$and $s$are related by:

$$-a^2 s'' + s = e(5.1)$$

5.1. We place ourselves in the conditions given in the definitions of the preamble.

a. Let us formally calculate the transfer function and deduce that, under these conditions, the filter is not causal.

After calculations, we find for the transfer function:

$$H(p) = \frac{1}{-a^2p^2+1} = -\frac{w^2}{p^2-w^2}(5.2)$$

The roots of the denominator being opposite $(w, -w)$, the condition of causality is therefore not verified.

b. By solving, in an elementary way, to the left, then to the right of 0, the differential equation $-a^2u'' + u = \varkappa$where $\varkappa$is the unitary step, we determine the continuous solution, with continuous derivative which in $\mathbb{L}^1$(a unique solution which is therefore the response to a step. We can pose: $w = \frac{1}{a}$).We find this result using the bilateral transformation. Let us deduce the impulse response.

To do this, let's start by solving the equation $-a^2u'' + u = 0$.on .$]-\infty, 0[$We find the following solution:

$$u(t) = A\exp(wt) + B\exp(-wt)(5.3)$$

while on $]0, +\infty[$, where we have to solve:$-a^2u'' + u = 1$,

we obtain:

$$u(t) = A'\exp(wt) + B'\exp(-wt) + 1.(5.4)$$

The condition that the solution must be summable requires:

$$A' = B = 0$$

The condition of continuity at point 0 then imposes:

$$-B' = A$$

Finally, the solution u is defined by:

$$\forall t > 0,\ u(t) = 1 - \frac{1}{2}\exp(-wt)(5.5a)$$

$$\forall t < 0,\ u(t) = \frac{1}{2}\exp(wt)(5.5b)$$

The impulse response is then given by the derivative:

$$h(t) = \frac{\omega}{2}\exp(-wt)(5.6)$$

confirming non-causality.

The transfer function being H, we must consider the domain of H as being that which contains the axis of imaginaries. By using the bilateral transformation and its inverse, we are led to decompose H. the transform of $h_+$is $\frac{w}{2}.\frac{1}{p+w}$, the causal part of $h$is therefore

$$t \longmapsto \frac{w}{2}\exp(-wt)(5.7)$$

The transformation of $h_$is

$$-\frac{w}{2}.\frac{1}{-p-w} = \frac{w}{2}.\frac{1}{p+w}(5.8)$$

Which brings us to the anti-causal part:

$$t \longmapsto \frac{w}{2}\exp(-wt) \quad (5.9)$$

We find the previous results.

5.2. Let us determine, elementary for example, the causal solution u of $-a^2u'' + u = \varkappa$ and show that the solution found is no longer transformable by $\mathcal{F}$, which explains why, in the classical framework, this solution must be rejected. However, if we take as admissible signals exponential type signals which remain in the space E, this solution becomes acceptable, the filter becomes causal. Is it stable?

To solve this application, we consider a modification of the admissible signals.

First, if we impose that the solution u is causal, the solution satisfies zero initial conditions at point 0. Using the general form of the solution on $]0, +\infty[$, namely:

$$u(t) = A' \exp(wt) + B' \exp(-wt) + 1, \quad (5.10)$$

We obtain:

$$A' + B' + 1 = 0 \;\; et \;\; w(A' - B') = 0$$

The step response is therefore:

$$u(t) = \varkappa(t)\big(1 - ch(wt)\big). \quad (5.11)$$

We deduce the impulse response, which we could have obtained by the unilateral Laplace transform:

$$h(t) = -w\,\varkappa(t)sh(wt) \quad (5.12)$$

Second, with admissible signals in the space E, the filter is causal, but the stability criterion on the poles is no longer verified, this filter is unstable in the strict sense or even in the broad sense, moreover the function h is neither integrable nor bounded.

6. “Resonator” filter

Let's go back to the previous questions for the filter governed by:

$$a^2s'' + s = e \quad (6.1)$$

Let us determine the transfer function and show that the filter is not causal. Let us place ourselves in the conditions where the admissible signals are in E (Laplace original space). Let us then calculate by elementary methods the step response and then the impulse response (we can also use $\mathcal{L}$the transformation directly to find h)a frequency response).

a. Considering this differential equation, the transfer function is defined by:

$$H(p) = \frac{1}{a^2p^2+1} = \frac{w^2}{p^2+w^2} \quad (6.2)$$

Its poles are located on the imaginary axis. There is no causality. This case is much more delicate than the previous one. If we solve the index equation by elementary methods, we find sinusoidal functions which are not Fourier transformable in the sense of functions.

b. If we impose that the solution $u$is causal, the function $u$being continuous at point 0, the initial conditions are zero at this point.

The solution being, on $]0, +\infty[$:

$$u(t) = Acos\,(wt) + B\sin(wt) + 1 \quad (6.3)$$

we deduce:

$$u(t) = \varkappa(t)(1 - \cos(wt)) \quad (6.4)$$

and its derivative which provides the impulse response:

$$h(t) = w\,\varkappa(t)\sin(wt) \quad (6.5)$$

We could just as well have used $\mathcal{L}$to find this function directly. h.We note that this function is not summable, which confirms that it does not admit a Fourier transform in the sense of functions. Its Fourier transform in the sense of distributions is calculated using that of $\mathcal{U}$(use of the relation of the function "sign ($[sgn]$)and the unit step $\mathcal{U}$)and the properties of multiplications by exponentials. We would thus find a principal value associated with the function.

$$H(2i\pi\lambda) = \frac{1}{1-4\pi^2\lambda^2a^2}, \quad (6.6)$$

Which verifies, a posteriori, the calculations made above.

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The details of the use of AI are given below:

1.

2.

3.

Conclusion

The notion of stability, in the study of systems, particularly in that of linear filters and in that of servo-controlled systems, is one of the most important. Generally speaking, a system is said to be stable if, removed from its equilibrium position, it tends to return; it is unstable if it tends to move away from it. The notion of transfer function for systems governed by linear differential equations with constant coefficients makes it possible to translate this stability by algebraic conditions (the transformation of the place) or graphical conditions.

It will often be necessary to go beyond this simple framework when the function of the second member is not continuous and especially when this function is replaced by a distribution. In either case, it will be imperative to modify the definition of the solutions of a differential equation while knowing that many filters are governed by such differential equations where the unknown function models the output of the filter, the second member representing the input signal.

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