Compartmental Model for Unemployment Dynamics in Ghana using Caputo Fractional Derivatives

ABSTRACT

The study develops a compartmental model for unemployment dynamics in Ghana. The compartmental modelis first developed through the method of compartmental designs under certain assumptionssuch as, all entrants into the labor force qualify for any job, some of the employed become unemployed with time, some of the unemployed go through skill training, among others.The compartmentalmodel isfurther represented in terms of integral form, and the value of the kernel is thereafter substituted as a power law correlation function to achieve a Caputo fractional derivative model. The Caputomodel is analyzed to ascertain its invariant and boundedness, its fixed points and its stability. The unemployment basic reproduction number $(\Re_0)$ which determines the threshold of recruitment is analyzed using the Next Generation Method Approach, and the global stability for the unemployment-free equilibriumis alsoanalyzed through the approach of Lyapunov function. The results from the analysis show that, there is attraction of all the solution of the model in $\mathbb{R}_+^4$ since the region is positively invariant. Again, the results indicates that, the model has two fixed point. Thus, unemployment-free equilibrium and risky-unemployment equilibrium. Notwithstanding, the analysis of both the local and global stabilityof the model show that, the unemployment free equilibrium is local asymptotically stable (LAS) for $\Re_0 < 1$, andglobal asymptotically stable (GAS). The results indicate that, the model can examine the unemploymentdynamics decision making.

Keywords: Compartmental; Caputo Fractional Derivative; Next Generation Method; UnemploymentFree Equilibrium; Basic Reproduction Number; Lyapunov Function; Local Stability; Global Stability

1. INTRODUCTION

According to [1], compartmental models (CM) are mathematical modelling which simplifies complex systems by monitoring the movements of several quantities or persons through a sequence of qualitatively diverse states in order to provide accurate and economical outcomes.

[2] noted that, compartmental models have gained increasing importance over the years in fields which study large population like sociology, ecology, epidemiology, demography, chemical kinetics, etc.

Additionally, compartmentalization offers a superior benefit in that it allows the modeler to build a model that is described by a fully deterministic system of differential or dynamic equations, which in turn allows the modeler to solve problems using simpler mathematical tools such as differential or analytical equations [1].

The area of mathematical modelling continues to play very important role in explaining complex systems with observable properties or behaviour in many fields like, economics, finance, biology, physics, etc [3].

Economics, which study the implications of the scarcity of resources, economic growth of production, production of goods and services, and well-being of the people with respect to time, also considers unemployment as one of the critical economic indicators that shows the health of every economy and the welfare of its citizens [4]. However, one of the major economic and social challenge encountered by many economies in the world is unemployment, for which Ghana is not an exemption.

There are many factors that cause unemployment, however, in Ghana, [5] underpinned that, lots of civil servants and the employees of the formal sector in Ghana have become unemployed as a result of economic crisis that bedevils the countrycoupled with the implementation of structural adjustments. Again, Ghana has experienced labor market pressure as a result of large growing population. This is evident in all the twelve times population census organized by [6] in Ghana since 1891 with the population moving from 18,912,079 in the population census conducted in the year 2000 to 24, 658,832 in the population census conducted in the year 2010. However, the population stands at a little over 30.8 million in the 2021 population census.

Both [4,7] underscored that, understanding and addressing the dynamics of unemployment in an economy is very necessary for researchers and policy makers because, it plays a vital role within every economy. In line with this, many research studies have been conducted in the area of unemployment in Ghana using econometric and traditional models to understand the unemployment dynamics in the country, notwithstanding, the problem still persist. Generally, in understanding unemployment dynamics, there have been many approaches like traditional econometric models, traditional integer models, etc. applied in modelling unemployment dynamics with economies specific. However, one of those approaches is through compartmentalization which are analyzed and solved with fractional derivatives due to its ability to add a very flexible and nuance view on the changes over time of the unemployment dynamics in an economy.

Fractional derivatives which have found applications in various fields like biology, economics, finance, etc., because of its ability to handle phenomenon which exhibits memory effects, long-range dependencies and anomalous behaviors, is classified to be a branch of mathematics which generalizes an idea of differentiation to a non-integer order [8]. The growing interest in applying fractional derivatives based mathematical modeling in recent times have not been limited to modeling unemployment dynamics to gain a deeper insight into the realities in the labor market in many economies. Fractional derivative based models in understanding the unemployment dynamics in the labor market give more advantages but not limited to the following: Memory effect, long range dependencies, flexibility, etc. Furthermore, an additional importance of developing Caputofractional derivative model is the ability of these models to incorporate the following element: Fractional differential equation, parameter(s) estimation, sensitivity analysis, forecasting, etc.

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2. LITERATURE REVIEW

The proper usage of the resources of an economy is an integral aspect within every economy. This is so because, the workers of every economy happen to be an important resource within an economy. Hence, the goal of all policymakers is to keep workersemployed or to increase the labor force participation rate in an economy [9]. Using data from 199 families across 84 countries between 1960 and 2015, [9] created a database of national jobless rates including nations of all socioeconomic levels. They emphasized that, when compared to Japan and the US, the average jobless rate in Western Europe is greater. They additionally noted that, the average unemployment rates in poor countries are lower than that of the rich countries. They further added that, the less educated in rich countries are the most affected by unemployment, but there is an average constant unemployment among the more educated across the countries. They concluded that, unemployment was less among the more educated in rich countries but high in poor countries, and the vice versa.

[10] posited that, job vacancies become limited thereby increasing unemployment during economic contractional periods or economic crisis, whiles more vacancies are created during economic expansion periods. The Beveridge curves or the “unemployment vacancy job curves” which measures the existing labor market [11] was used among the European Union Countries and on the European Area. It showed that, in 2006 to 2019, the labor market for the European Union countries and the European Area had undergone significant increase in unemployment rate, and reduction in job vacancies as a result of the 2008 economic and financial crisis. Their findings with the Beveridge curves indicated a significant between the years 2010 and 2013 with emphasis on the period between 2010 and 2011 pinpointing a clear difference between the respective member states, in the sense that, there were more vacancies in countries that had low unemployment rates. The findings with the curve further unraveled that, there were changes in both the European Union and the European area which have been evident since 2014 and were brought about by the rise in the unemployment rate on the background of the reduction in the unemployment rate.

In Africa, many research studies have been conducted to understand the factors causing the rise in unemployment in Sub-Saharan Africa. Most of these studies considered and examined both the demand and supply factors of unemployment, however, this canker still persists [12].

Ghana as the beacon of Africa for its first role in gaining independence has consistently suffered the fate of unemployment. The population of Ghana has increased drastically,leading to anincrease in the labor force. However, the country's unemployment problems stem from the fact that new work prospects have not expanded proportionately.

[13] posited that, economic problems form part of dynamical problems, and can be studied with dynamic processes. Unemployment is an economic development problem which is among the economic problems that can also be modelled with dynamic systems. In an economy, unemployment growth and its control models which aremodelled with compartmental models have shown a very high accuracy in understanding the dynamics and how to strategically control or mange this social canker as shown in the work of [14]. [14] developed and analyzed a seven compartmental non-linear ODE model for unemployment dynamics in Ghana. In developing their model, they assumed that all entrants in the labor

force were between the ages of 18 to 60 years, and were all qualified to work, but were all either employed or unemployed at any time t. They further assumed that, the three main sectors of the economy (Agricultural, Industry and Servies) had different entrants rates, andthe creation of vacanciesis dependent on the number of employed and unemployed labor force in the respective sectors. They established the behaviour of the developed compartmental model over time by applying stability analysis to study the developed thesystems of equations, and found that, there was a positive and bounded solution to the system. Also, the systems of equations hada non-negative equilibrium point. Furthermore, they established that the equilibrium point was local asymptotically stable after using the Routh-Hurwitz criterion, and they ultimately noticed the existence of a workable stability of the developed compartmental model under specific circumstances.Notwithstanding, the model developed by [14] did not allow movements across different sectors thereby constraining a switch in profession. Additionally, the model did not factor skill training in any of the sectors which is an integral variable in creating and empowering the private sector to create their own vacancies and not to solely depend on the government. Finally, the model developed was an integer order model which lacks the ability of memory effects unlike fractional derivatives.

Again, [15] did a similar work to understand the problem of unemployment by proposing a two simplecompartmental model for unemployment with two systems of equation. The two compartments considered were the number of employed and unemployed persons. They considered the assumptions that, employment may directly be gotten by some unemployed persons, whiles others from the employed compartment may be unemployed as a result of resignation or a sack from the job. Other additional assumptions to the construction of their model were that, persons that entered the unemployment compartment were fully qualified to be employed, with the number of unemployed people continuously increasing as a result of migration of those persons from their locality to other places in search of job, with a limited number of vacancies being constant. They employed the Routh-Hurwitz criterionto show that the suggested unemployment model has just one non-negative equilibrium point and islocal asymptomatically stable. Additionally, they suggested that, the model of unemployment can generally be improved by introducing more compartments like inflation, salary, etc into the model as also suggested by [16].However, the model of [15] also failed to consider the impact of skill training that would take care of unemployment caused by skill mismatch. Furthermore, the study failed to consider the impact of startup capital for skill trainee graduates to support them to create their own vacancies. This work also used integer-order model and did not also explore the hereditary traits of the model using fractional derivatives to ascertain the nuance nature of the unemployment dynamics.

Fractional derivatives models have proven to be very effective and superior than traditional or integer modelsdue totheir memory effect [8]. Though there have been many works on nonlinear unemployment dynamics like the works by [17,18, 19, 20, 21, 22], etc, however, Caputofractional derivatives as applied to compartmental unemployment models have not been extensively explored despite its superiority over traditional and integer models.

3. DEVELOPMENT OF THE COMPARTMENTAL MODEL

Compartmental modelling used to understand the dynamics of dynamical systems is a great tool to understanding unemployment in an economy as used by many researchers. Compartmental models are mostly built on assumptions which are consistent with literature and the realities surrounding the complexities of the system under study.

3.1 Description of the Model

This proposed unemployment model has four compartments. These are, the unemployment (U), Employed (E), Skill Training (A) and New Created Vacancies (V) created exclusively from existing firms. U is the chamber through which new entrants (Λ) enter the labor market. E is the number of existing vacancies which are fully filled by the employed labor force. Again, A is the skill training compartment to train interested people through formal and non-formal education (students in the Technical and Vocational Education Training (TVET), and those who are not in TVET but are going through any form of skill training) whose skills have lost relevance or lack the requisite skills to be employed in a specialized sector within the economy over time, and V is the number of new vacancies created by the private sector and the government. In creating V, government invests in skill training cum tax cuts/holidays policy to help increase investments by the private sector, thereby making a fraction of the already employed private sector to invest in areas that creates more vacancies.In compartment E, vacancies are only created as a result of death, retirement, migration to foreign countries, incapacitation, resignation and/or being sacked by employers.

The U persons can migrate into the E, A and V. However, A persons can only move into U if the rate of revenue r of government and corporate bodies do not cover a lot of the graduated skill trainees and/or those graduated trainees are not highly skilled and/or the trainees dropped outduring their skill acquisition. In this case, a highly skilled trainee without funding falls back into U and could be employed into E or V.

With respect to compartment V, this will be achieved with an associated incentive packages like tax cuts or tax holidays, reduction of utilities billslike, the consumption of electricity, water, etc for industries while making the economy very investor friendly with a conscious effort of the government to create emergency vacancies like the National Builders Corpse (NABCO), etc.

Additionally, the rate of successful training and skill trainee becoming unemployed after their skill training is ϕ, ρ respectively, with r, γ being the rate of effective government revenue and diminution rate due to limited support for more highly skilled graduates. Also, the rate of U persons moving into E, rate of E persons moving into U, the rate of desire of the unemployed to be trained, rate of logistics provision by government and corporate institutions, rate at which the private sector creates new vacancies as a result of tax cuts/holidays, rate at which government creates emergency vacancies, and the rate of startup capital is $\eta, \varepsilon, \omega, \psi, \sigma, \tau, \theta$ respectively.

To represent that the number of labor force in the compartments U, E, A, V may change over time, while the specific numbers made a function of time (t) as $U(t), E(t), A(t), V(t)$. then, Letting Q be the total labor force in Ghana, the total labor force Q is then given by:

$$Q = U + E + A + V \tag{1}$$

In this model, though the system is nonlinear, however, it is noted that:

$$\frac{dU(t)}{dt} + \frac{dE(t)}{dt} + \frac{dA(t)}{dt} + \frac{dV(t)}{dt} = 0 \tag{2}$$

This implies that, at any given time,

$$U + E + A + V = cons\tan t = Q(t) = Q \text{ for all } t \in [0, \infty) \tag{3}$$

The unemployment free state is represented as:

$$U(t) = 0, E(t) = 0, A(t) = 0, V(t) = 0 \; \forall t \in [0, \infty) \tag{4}$$

3.2Assumptions of the Model

In developing the compartmental model to study the unemployment dynamics in Ghana, there are specific assumptions in relation to Ghana's economy. These are:

1. All the entrants of the labor force in Ghana to the unemployed class U fully qualify to perform any job. However, their skills loose relevance with respect to time.
2. Some of the employed in the existing vacancies may become unemployed as a result of incapacitation, resignation or firing by bosses as a result of loss of their relevant skills over time.
3. There are two categories of skilled trainees. Thus, low skilled trainees, and highly skilled trainees.
4. Unemployed persons with the desire to be trained can enrol into skill training. The desire to be trained represented as ω, operates on a probability with $\omega \in (0,1]$.
5. The number of U people moving into A is jointly proportional to the number of people in U and the number of opportunities available for skill training.
6. The average of death rate, migration rate, retirement rate, and the incapacitation rate represented by, κ, is the same for all the compartments, and $\kappa > 0$.
7. The rate of diminution (γ) of vacancies available due to financial crisis in the country is equal to κ.
8. Only highly skilled graduates can be supported with startup capital by government, and other corporate institutions like (non-governmental organizations (NGOs), banks, etc.). to enable themcreate vacancies for themselves and others.
9. New vacancies are created due to death, retirement, immigration to foreign countries, and incapacitation.
10. Low skilled trainees and few high skilled trainees become unemployed, but can be employed into E or V.
11. Tax cuts or tax holidays increases investment through creation of vacancies by a proportion of persons in E and V.

12. Government revenue impacts logistic provision for skill training, and provision of startup capital.
13. Governments creates emergency vacancies through the National Youth Authority to control unemployment.
14. Financial crisis in the economy reduces the number of vacancies to be created.

Considering the assumptions and facts for developing the model, these conceptualization of the evolutionary dynamics of unemployed labor force in Ghana is described through these nonlinear systems differential equations in (5) with the flow of description of the parameters and its variables as shown in Figure 1.

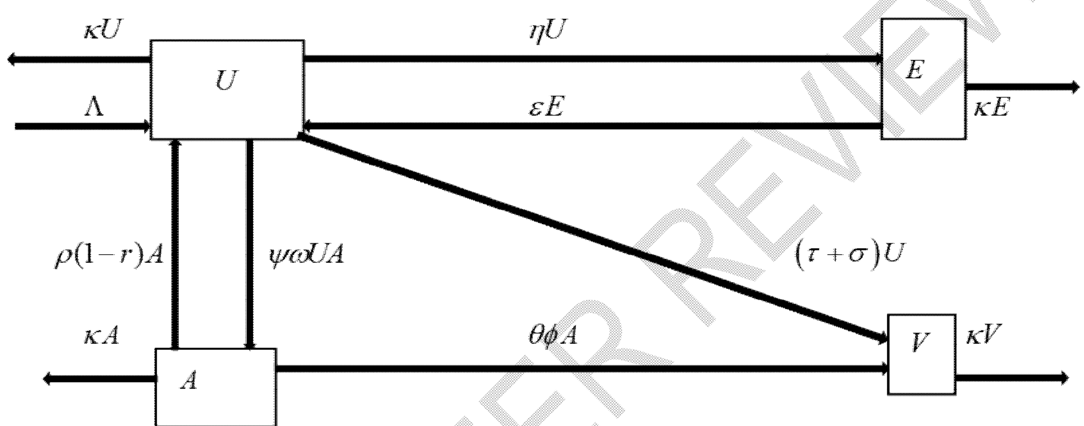

Figure 1 A compartmental diagram for unemployment control model

The set of differential equations deduced from figure 1 are displayed as:

$$\left.\begin{aligned}
\frac{dU}{dt} &= \Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - (\eta+\tau+\sigma+\kappa)U \\
\frac{dE}{dt} &= \eta U - (\varepsilon+\kappa)E \\
\frac{dA}{dt} &= \psi\omega UA - \rho(1-r)A - (\theta\phi+\kappa)A \\
\frac{dV}{dt} &= \theta\phi A + (\tau+\sigma)U - \kappa V
\end{aligned}\right\} \tag{5}$$

With the initial conditions for the unemployment model variables given by:

$$U_0 = U(0), E_0 = E(0), A_0 = A(0), V_0 = V(0) \tag{6}$$

Model (5) has a biological feasible region on Ξ_1 with

$$\Xi_1 = \left\{ (U, E, A, V) \in \mathbb{R}_+^4 : 0 \le Q \le \frac{\Lambda}{\kappa} \right\} \tag{7}$$

This means that, the model isbiologically feasible with the associated results holding for it. Also, since all the variables in model (5) are all associated to human beings, it is therefore nonnegative for which the region shown by Ξ_1 is positive invariant for model (5). Hence, the region shown for model (5) is very well posed, and all the initial values solutions belonging to Ξ_1 remain in Ξ_1 for every time $t \ge 0$.

4THE FRACTIONAL DERIVATIVE UNEMPLOYMENT MODEL

This section presents the fractional derivative version of model (5) through the Caputo fractional derivative operator. In developing the Caputo unemployment dynamics model, model (5) is represented in terms of integral form, and the value of the kernel is thereafter substituted as a power law correlation function.

Following the said approach, model (5) gives equation (8) after the introduction of the Caputo fractional derivative of order $\alpha - 1$as:

$$\left.\begin{aligned}
{}^{C}D_t^{\alpha}\frac{dU}{dt} &= {}^{C}D_t^{\alpha-1}I_t^{-(\alpha-1)}[\Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - (\eta + \tau + \sigma + \kappa)U] \\
{}^{C}D_t^{\alpha}\frac{dE}{dt} &= {}^{C}D_t^{\alpha-1}I_t^{-(\alpha-1)}[\eta U - (\varepsilon + \kappa)E] \\
{}^{C}D_t^{\alpha}\frac{dA}{dt} &= {}^{C}D_t^{\alpha-1}I_t^{-(\alpha-1)}[\psi\omega UA - \rho(1-r)A - (\theta\phi + \kappa)A] \\
{}^{C}D_t^{\alpha}\frac{dV}{dt} &= {}^{C}D_t^{\alpha-1}I_t^{-(\alpha-1)}[\theta\phi A + (\tau + \sigma)U - \kappa V]
\end{aligned}\right\} \tag{8}$$

From model (8) which are both the inverse operators, the fractional order model of unemployment in Ghana with skill training, government revenue, logistics provision, startup capital provision applied in the Caputo fractional operator sense is given by:

$$\left.\begin{aligned}
{}^{C}D_t^{\alpha}\frac{dU}{dt} &= \Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - (\eta + \tau + \sigma + \kappa)U \\
{}^{C}D_t^{\alpha}\frac{dE}{dt} &= \eta U - (\varepsilon + \kappa)E \\
{}^{C}D_t^{\alpha}\frac{dA}{dt} &= \psi\omega UA - \rho(1-r)A - (\theta\phi + \kappa)A \\
{}^{C}D_t^{\alpha}\frac{dV}{dt} &= \theta\phi A + (\tau + \sigma)U - \kappa V
\end{aligned}\right\} \tag{9}$$

$\forall \alpha$ being a fractional order and $\alpha \in (0,1]$.

Also, the initial values or conditions associated to the model variables in the fractional order model in equation (9) is given by:

$$U_0 = U(0) \geq 0, E_0 = E(0) \geq 0, A_0 = A(0) \geq 0, V_0 = V(0) \geq 0 \qquad (10)$$

5RESULTS AND DISCUSSIONS

5.1The Invariant Region of the Caputo Unemployment Model

The dynamics of the Caputo unemployment model in equation (9) is explored in a feasible region given by:

$$\Xi_2 \subset \mathbb{R}^4_+ : \Xi_2 = \left\{ (U, E, A, V) \in \mathbb{R}^4_+ : Q \leq \frac{\Lambda}{\kappa} \right\} \qquad (11)$$

Lemma 1 The region $\Xi_2 \subset \mathbb{R}^4_+$ is positive invariant with non-negative initial conditions for the Caputo unemployment model in equation (9) in $\mathbb{R}^4_+$.

Proof: after adding the components of the labor force population in model (9), the total labor force is obtained as follows:

$$^C D_t^\alpha Q(t) = {^C D_t^\alpha} U(t) + {^C D_t^\alpha} E(t) + {^C D_t^\alpha} A(t) + {^C D_t^\alpha} V(t) \qquad (12)$$

After applying basic mathematics on equation (12) gives

$$^C D_t^\alpha Q(t) = \Lambda - \kappa (U + E + L + H + V) \qquad (13)$$

From equation (1)

$$Q(t) = U + E + A + V$$

Hence, equation (13) gives

$$^C D_t^\alpha Q(t) + \kappa Q(t) \leq \Lambda \qquad (14)$$

Again, after applying Laplace transform on equation (14), gives,

$$Q(\varsigma) \leq \frac{\Lambda}{\varsigma(\varsigma^\alpha + \kappa)} + Q(0) \frac{\varsigma^{\alpha - 1}}{\varsigma^\alpha + \kappa} \qquad (15)$$

In considering the inverse Laplace, equation (15) becomes:

$$Q(t) \le Q(0)E_{\alpha,1}(\kappa t^{\alpha}) + \Lambda t^{\alpha} E_{\alpha,\alpha+1}(\kappa t^{\alpha}) \tag{16}$$

Where the equation's Mittag-Leffler function is denoted by

$$E_{\alpha,\beta(z)} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \tag{17}$$

And, also the Laplace transform is

$$L\left[t^{\beta-1}E_{\alpha,\beta}(\pm\alpha t^{\alpha})\right] = \frac{\varsigma^{\alpha-\beta}}{\varsigma^{\alpha} \pm \alpha} \tag{18}$$

This means that, Ξ_2 remains in Ξ_2 of the Caputo unemployment model solution with the non-negative criteria or conditions. Therefore, there is attraction of all the solutions in $\mathbb{R}_+^4$ since Ξ_2 region is positively invariant.

5.2 The Unemployment Model Fixed Points

This part indicates the possible fixed points of the unemployment model specified by model (9). The possible equilibrium points of the stability analysis are obtained at this juncture. In the Caputo unemployment model (9), there are two possible equilibrium points that exists. Thus, the Unemployment Free Equilibrium U_f and the Endemic Unemployment Equilibrium U_e.

5.3 The Unemployment-Free Equilibrium

The equisetum points of the model are achieved by setting the time rate of change of the Caputo unemployment model (9) to zero. This is given by:

$${}^{C}D_t^{\alpha}U\,|_{U=0} = {}^{C}D_t^{\alpha}E\,|_{E=0} = {}^{C}D_t^{\alpha}A\,|_{A=0} = {}^{C}D_t^{\alpha}V\,|_{V=0} = 0 \tag{19}$$

Hence, equation (20) is the new expressions obtained as a result of setting the time rate of change to zero:

$$\left.\begin{aligned}
&\Lambda + \varepsilon E + \rho(1-r)A - \psi\omega UA - \left(\eta + \tau + \sigma + \kappa\right)U = 0\\
&\eta U - \left(\varepsilon + \kappa\right)E = 0\\
&\psi\omega UA - \rho(1-r)A - \left(\theta\phi + \kappa\right)A = 0\\
&\theta\phi A + \left(\tau + \sigma\right)U - \kappa V = 0
\end{aligned}\right\} \tag{20}$$

The risk-free unemployment equilibrium $\left(U_f\right)$ is that stable state where there is no unemployment in the labor force. At this risk-free unemployment equilibrium U_f ,as denoted by $U_f=\left(U^0,E^0,A^0,V^0\right)$, in equation (20),this gives the results of the unemployment free equilibrium as stated in equation (21):

$$U_f=(U^o,E^o,A^o,V^o)=\left(\frac{\Lambda d_7}{d_0},\frac{\Lambda\eta}{d_0},0,\frac{\Lambda d_7 d_8}{\kappa d_0}\right) \quad (21)$$

where,

$$d_0=\kappa^2+\left(\eta+\tau+\sigma+\varepsilon\right)\kappa+\varepsilon d_8, d_7=\kappa+\varepsilon, d_8=\tau+\sigma$$

5.4 Existence of Risky Equilibrium Points

Also, at the endemic equilibrium of the Caputo unemployment model (4.6), the endemic equilibrium of the model is given by $U_e=(U^*,E^*,A^*,V^*)$.

Where;

$$U^*=\frac{\Lambda+\varepsilon E^*+\rho(1-r)A^*}{\psi\omega A^*+\eta+\tau+\sigma+\kappa} \quad (22)$$

$$E^*=\frac{\eta U^*}{d_7} \quad (23)$$

$$V^*=\frac{\theta\phi A^*+d_8 U^*}{\kappa} \quad (24)$$

5.5 The Unemployment Basic Reproduction Number (UBR)

To derive the $\Re_0$ or (UBR) for the unemployment dynamics, the next generation technique in the work of Diekmann *et al.,* (2010) is applied.

In calculating $\Re_0$, we begin with the equations of equation (9), which explain how fresh unemployment is created and how the conditions in state of unemployed people change. However, in this contest, $\Re_0$ refers to the threshold for recruitment into unemployment labor force or compartments U. We consider the E, A, V compartments, and use the approach of the next generation matrix (NGM). The equations are:

$$\left.\begin{aligned}
{}^{C}D_t^{\alpha}\left[\frac{dE}{dt}\right] &= \eta U-(\varepsilon+\kappa)E \\
{}^{C}D_t^{\alpha}\left[\frac{dA}{dt}\right] &= \psi\omega UA-\rho(1-r)A-(\theta\phi+\kappa)A \\
{}^{C}D_t^{\alpha}\left[\frac{dV}{dt}\right] &= \theta\phi A+(\tau+\sigma)U-\kappa V
\end{aligned}\right\} \quad (25)$$

From equation (9), Let $F_1 = {}^{C}D_t^{\alpha}\left[\frac{dE}{dt}\right]$, $F_2 = {}^{C}D_t^{\alpha}\left[\frac{dA}{dt}\right]$, and $F_3 = {}^{C}D_t^{\alpha}\left[\frac{dV}{dt}\right]$. Using the approach of the Next Generation Matrix (NGM), let $x=(E,A,V)^T$.Decomposing equation (25) as $\frac{dx}{dt}=T-M$ where T is the transmission part, and M is the transition part gives.

$$\frac{d}{dt}\begin{bmatrix}E\\A\\V\end{bmatrix}=\left[\begin{pmatrix}\eta U\\ \psi\omega UA\\ \theta\phi A+d_8U\end{pmatrix}-\begin{pmatrix}d_7E\\ \rho(1-r)A+(\theta\phi+\kappa)A\\ \kappa V\end{pmatrix}\right] \quad (26)$$

For which T and M are gotten as:

$$T=\begin{pmatrix}0&0&0\\0&\psi\omega\Pi&0\\0&\theta\phi&0\end{pmatrix}, \text{ and } M=\begin{pmatrix}d_7&0&0\\0&\rho(1-r)+\theta\phi+\kappa&0\\0&0&\kappa\end{pmatrix}, \text{ and } \Pi=U$$

Finding the inverse of M gives:

$$M^{-1}=\begin{pmatrix}\frac{1}{d_7}&0&0\\0&\frac{1}{\rho(1-r)+\theta\phi+\kappa}&0\\0&0&\frac{1}{\kappa}\end{pmatrix} \quad (27)$$

In obtaining the threshold for recruitment, it represents the spectral radius of the NGM which is mathematically written as $\Re_0=\rho(TM^{-1})$.

Finding TM^{-1} gives:

$$TM^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \dfrac{\psi\omega\Pi}{\rho(1-r)+\theta\phi+\kappa} & 0 \\ 0 & \dfrac{\theta\phi}{\rho(1-r)+\theta\phi+\kappa} & 1 \end{pmatrix} \tag{28}$$

The dominant eigenvalue of TM^{-1} is

$$TM^{-1} = \frac{\psi\omega\Pi}{\rho(1-r)+\theta\phi+\kappa} \tag{29}$$

So,

$$\Re_0 = \rho(TM^{-1}) = \frac{\psi\omega\Pi}{\rho(1-r)+\theta\phi+\kappa} \tag{30}$$

It is noted that, $\Re_0$ controls the unemployment labor force. When $\Re_0 < 1$, it becomes the situation where E, A, V are more than the unemployed labor force, and this is closely monitored, else Ghana's economy will seriously be affected.

5.6 Local Stability

Theorem2 For any $a,b \in \mathbb{Z}$ such that $\gcd(a,b)=1$ for $\alpha = \dfrac{a}{b}$ and $M = b$, theunemployment free equilibrium of the system (9) is locally Asymptotically stable (LAS) if $|\arg(\lambda)| > \dfrac{\pi}{2M}$ for every root λ belonging to the characteristic equation (31) of matrix $J(U_f)$.

$$\det\left(diag[\lambda^{q_1}\lambda^{q_1}\lambda^{q_1}\lambda^{q_1}]\right) - J(U_f) \tag{31}$$

The Jacobian matrix J of equation (9) which is evaluated at the unemployment free equilibrium of equation (21) is:

$$J(U_f) = \begin{pmatrix} -d_5 & \varepsilon & d_6 & 0 \\ \eta & -d_7 & 0 & 0 \\ d_1 & 0 & -d_{10} & 0 \\ 0 & d_8 & d_9 & -\kappa \end{pmatrix} \tag{32}$$

Where,

$$d_1 = \psi\omega A^*; d_2 = \psi\omega U; d_3 = \rho(1-r) + \theta\phi + \kappa; d_4 = \eta + \kappa + d_8; d_5 = d_1 + d_4$$

$$d_6 = \rho(1-r) - d_2; d_9 = \theta\phi; d_{10} = d_3 - d_2$$

The obtained associated characteristic equation of equation (32) is as:

$$\left(\lambda^{q_1} + \kappa\right)\left[y_0\lambda^{3q_1} + y_1\lambda^{2q_1} + y_2\lambda^{q_1} + y_3\right] = 0 \tag{33}$$

From equation (33), the argument of the root of equation $\lambda^{q_1} + \kappa = 0$ is indicated below:

$$\arg(\lambda_k) = \frac{\pi}{q_1} + k\frac{2\pi}{q_1} > \frac{\pi}{M} > \frac{\pi}{2M}, \forall k = 0,1,...,(q_1 - 1) \tag{34}$$

It is seen from equation (33) that, the root or eigenvalues have negative real part. Thus, $-\kappa$, and the remaining roots are computed from equation (35)

$$G(\lambda) = \left[y_0\lambda^{3q_1} + y_1\lambda^{2q_1} + y_2\lambda^{q_1} + y_3\right] = 0 \tag{35}$$

The coefficients of equation (35) are of the form:

$$y_0 = 1; y_1 = d_5 + d_7 + d_{10}; y_2 = (d_5 + d_7)d_{10} + d_5d_7 - d_1d_6 - \eta\varepsilon; y_3 = (d_5d_7 - \eta\varepsilon)d_{10} - d_1d_6d_7$$

Hence,

$$y_1 = d_5 + d_7 + d_3(1 - \Re_0)$$

$$y_2 = d_3(d_5 + d_7)(1 - \Re_0) + d_5d_7 - d_1d_6d_7 - \eta\varepsilon$$

$$y_3 = d_3(d_5d_7 - \eta\varepsilon)(1 - \Re_0) - d_1d_6d_7$$

It can be clearly seen that, for y_i where $y_i = 1,2,3$ give positive values if $\Re_0 < 0$.
In the same way, it is therefore deduced that, the arguments of the equation $(y_0\lambda^{3q_1} + y_1\lambda^{2q_1} + y_2\lambda^{q_1} + y_3) = 0$ are all bigger than $\frac{\pi}{2M}$ if $\Re_0 < 1$, and has an argument of less than $\frac{\pi}{2M}$ for $\Re_0 > 1$. Hence, the unemploymentfree equilibrium is LAS for $\Re_0 < 1$.

5.7Global Stability

The suggested Caputo unemployment model's global asymptotical stability in the unemployment free equilibrium state is examined in this part using the Lyapunov function approach. This is made possible by the subsequent theorem.

Theorem 3 The unemployment free equilibrium of the Caputo unemployment model is global asymptotically stable (GAS) if $\Re_0 < 0$.

Proofconsider the following appropriate Lyapunov function:

$$H(t) = p_1 E + p_2 A + p_3 V \tag{36}$$

Where the coefficients, p_i for $i = 1, 2, 3$ are unknown constants. However, they will be later selected. The time Caputo fraction derivative of $H(t)$ along model (9) gives:

$${}^C D_t^\alpha H(t) = p_1 {}^C D_t^\alpha E(t) + p_2 {}^C D_t^\alpha A(t) + p_3 {}^C D_t^\alpha V(t) \tag{37}$$

Considering the Caputo model in (9), we have:

$${}^C D_t^\alpha H(E,A,V) = \left.\begin{matrix} p_1\left(\eta U - (\varepsilon+\kappa)E\right) + p_2\left(\psi\omega UA - \rho(1-r)A - (\theta\phi+\kappa)A\right) \\ +p_3\left(\theta\phi A + (\tau+\sigma)U - \kappa V\right) \end{matrix}\right\} \tag{38}$$

$$\leq \left(p_2\left(\psi\omega U - d_3\right) + p_3\left(\theta\phi\right)\right)A + \left(p_1\eta + p_3 d_8\right)U - p_1 d_7 E - p_3\kappa V \tag{39}$$

At this point, we select

$p_1 = p_3 = 0; p_2 = 1$, which results into:

$${}^C D_t^\alpha H(t) \leq \left(d_3\Re_0 - \left(\rho(1-r) + \theta\phi + \kappa\right)\right)A \tag{40}$$

But, $d_3 = \rho(1-r) + \theta\phi + \kappa$ into equation (40) gives:

$${}^C D_t^\alpha H(t) \leq \left(\rho(1-r) + \theta\phi + \kappa\right)\left[\Re_0 - 1\right]A = d_3\left(\Re_0 - 1\right)A \tag{41}$$

From equation (41), at $\Re_0 \leq 0$ gives ${}^C D_t^\alpha H(t) \leq 0$. It implies that, the parameters arenon-negative with ${}^C D_t^\alpha H(t) \leq 0$ if and only if $E = A = V = 0$.Hence, $(E, A, V) \to (0,0,0)$ with $t \to \infty$.

Additionally, the solutions with $t \to \infty$ approaches the U_f according to the Caputo unemployment model as seen in the feasible region. It therefore means that the risk-free unemployment equilibrium is global asymptotically stable.

6CONCLUSIONS

In this research work, compartmental model for the unemployment dynamics using Caputo fractional order derivatives is developed.The developed model has four compartments with a bi-directional flow between the unemployment compartment and the employed compartment, the unemployment compartment and the skill training compartment. Also, the model has a unidirectional flow from the unemployment compartment to the new vacancies compartment, and a unidirectional flow from the skill training compartment to the new vacancies compartment. Additionally, the model considered interventions like government revenue, logistics provision for skill training institutions, startup capital, among others. The analysis of the developed model proved that; the model is bounded within an invariant feasible region with a non-negative solution. Additionally, the model has two fixed points. Thus, the unemployment free equilibrium and the risky unemployment equilibrium. Further analysis with the Next Generation Method and the application of the Lyapunov function showed that, the unemploymentfree equilibrium is both local asymptotically stable (LAS) and global asymptotically stable (GAS). This developed Caputo fractional model is very robust in capturing the memory dynamics of the labor market as compared to integer models.

Disclaimer (Artificial intelligence)

Authors hereby declare that, NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used duringthe writing or editing of manuscript.

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