Original Research Article

"ON THE IMPACT OF DIFFERENTIAL EQUATIONS ON INDEX RESPONSE, TRANSFER FUNCTION IMPULSE RESPONSE AND FREQUENCY RESPONSE"

Abstract

For some examples of filters, defined by the "input-output" correspondence, a correspondence which can be translated in many cases by linear differential equations (first or second order filters), we specify the notions of step response, impulse response, transfer function, frequency response. We also specify the qualities of the filters considered from the point of view of causality, stability and rationality. To avoid any risk of confusion, some usual definitions and criteria are repeated below.

Keywords: Differential equations, step response, impulse response, transfer function, frequency response.

INTRODUCTION

I. FILTER CONCEPT

Consider a physical system that receives input signals $t \longmapsto e(t)$ and delivers output signals: $t \longmapsto s(t)$.

To represent this correspondence between input signal and output signal, we can define a mathematical operator ϕ acting on admissible signals, such that for any input e and the s associated output verifies $s = \phi(e)$.

Assuming that the set $\mathcal{A}$ of admissible signals is a vector space of functions or distributions also equipped with a notion of convergence for the sequences (convergence in the sense of functions or in the sense of distributions), this operator ϕ can present some remarkable properties:

- linearity: the input signal $\alpha e_1 + \beta e_2$ is transformed into an output signal $\alpha\phi(e_1) + \beta\phi(e_2)$.
- of continuity in the following sense: if, when n tends towards $+\infty$, an input signal of type $\sum_0^n e_k$ admits a limit e in $\mathcal{A}$, then the output signal $\sum_0^n \phi(e_k)$ also admits a limit s and $s = \phi(e)$.4

It is the same to say that the sum of a converging series of admissible signals is transformed into the sum of the series of associated outputs.

- of time invariance: if $s = \phi(e)$then the translated signal

$e_t: t \longmapsto e(t-\tau)$transforms into the translated $s_t: t \longmapsto s(t-\tau)$, in other words$s_t = \phi(e_t)$

When a system has these three properties, it is said to be a "filter" or "linear filter".

II. DEFINITION

$i)$a priori $\mathbb{L}^1$function in $\varkappa$,or in $\mathbb{L}^2$, response of the echelon system$\varkappa$.

$ii)$Transfer function and frequency response: if, formally, we denote $E$and $S$the bilateral Laplace transforms of the input $e$and output ,$s$the "input-output" relationship in the filter translates to:

$$Q(z)S(z) = P(z)S(z)$$

from where $$S(z) = \frac{P(z)}{Q(z)}E(z).$$

The function H defined by :$H(z) = \frac{P(z)}{Q(z)}$, to which we possibly associate its domain of validity, is called the transfer function of the filter. Often; H is rational, $P\ et\ Q$are then polynomials. We will mainly examine the case where $\deg(P) \leq \deg(Q)$.When, on the axis of imaginaries, there is no pole of H, we can define the frequency response $G$by

$$G(\lambda) = H(2i\pi\lambda).$$

$iii)$Impulse response: it is the function or the h,a priori distribution transformable by $\mathcal{F}$,which constitutes the response of the filter to the input. δ.It is also the derivative, in the sense of distributions, of the step response $\varkappa$and the Fourier transform of $h$is the frequency response.

$iv)$Causality: we have been able to see that this property depends on the class of signals admissible at the input of the filter. In all cases, causality is expressed, when $h$is known, by $\forall t > 0, h(t) = 0$;in other words, by the causality of h.In the case where H is a rational fraction with $\deg(P) \leq \deg(Q)$,this is equivalent to saying that H is a unilateral transform whose domain of validity contains the axis of imaginaries, therefore that all the poles of H are of strictly negative real parts. When this property is verified, the bilateral Laplace transform of $h$merges with its unilateral transform; it is H with the summability domain being the half-plane to the right of the pole of the smallest real part.

$v)$Stability: In the case where H is a rational fraction, we use the condition on the poles, otherwise, in the general case, there remains the definition which imposes bounded outputs for bounded inputs.

$vi)$Dynamic filter: it is a causal and stable filter whose transfer function is also rational.

III. Applications

1. " Time average " filter

We consider a system which, for any functional input $t \longmapsto e(t)$, matches the output $s$defined by:

$$s(t) = \frac{1}{A}\int_{t-A}^{t} e(\theta)d\theta \quad avec\ A > 0$$

a. Let us show that this system has the properties of linearity and time invariance.

From the linearity of the integral on each of the intervals $[t-A, t]$, we show the evidence that the system is linear. Consider a translated $e_a : t \longmapsto e(t-a)$. The corresponding output being provisionally noted y,we have:

$$y(t) = \frac{1}{A}\int_{t-A}^{t} e(\theta - a)d\theta$$

A simple change of variable provides, $s$being the output associated with e,the desired result:

$$y(t) = \frac{1}{A}\int_{t-a-A}^{t-a} e(u)du = s(t-a),$$

Namely the translated index a of the output e.Let us conclude that the system is invariant. The system is therefore a linear filter. We do not examine any questions of continuity for the filters studied here.

b. Let us determine the step response of this filter. Let us deduce its impulse response $h$and then the transfer function H. Let us show that h can be extended, by means of an entire power series, to the entire complex field and deduce the frequency response .$G(\lambda)$Is this filter causal?

Let's start by using a graphical representation:

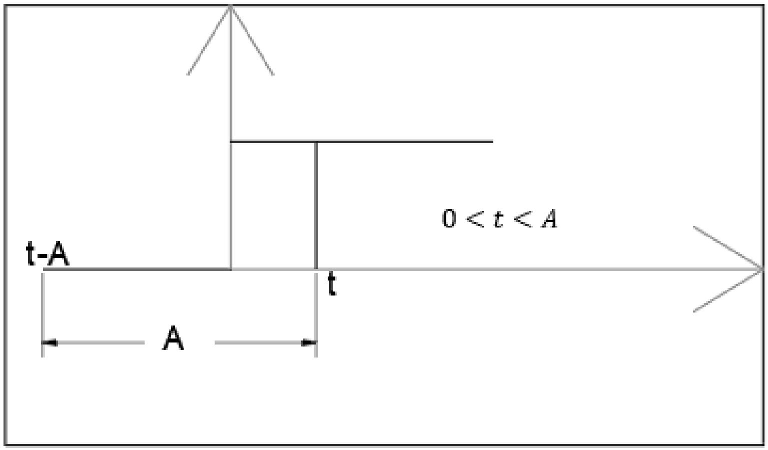

Figure 1. Graphical representation of Time average filter

This is to calculate the area of a rectangle. We find:

$$\begin{cases} si\ t < 0, u(t) = 0 \\ si\ 0 \leq t \leq A, u(t) = \dfrac{t}{A} \\ si\ t > A, u(t) = 1 \end{cases}$$

This response being causal, the filter studied is causal. The impulse response is the derivative, in the sense of distributions, of this step response. The function being continuous, it suffices to calculate in the sense of functions this derivative which is easily expressed using the step:

$$h(t) = \frac{1}{A}\ (\varkappa(t) - \varkappa(t - A))$$

The Laplace transform of this causal function $h$provides us

$$H(p) = \frac{1}{AP}(1 - e^{-PA})$$

This formula is not valid a priori at point 0, but, using the entire series development of the exponential which is valid in the entire complex plane, we can extend by the series:

$$H(p) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}(pA)^{n-1}}{n!},$$

valid everywhere. No difficulty in replacing $p$by $2i\pi\lambda$to obtain the frequency response from which:

$$G(\lambda) = \frac{1 - e^{-2i\pi\lambda A}}{2i\pi\lambda A}$$

It should be noted that the direct calculation of the Fourier transform is not simple since it would involve the unit step transform. We would find:

$$G(\lambda) = \frac{1}{A}(1 - \exp(-2i\pi A\lambda)\mathcal{F}(\varkappa)(\lambda)$$

$$= \frac{1 - \exp(-2i\pi A\lambda)}{A}\left(\frac{\delta}{2} + \frac{1}{2i\pi}V_p\left(\frac{1}{\lambda}\right)\right)$$

c. Let us check that this filter is well characterized by the relation:

$$s = e * h$$

Let e.us perform the convolution $e * h$, defined by the classical formula, since these are functions:

$$(e * h)(t) = \frac{1}{A}\int_{-\infty}^{+\infty} e(t-u)\big(\varkappa(u) - \varkappa(u-A)\big)du$$

ou $$(e * h)(t) = \frac{1}{A}\int_{0}^{A} e(t-u)du = \frac{1}{A}\int_{t-A}^{t} e(\theta)d\theta$$

which gives the output $s$associated with$e$.

d. Is this filter stable (in the strict sense, in the broad sense)? Is it a dynamic filter?

Since the function H is not rational, we cannot use the criterion on the poles. However, the definition of stability applies since $h$being of bounded support is summable. The function $h$being bounded, stability in the broad sense results. Since the function H is not rational, the filter is not dynamic.

2. Filter "translator"

Let an 'input-output' system be such that $e$is related to $s$by:

$$s(t) = e(t - a)$$

where $a$is a fixed number > 0. Let's deal with the previous questions for this filter.

a. It is simple to demonstrate the properties of linearity, continuity (while retaining the same notion of convergence) and invariance in the face of an integral and the domain of validity.
b. We take the bilateral transforms of two members of the "input-output" relation, we obtain

$S(p) = \exp(-pa)E(p)$,hence $H(p) = \exp(-pa)$the frequency response is therefore $\lambda \longmapsto \exp(-2i\pi a\lambda)$.

The impulse response is therefore the Dirac distribution: $\delta_a$which is causal only if$a > 0$.

c. We note that the filter is stable in the broad sense since $h$it contains only a Dirac distribution. Stability in the strict sense is obvious because if an input is bounded, the output which is a translated one is also bounded. Finally, the filter is not dynamic since it is not stable in the strict sense.

3. Filter "R – C"

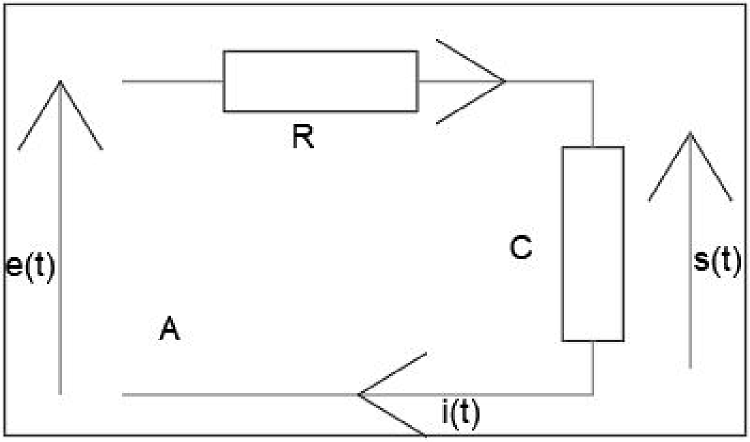

Figure 2Graphical representation of Time average Filter "R – C"

From this diagram, we are led to the "input-output" relationship which is expressed by the relationships:

$$\begin{cases} e(t) = Ri(t) + s(t) \\ i(t) = C\,\dfrac{ds}{dt}\,(t) \end{cases}$$

a. Let us find the output $s$as a function of $e$assumed causal, by expressing it with an integral with one of the limits zero and an arbitrary constant. Let us calculate this constant. Let us show that the expression of $s$as an integral is the translation of a convolution product of by a $e$causal function that we will make explicit. Let us deduce the impulse response and the frequency response.

From this classical scheme we obtain the following differential equation link$e$ And s.

$$RC\,s' + s = e$$

Let us pose classically:

$$s(t) = v\exp\left(-\frac{t}{RC}\right),$$

The equation deviates

$$v'(t) = \frac{1}{RC}\exp\left(\frac{t}{RC}\right)e(t),$$

from where :

$$v(t) = K + \frac{1}{RC}\int_0^t \exp\left(\frac{\theta}{RC}\right)e(\theta)d\theta,$$

which finally gives

$$s(t) = \exp\left(-\frac{t}{RC}\right)\left(K + \frac{1}{RC}\int_0^t \exp\left(\frac{\theta}{RC}\right)\right)e(\theta)d\theta.$$

The initial condition which is given imposes, $s(\theta) = 0$and by continuity, from where $K = 0$. We can, by sliding the exponential under the integral, make a convolution appear:

$$s(t) = \frac{1}{RC}\int_0^t \exp\left(-\frac{t-\theta}{RC}\right)e(\theta)d\theta$$

$$= \frac{1}{RC}\int_{-\infty}^{+\infty} e(\theta)\varkappa(t-\theta)\exp\left(-\frac{t-\theta}{RC}\right)d\theta$$

We deduce that the impulse response $h$is defined by the causal function:

$$h(t) = \frac{1}{RC}\varkappa(t)\exp\left(-\frac{t}{RC}\right)$$

The frequency response is then

$$G(\lambda) = \frac{1}{1 + 2i\pi\lambda RC}$$

by direct calculation

b. Let us resume the calculation of the transfer function by directly transforming the equations of the system. Let us then determine its impulse response.

With our assumptions, we transform the two equations of the circuit by; the capital letters designate the Fourier transforms, we have:

$$E = RI + S \quad et\ I = 2i\pi\lambda CS,$$

from where

$$E(\lambda) = (RC(2i\pi\lambda) + 1)S(\lambda).$$

The advantage of this method is to obtain the frequency response immediately.

$$G(\lambda) = \frac{1}{1 + 2i\pi\lambda RC}$$

It is still necessary to show that this function is indeed a Fourier transform, but this is true, at least in the sense of distributions. It is, in fact, a continuous and bounded function. We use the transformation $\bar{\mathcal{F}}$, we then obtain the impulse response above.

By transforming the 'input-output' link equation using the bilateral transformation, we obtain:

$$(RCp + 1)S(p) = E(p)$$

from where

$$H(p) = \frac{1}{RCp + 1}$$

The only pole of this fraction has affix $-\frac{1}{RC}$, therefore a strictly negative real number.

We deduce that the filter is causal. The Laplace transform H is then the unilateral transform and we find the impulse response.

$$h(t) = \varkappa(t) \exp\left(-\frac{t}{RC}\right)$$

c. Let's check if the filter is stable and dynamic

In fact, the filter is stable in the broad and strict sense and it is a dynamic filter.

4. Filtered$''\boldsymbol{R - L - C}''$

This filter translates the 'input-output' relationship into the circuit below:

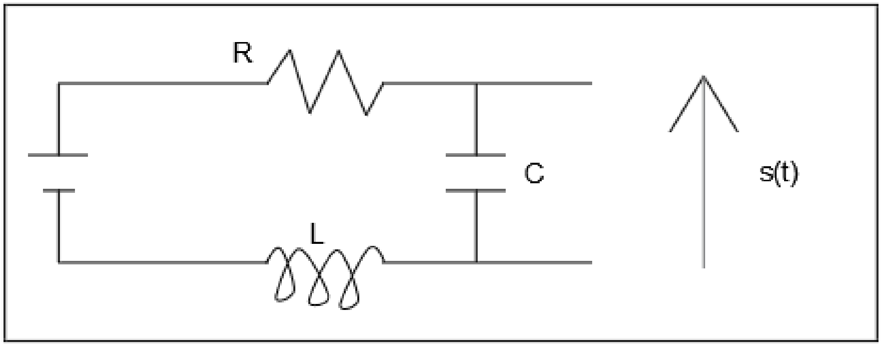

Figure 3

This relationship being governed by a second order differential equation:

$$LCs'' + RC\,s' + s = e$$

From this differential equation we say that the filter is second order.

a. Using the direct method, let us calculate the transfer function and the frequency response. Is the Filter causal?
   Using the $c$previous (3°), the same method leads to the transfer function H.

We find

$$H(p) = \frac{1}{LCp^2 + RC\,p + 1}$$

Due to the sign of the coefficients, the roots of the denominator are either strictly negative real or complex with strictly negative real parts. We deduce that the filter is causal. The frequency response is defined by:

$$G(\lambda) = \frac{E}{S} = \frac{1}{-4\pi^2\lambda^2 LC + 2i\pi\lambda RC + 1}$$

b. Let us determine, in all cases, the impulse response. Without doing all the calculations, we will give the various forms of this impulse response by passing either by an inverse Fourier transform or by an inverse Laplace transform.

We factorize the denominator into a product of two factors, and we easily arrive at the impulse response using a dictionary of Fourier images. We can also use the residue theorem to calculate the Fourier transforms of rational fractions. In the case of two distinct roots, we obtain a function of the type:

$$h(t) = \varkappa(t)(Aexp\,(z_1 t) + b\exp(z_2 t);$$

in the case of a double root, a function of the type:

$$h(t) = At\,\varkappa(t)\exp\left(-\frac{R}{2L}\right)$$

c. Let's study stability in the strict sense and in the broad sense. Is it a dynamic filter?

From our above assumptions and results, the filter is stable and it is a dynamic filter.

5. Study of another second order filter

Given $a > 0$,we consider the system in which $e$and $s$are related by:

$$-a^2 s'' + s = e$$

5.1. We place ourselves in the conditions given in the definitions of the preamble.

a. Let us formally calculate the transfer function and deduce that, under these conditions, the filter is not causal.

After calculations, we find for the transfer function:

$$H(p) = \frac{1}{-a^2p^2 + 1} = -\frac{w^2}{p^2 - w^2}$$

Since the roots of the denominator are opposites $(w, -w)$, therefore, the condition of causality is not verified.

b. By solving, in an elementary manner, to the left, then to the right of 0, the differential equation $-a^2u'' + u = \varkappa$where $\varkappa$is the unit step, we determine the continuous solution, with continuous derivative which in $\mathbb{L}^1$(a unique solution which is therefore the step response. We can pose: $w = \frac{1}{a}$).We find this result using the bilateral transformation. Let us deduce the impulse response.

To do this, let's start by solving the equation $-a^2u'' + u = 0$.on .$]-\infty, 0[$We find the following solution:

$$u(t) = A\exp(wt) + B\exp(-wt)$$

while on $]0, +\infty[$, where we have to solve:$-a^2u'' + u = 1$,

we get:

$$u(t) = A'\exp(wt) + B'\exp(-wt) + 1.$$

The condition that the solution must be summable requires:

$$A' = B = 0$$

The continuity condition at point 0 then imposes:

$$-B' = A$$

Finally, the solution u is defined by:

$$\forall t > 0, \qquad u(t) = 1 - \frac{1}{2}\exp(-wt)$$

$$\forall t < 0, \qquad u(t) = \frac{1}{2}\exp(wt)$$

The impulse response is then given by the derivative:

$$h(t) = \frac{\omega}{2}\exp(-wt)$$

confirming the non-causality.

The transfer function being H, we must consider the domain of H as being that which contains the axis of imaginaries. Using the bilateral transformation and its inverse, we are led to decompose H. the transform ofh_+ is $\frac{w}{2}.\frac{1}{p+w}$, the causal part of $h$is therefore

$$t \longmapsto \frac{w}{2}\exp(-wt)$$

The transform of $h_-$is

$$-\frac{w}{2}.\frac{1}{-p-w} = \frac{w}{2}.\frac{1}{p+w}$$

Which brings us to the anti-causal part:

$$t \longmapsto \frac{w}{2}\exp(-wt)$$

We find the previous results again.

5.2. Let us determine, elementary for example, the causal solution $u$of $-a^2u'' + u = \varkappa$and show that the solution found is no longer transformable by $\mathcal{F}$, which explains why, in the classical framework, this solution must be rejected. However, if we take as admissible signals exponential type signals which remain in the space E, this solution becomes acceptable, the filter becomes causal. Is it stable?

To solve this application, we consider a modification of the admissible signals.

First, if we impose the solution $u$to be causal, the solution satisfies zero initial conditions at point 0. Using the general form of the solution on $]0,+\infty[$, namely:

$$u(t) = A'\exp(wt) + B'\exp(-wt) + 1,$$

We obtain:

$$A' + B' + 1 = 0 \;\; et \;\; w(A' - B') = 0$$

The step response is therefore:

$$u(t) = \varkappa(t)\big(1 - ch(wt)\big).$$

We deduce the impulse response, which we could have obtained by the unilateral Laplace transform:

$$h(t) = -w\,\varkappa(t)sh(wt)$$

Second, with admissible signals in space E, the filter is causal, but the stability criterion on the poles is no longer verified, this filter is unstable in the strict sense or even in the broad sense, moreover the function $h$is neither integrable nor bounded.

6. “Resonator” filter

Let's revisit the previous questions for the filter governed by:

$$a^2s'' + s = e$$

Let us determine the transfer function and show that the filter is not causal. We place ourselves in the conditions where the admissible signals are in E (space of Laplace originals). Let us then calculate by elementary methods the step response then the impulse response (we can also use the transformation $\mathcal{L}$directly to find h)and frequency response.

a. Considering this differential equation, the transfer function is defined by:

$$H(p) = \frac{1}{a^2p^2 + 1} = \frac{w^2}{p^2 + w^2}$$

Its poles are located on the imaginary axis. There is no causality. This case is much more delicate than the previous one. If we solve the index equation by elementary methods, we find sinusoidal functions which are not Fourier transformable in the sense of functions.

b. If we impose the solution $u$to be causal, the function $u$being continuous at point 0, the initial conditions are zero at this point.

The solution being, on $]0, +\infty[$:

$$u(t) = A\cos(wt) + B\sin(wt) + 1$$

we deduce from this:

$$u(t) = \varkappa(t)(1 - \cos(wt))$$

and its derivative which provides the impulse response:

$$h(t) = w\,\varkappa(t)\sin(wt)$$

We could just as well have used $\mathcal{L}$to find this function directly h.We note that this function is not summable, which confirms that it does not admit a Fourier transform in the sense of functions. Its Fourier transform in the sense of distributions is calculated using that of $\mathcal{U}$(use of the relation of the function "sign $([sgn])$and the unit step $\mathcal{U})$and the properties of multiplications by exponentials. We would thus find a principal value associated with the function.

$$H(2i\pi\lambda) = \frac{1}{1 - 4\pi^2\lambda^2a^2},$$

Which verifies, a posteriori, the calculations made above.

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