A Generalized Suja Distribution with application to lifetime data

ABSTRACT

This paper introduces a new lifetime distribution called the Kumaraswamy-Suja (KW-Suja) distribution using the Kumaraswamy generator. The proposed distribution has Suja distribution as a special case. Some statistical and reliability properties of the new distribution were derived and the method of maximum likelihood was employed for estimating the model parameters. The usefulness and flexibility of the KW-Suja distribution were illustrated with a real lifetime data. Results based on the log-likelihood and goodness of fit statistics values showed that the KW-Suja distribution provides a better fit to the data than the other competing distributions considered in this study. The KW-Suja distribution is therefore recommended for effective modelling of the unimodal or bimodal continuous lifetime data with non-decreasing shape and bathtub-shaped failure rate.

Keywords: Bimodal data, hazard rate function, maximum likelihood method, Kumaraswamy generator, Suja distribution

1. Introduction

One of the activities of statisticians is to make informed decisions about a population on the basis of a sample drawn from that population. Obviously, several phenomena upon which decisions are taken often occur by chance and the best way to account for uncertainties surrounding them is to adopt probabilistic models. Probability models serve as mathematical structures for describing physical phenomena. A necessary step in the use of probabilistic models for modelling real-life problems is to ensure that the observed sample data follow certain probability distribution(s). Standard probability distributions commonly used for modelling several real-life problems include exponential, Weibull, gamma, two-parameter Odoma (Enogwe, *et al.*, 2020), Beta-Exponentiated Ishita (Enogwe and Ibeh, 2021), Inverse Power Akash (Engowe *et al.*, 2020), Generalized Weighted Rama (Enogwe et al., 2021) and so on. Unfortunately, so many datasets do not come from the existing probability distributions and this has engendered a demand for alternative distributions, especially for the extension of the existing distributions which can be more appropriate for fitting real-life data.

Recently, Shanker (2017) introduced and studied a new distribution, called the Suja distribution (SD) with probability density function (pdf) and cumulative distribution function(cdf) given, respectively, by

$$g(x;\theta)=\frac{\theta^5}{\theta^4+24}\left(1+x^4\right)e^{-\theta x};x>0,\theta>0 \tag{1}$$

and

$$G(x;\theta)=1-\left[1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right]e^{-\theta x};x>0,\theta>0. \qquad (2)$$

The parameter θ in (1) and (2) is a scale parameter. Shanker (2017) utilised the Suja distribution for lifetime analysis of engineering data and the findings showed that the Suja distribution performed better than the Akash, Shanker, Amarendra, Aradhana, Sujatha, Devya, Lindley and exponential distributions, respectively.

In spite of the utility of the Suja distribution, it cannot be used for statistical modelling of datasets with varieties of tails due its dependency on only one parameter. This limitation of the Suja distribution can be overcome by obtaining some of its generalizations so as to provide greater flexibility in modelling observed data (Terzieva et al., 2020). The work of Al-Omari and Alsmairan (2019) introduced a length-biased Suja distribution. Also, a power length-biased Suja distribution was developed by Al-Omari *et al.* (2019). Further, Alsmairan and Al-Omari (2020) used the weighted method to extend the Suja distribution, which was applied to ball bearing data to show that weighted Suja is better than Suja distribution. The limitation of these extensions of Suja distribution is that they cannot be used to model data with right-skewed and bimodal shape. However, Enogwe *et al.* (2022) introduced a transmuted Suja distribution for modelling data with bimodal shape.

In recent years, many methods have been proposed and utilized for generating new probability distributions; among them is the Kumaraswamy distribution (Kumaraswamy, 1980). Jones (2009) explored the background and genesis of the Kumaraswamy distribution and mentioned that the Kumaraswamy densities are unimodal, uniantimodal, increasing, decreasing or constant depending on the values of its parameters. In other words, Jones (2009) posits that the Kumaraswamy distribution provides distributions that are more flexible than baseline distributions in modelling real-life datasets. Further, Cordeiro and de Castro (2011) combined the works of Eugene *et al.* (2002) and Jones (2009) to construct a new class of Kumaraswamy generalized (Kw-G) distributions. The cumulative distribution function (cdf) and probability density function (PDF) of the of the Kumaraswamy generalized distributions (KW-G) is given by

$$F(x)=1-\left[1-G(x)^{\alpha}\right]^{\beta} \qquad (3)$$

and

$$f(x)=\alpha\beta g(x)\left(G(x)\right)^{\alpha-1}\left[1-\left(G(x)\right)^{\alpha}\right]^{\beta-1} \qquad (4)$$

respectively, where $0<x<1$, $\alpha>0$, $\beta>0$ are shape parameters, $G(x)$ is the baseline pdf of X and $g(x)=dG(x)/dx$, the baseline pdf of X. Observe from (3) and (4) that if $\alpha=\beta=1$, the Kumaraswamy family of distributions reduces to the baseline distribution.

Several generalized distributions from (3) and (4) have been studied in the literature including KW-Weibull distribution (Cordeiro, *et al.*, 2010), KW-Gumbel (Cordeiro, *et al.*, 2010), Kw-generalized gamma distribution (de Castro et al., 2011), KW-Birnbaum-Saunders (Saulo, *et* *al.*), the KW-generalized half-normal distribution (Cordeiro *et al.*, 2012), KW-Pareto distribution (Bourguignon, *et al.*, 2012), among others.

The aim of this article is to propose a Kumaraswamy Suja (KW-Suja) distribution, which is more flexible than the Suja distribution and some other competing lifetime distributions in modelling complex lifetime datasets. Specifically, this study reveals that the Kumaraswamy generator can be used to generalize a one-parameter continuous distribution to obtain a bimodal two-parameter distribution that has a monotone or non-monotone hazard rate function, especially the bathtub shape. In Section 2, we define the expressions for the pdf and cdf of the KW-Suja distribution. The statistical and reliability properties of the KW-Suja distribution are discussed in Section 3. The quantile function and entropies of the KW-Suja distribution are given in Section 4. Section 5 Provides the distribution of order statistics. In Section 6, the parameters of the KW-Suja distribution are estimated through the method of maximum likelihood estimation. Section 7 discusses the **a**symptotic confidence intervals of the parameters of KW-Suja distribution. A simulation study is conducted in Section 8. In Section 9, two real datasets, methods of model selection, applications of the KW-Suja distribution to the data sets and the results are presented. In Section 10, we give the concluding remarks.

2. The KW-Suja Distribution

Inserting (2) into (3), we get the cdf of the new distribution. Also, inserting (1) and (2) into (4), we obtain the pdf of the new distribution. Consequently, a random variable X is said to have the KW-Suja distribution if its cdf and pdf are defined as

$$F(x;\theta,\alpha,\beta)=1-\left[1-\left(1-\left(1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha}\right]^{\beta}, \quad (4)$$

and

$$\begin{aligned} f(x;\theta,\alpha,\beta)=&\frac{\alpha\beta\theta^5\left(1+x^4\right)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}\\ &\times\left[1-\left(1-\left(1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha}\right]^{\beta-1} \end{aligned} \quad (5)$$

respectively, for $\alpha,\beta,\theta>0,\ 0<x<1$. The KW-Suja distribution reduces to the Suja distribution when $\alpha=\beta=1$.

Figure 1 shows the plots of the pdf of the KW-Suja variable based on several sets of values of the parameters of the distribution. As can be seen in Figure 1, the KW-Suja has both unimodal, heavy-tailed and upside-down bathtub shapes.

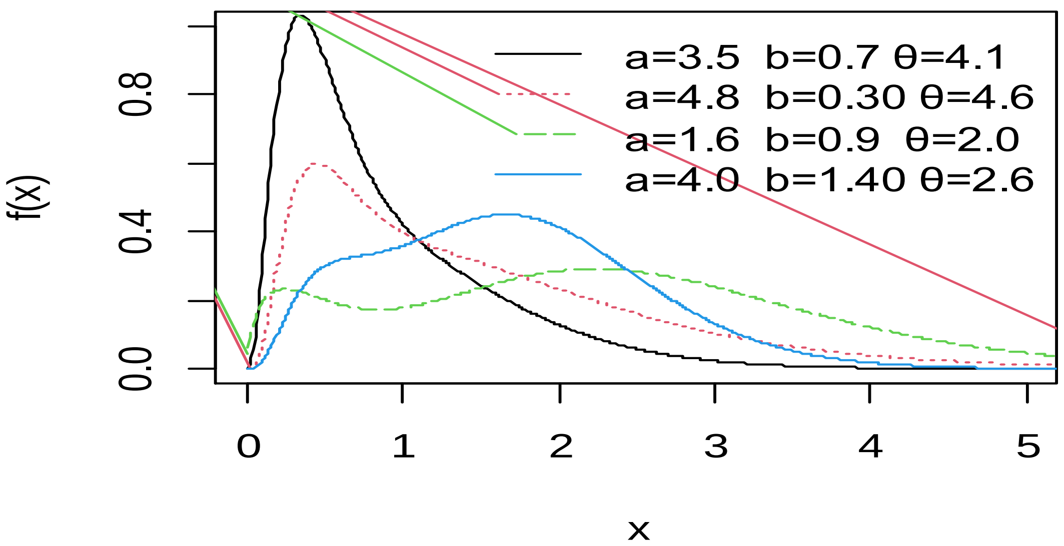

Figure 1: Various shapes of the pdf of KW-Suja for some parameter values

2.1 Expansions for the cumulative and density functions

In this sub-section, simple expansions for the KW-Suja cumulative distribution as well as that of its density function are given. By using the generalized binomial theorem (for $0<\tau<1$)

$$(1+\tau)^{\eta}=\sum_{i=0}^{\infty}\binom{\eta}{i}\tau^{i} \qquad (6)$$

and

$$(1-\tau)^{\eta}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\eta}{i}\tau^{i} \qquad (7)$$

From (3), we can write

$$\left[1-\left(1-\left(1+\frac{\theta x\left(\theta^{3}x^{3}+4\theta^{2}x^{2}+12\theta x+24\right)}{\theta^{4}+24}\right)e^{-\theta x}\right)^{\alpha}\right]^{\beta-1}$$

$$=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}\left(1-\left(1+\frac{\theta x\left(\theta^{3}x^{3}+4\theta^{2}x^{2}+12\theta x+24\right)}{\theta^{4}+24}\right)e^{-\theta x}\right)^{\alpha i} \qquad (8)$$

Putting (8) into (5) gives

$$f\left(x;\theta,\alpha,\beta\right)=\frac{\alpha\beta\theta^{5}\left(1+x^{4}\right)e^{-\theta x}}{\theta^{4}+24}\sum_{i=0}^{\infty}\left(-1\right)^{i}\binom{\beta-1}{i}\left(1-\left(1+\frac{\theta x\left(\theta^{3}x^{3}+4\theta^{2}x^{2}+12\theta x+24\right)}{\theta^{4}+24}\right)e^{-\theta x}\right)^{\alpha\left(i+1\right)-1} \quad (9)$$

Also, from (9), we get

$$\left(1-\left(1+\frac{\theta x\left(\theta^{3}x^{3}+4\theta^{2}x^{2}+12\theta x+24\right)}{\theta^{4}+24}\right)e^{-\theta x}\right)^{\alpha\left(i+1\right)-1}$$
$$=\sum_{j=0}^{\infty}\left(-1\right)^{j}\binom{\alpha\left(i+1\right)-1}{j}\left(1+\frac{\theta x\left(\theta^{3}x^{3}+4\theta^{2}x^{2}+12\theta x+24\right)}{\theta^{4}+24}\right)^{j}e^{-j\theta x} \quad (10)$$

Putting (10) into (9) yields

$$f\left(x;\theta,\alpha,\beta\right)=\frac{\alpha\beta\theta^{5}\left(1+x^{4}\right)}{\theta^{4}+24}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(-1\right)^{i+j}\binom{\beta-1}{i}\binom{\alpha\left(i+1\right)-1}{j}\left(1+\frac{\theta x\left(\theta^{3}x^{3}+4\theta^{2}x^{2}+12\theta x+24\right)}{\theta^{4}+24}\right)^{j}e^{-\theta\left(j+1\right)x} \quad (11)$$

Again, from (11), we get

$$\left(1+\frac{\theta x\left(\theta^{3}x^{3}+4\theta^{2}x^{2}+12\theta x+24\right)}{\theta^{4}+24}\right)^{j}=\sum_{k=0}^{\infty}\binom{j}{k}\left(\frac{\theta x\left(\theta^{3}x^{3}+4\theta^{2}x^{2}+12\theta x+24\right)}{\theta^{4}+24}\right)^{k}$$
$$=\sum_{k=0}^{\infty}\binom{j}{k}\left(\frac{1}{\theta^{4}+24}\right)^{k}\left(\theta^{4}x^{4}+4\theta^{3}x^{3}+12\theta^{2}x^{2}+24\theta x\right)^{k} \quad (12)$$
$$=\sum_{k=0}^{\infty}\binom{j}{k}\left(\frac{1}{\theta^{4}+24}\right)^{k}\left(24\sum_{l=1}^{4}\frac{\left(\theta x\right)^{l}}{l!}\right)^{k}=\sum_{k=0}^{\infty}\binom{j}{k}\left(\frac{24}{\theta^{4}+24}\right)^{k}\left(\sum_{l=1}^{4}\frac{\theta^{l}}{l!}\right)^{k}x^{lk}$$

On putting (12) into (11), we obtain

$$f\left(x;\theta,\alpha,\beta\right)=\frac{\alpha\beta\theta^{5}\left(1+x^{4}\right)}{\theta^{4}+24}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(-1\right)^{i+j}\binom{\beta-1}{i}\binom{\alpha\left(i+1\right)-1}{j}\binom{j}{k}\left(\sum_{l=1}^{4}\frac{\theta^{l}}{l!}\right)^{k}\left(\frac{24}{\theta^{4}+24}\right)^{k}x^{lk}e^{-\theta\left(j+1\right)x}$$

$$f\left(x;\theta,\alpha,\beta\right)=W_{ijkl}\left(1+x^{4}\right)x^{lk}e^{-\theta\left(j+1\right)x} \quad (13)$$

where

$$W_{ijkl}=\frac{\alpha\beta\theta^{5}}{\theta^{4}+24}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(-1\right)^{i+j}\binom{\beta-1}{i}\binom{\alpha\left(i+1\right)-1}{j}\binom{j}{k}\left(\sum_{l=1}^{4}\frac{\theta^{l}}{l!}\right)^{k}\left(\frac{24}{\theta^{4}+24}\right)^{k} \quad (14)$$

3. Statistical and reliability properties of KW-Suja distribution

3.1 Statistical properties

In line with Enogwe and Ibeh (2021), the rth non-central moment of KW-Suja distribution is given by

$$\mu_r' = E\left(X^r\right) = \int_0^\infty x^r f\left(x;\theta,\alpha,\beta\right)dx$$

$$= \int_0^\infty x^r W_{ijkl}\left(1+x^4\right)x^{lk}e^{-\theta(j+1)x}dx$$

$$= W_{ijkl}\int_0^\infty x^{r+lk}\, e^{-\theta(j+1)x}dx + W_{ijkl}\int_0^\infty x^{r+lk+4}\, e^{-\theta(j+1)x}dx$$

$$= \frac{W_{ijkl}}{\left[\theta\left(j+1\right)\right]^{r+lk+1}}\left[\Gamma\left(r+lk+1\right) + \frac{\Gamma\left(r+lk+5\right)}{\left[\theta\left(j+1\right)\right]^{r+lk+3}}\right] \quad (15)$$

According to Enogwe *et al.* (2020), the rth central moment of KW-Suja distribution can be obtained from the relation

$$\mu_r = \sum_{j=0}^{r}(-1)^j\binom{r}{j}\mu_j'(\mu)^{r-j} \quad (16)$$

where μ_j' is deduced from (15) by replacing r with j and μ is obtained from (15) by letting $r=1$. The following central moments are obtained by letting $r = 2,3,4$ in (16):

$$\mu_2 = \sum_{j=0}^{2}(-1)^j\binom{2}{j}\mu_j'\left(\mu\right)^{2-j} \quad (17)$$

$$\mu_3 = \sum_{j=0}^{3}(-1)^j\binom{3}{j}\mu_j'\left(\mu\right)^{3-j} \quad (18)$$

$$\mu_4 = \sum_{j=0}^{4}(-1)^j\binom{4}{j}\mu_j'\left(\mu\right)^{4-j} \quad (19)$$

The coefficient of variation $\left(\gamma_0\right)$, skewness $\left(\gamma_1\right)$ and kurtosis $\left(\gamma_2\right)$ of the KW-Suja distribution could be obtained by evaluating

$$\gamma_0 = \frac{\left(\mu_2\right)^{\frac{1}{2}}}{\mu} \quad (20)$$

$$\gamma_1 = \frac{\mu_3}{\left(\mu_2\right)^{\frac{3}{2}}} \quad (21)$$

$$\gamma_2 = \frac{\mu_4}{(\mu_2)^2} \tag{22}$$

Following the work of Enogwe *et al.* (2022), the moment generating function of KW-Suja distribution is defined as

$$M_X(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx} f(x;\theta,\alpha,\beta)\,dx$$

$$= W_{ijkl}\int_0^\infty \left(1+x^4\right) x^{lk} e^{-(\theta(j+1)-t)x}dx$$

$$= W_{ijkl}\int_0^\infty x^{lk} e^{-(\theta(j+1)-t)x}dx + W_{ijkl}\int_0^\infty x^{lk+4} e^{-(\theta(j+1)-t)x}dx$$

$$= \frac{W_{ijkl}}{\left(\theta(j+1)-t\right)^{lk+1}}\left[\Gamma(lk+1) + \frac{\Gamma(lk+5)}{\left(\theta(j+1)-t\right)^{lk+4}}\right] \tag{23}$$

3.2 Reliability properties

The survival function of the KW-Suja distribution is given by

$$S(x) = 1 - F(x;\theta,\alpha,\beta) = \left[1-\left(1-\left(1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha}\right]^{\beta} \tag{24}$$

The hazard function of the KW-Suja distribution is given by

$$h(x) = \frac{f(x;\theta,\alpha,\beta)}{S(x)} = \frac{\dfrac{\alpha\beta\theta^5\left(1+x^4\right)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\dfrac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1}}{1-\left(1-\left(1+\dfrac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha}} \tag{25}$$

The graph of the survival and hazard rate functions of the KW-Suja are shown in Figures 2 and 3 respectively.

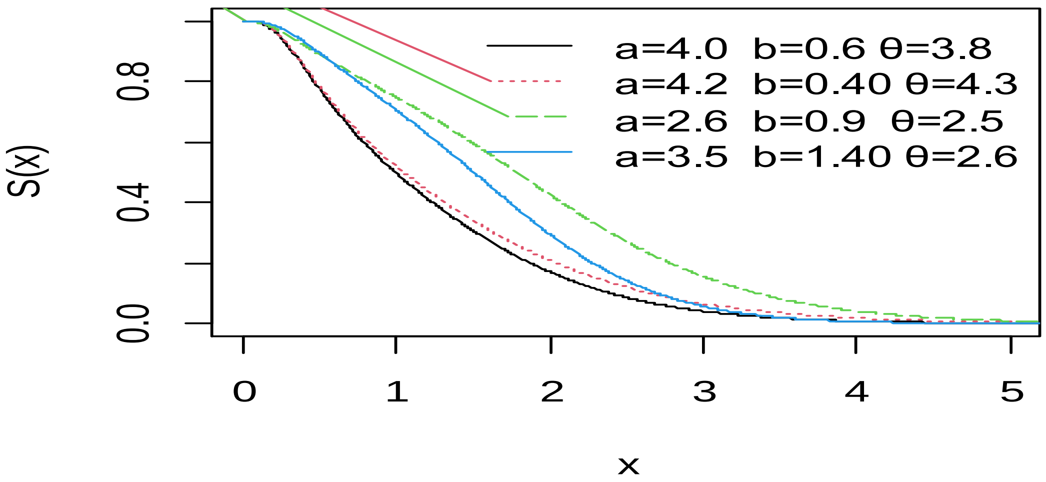

Figure 2: The Survival function the KW-Suja distribution for different values of its parameters.

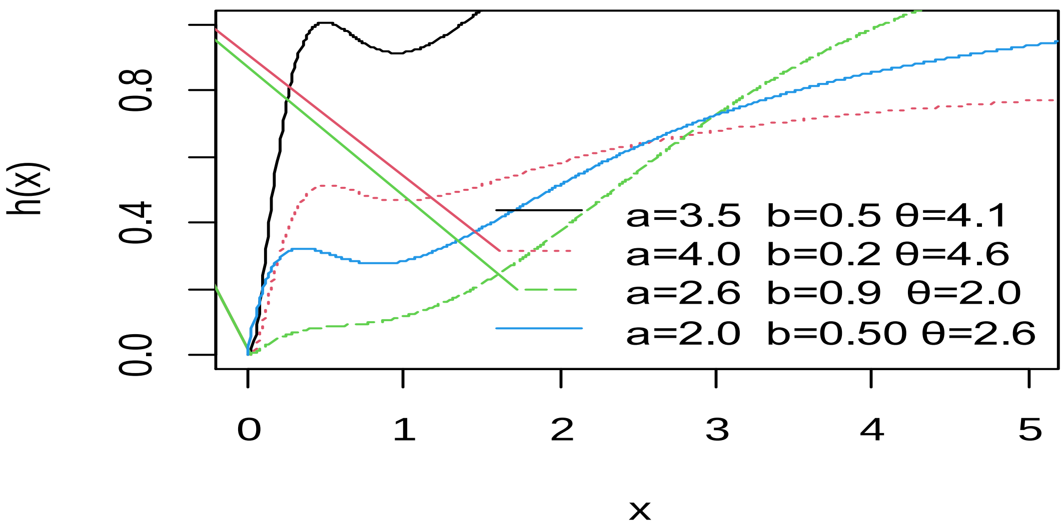

Figure 3: Various shapes of the Hazard Function of the KW-Suja distribution

4. Quantile function and entropy measures of KW-Suja Distribution

4.1 Quantile function of the KW-Suja distribution

The x_ω quantile function of the KW-Suja distribution satisfies the equation

$$F\left(x;\theta,\alpha,\beta\right)=q \ \ , \ 0<q<1 \tag{26}$$

Plugging (3) into (26), we have

$$1-\left[1-\left(1-\left(1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha}\right]^{\beta}=q \tag{27}$$

$$\left(1-\left(1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha}=1-\left(1-q\right)^{1/\beta}$$

$$\left(1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}=1-\left[1-\left(1-q\right)^{1/\beta}\right]^{1/\alpha}$$

$$e^{-\theta x}=\frac{\left(\theta^4+24\right)\left[1-\left(1-\left(1-q\right)^{1/\beta}\right)^{1/\alpha}\right]}{\left(\theta^4+24+\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)\right)}$$

Hence,

$$x_q=-\frac{1}{\theta}\ln\left\{\frac{\left(\theta^4+24\right)\left[1-\left(1-\left(1-q\right)^{1/\beta}\right)^{1/\alpha}\right]}{\theta x_q\left(\theta^3x_q^3+4\theta^2x_q^2+12\theta x_q+24\right)+\theta^4+24}\right\} \tag{28}$$

Therefore, the qth quantile, denoted by x_q, for KW-Suja distribution, is a positive solution of (28), which can be found by numerical method.

4.2 Entropy measures of the KW-Suja distribution

The Rényi entropy may be defined for the KW-Suja as

$$E_R=\frac{1}{1-\gamma}\log\left(\int_0^\infty f^{\gamma}\left(x;\theta,\alpha,\beta\right)dx\right),\ \gamma\neq 1,\ \gamma>0$$

$$E_R = \frac{1}{1-\gamma}\log\left(\int_0^{\infty}\frac{\alpha\beta\theta^5\left(1+x^4\right)e^{-\theta x}}{\theta^4+24}\left(1-\left(1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha-1} \times\left[1-\left(1-\left(1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha}\right]^{\beta-1}\right)^{\gamma} dx \quad (29)$$

Applying binomial expansion to the terms in (29) and simplifying, one gets

$$E_R = \frac{1}{1-\gamma}\log\left(W_{ijklm}\frac{\Gamma\left(lk+4m+1\right)}{\left[\theta\left(\gamma+i\right)\right]^{lk+4m+1}}\right);\ \gamma\neq 1, \gamma>0 \quad (30)$$

where

$$W_{ijklm} = \left(\frac{\alpha\beta\theta^5}{\theta^4+24}\right)^{\gamma}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{j}\sum_{m=0}^{\gamma}(-1)^{i+j}\binom{\gamma(\beta-1)}{i}\binom{\alpha(\gamma+1)-\gamma}{j}\binom{j}{k}\binom{\gamma}{m}\left(\sum_{l=1}^{4}\frac{\theta^l}{l!}\right)^k\left(\frac{24}{\theta^4+24}\right)^k \quad (31)$$

5. Distributions of order statistics of KW-Suja **Distribution**

The pdf of the rth order statistic for KW-Suja is given by

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!}\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} f(x)$$

$$f_{X_{(r)}}(x) = \frac{n!\alpha\beta\theta^5\left(1+x^4\right)e^{-\theta x}}{(r-1)!(n-r)!\left(\theta^4+24\right)}\left[1-\left(1-\left(1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha}\right]^{\beta r-1} \times\left(1-\left(1+\frac{\theta x\left(\theta^3x^3+4\theta^2x^2+12\theta x+24\right)}{\theta^4+24}\right)e^{-\theta x}\right)^{\alpha\left(\beta(n-r)+1\right)-1} \quad (32)$$

6. Maximum likelihood estimates of parameters of the KW-Suja distribution

Consider a random sample $X_1, X_2, ..., X_n$ drawn from a KW-Suja distribution. Obviously, the likelihood function of the random sample is

$$L(\theta,\alpha,\beta) = \prod_{i=1}^{n}\left\{\frac{\alpha\beta\theta^5\left(1+x_i^4\right)e^{-\theta x_i}}{\theta^4+24}\left(1-\left(1+\frac{\theta x_i\left(\theta^3x_i^3+4\theta^2x_i^2+12\theta x_i+24\right)}{\theta^4+24}\right)e^{-\theta x_i}\right)^{\alpha-1} \times\left[1-\left(1-\left(1+\frac{\theta x_i\left(\theta^3x_i^3+4\theta^2x_i^2+12\theta x_i+24\right)}{\theta^4+24}\right)e^{-\theta x_i}\right)^{\alpha}\right]^{\beta-1}\right\} \quad (33)$$

The log-likelihood function is

$$\begin{aligned}\ln L(\theta,\alpha,\beta) &= n\left[\ln(\alpha)+(\beta)+5\ln(\theta)-\left(\theta^4+24\right)\right]+\sum_{i=1}^{n}\ln\left(1+x_i^4\right)-\theta\sum_{i=1}^{n}x_i \\ &+(\alpha-1)\sum_{i=1}^{n}\ln\left(1-\left(1+\frac{\theta x_i\left(\theta^3x_i^3+4\theta^2x_i^2+12\theta x_i+24\right)}{\theta^4+24}\right)e^{-\theta x_i}\right) \\ &+(\beta-1)\sum_{i=1}^{n}\ln\left[1-\left(1-\left(1+\frac{\theta x_i\left(\theta^3x_i^3+4\theta^2x_i^2+12\theta x_i+24\right)}{\theta^4+24}\right)e^{-\theta x_i}\right)^{\alpha}\right]\end{aligned} \tag{34}$$

Taking the partial derivatives of (34) with respect to η and λ, and equating the results to zero, yields

$$\begin{aligned}\frac{\partial \ln L(\theta,\alpha,\beta)}{\partial\theta} &= \frac{5n}{\theta}-\frac{4n\theta^3}{\theta^3+24}-\sum_{i=1}^{n}x_i+(\alpha-1)\sum_{i=1}^{n}\frac{1}{Z_i}\left[(1-Z_i)x_i-Z_{1i}e^{-\theta x_i}\right] \\ &+\alpha(\beta-1)\sum_{i=1}^{n}\frac{Z_i^{\alpha-1}Z_{1i}e^{-\theta x_i}}{1-Z_i^{\alpha}}\end{aligned} \tag{35}$$

$$\frac{\partial \ln L(\theta,\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha}+\sum_{i=1}^{n}\ln Z_i+(\beta-1)\sum_{i=1}^{n}\frac{Z_i^{\alpha}\ln Z_i}{Z_i^{\alpha}-1} \tag{36}$$

$$\frac{\partial \ln L(\theta,\alpha,\beta)}{\partial\beta} = \frac{n}{\beta}+\sum_{i=1}^{n}\ln\left(1-Z_i^{\alpha}\right) \tag{37}$$

where

$$Z_i = 1-\left(1+\frac{\theta x_i\left(\theta^3x_i^3+4\theta^2x_i^2+12\theta x_i+24\right)}{\theta^4+24}\right)e^{-\theta x_i} \tag{38}$$

$$Z_{1i} = \frac{\begin{array}{c}-6\theta^6x_i^3-24\theta^5x_i^2-72\theta^4x_i+96\theta^3x_i^4+288\theta^2x_i^3 \\ +576\theta x_i^2+576x_i+24\theta^4x_i^4+96\theta^3x_i^3+288\theta^2x_i^2+576\theta x_i\end{array}}{\left(\theta^4+24\right)^2} \tag{39}$$

Due to the complex nature of (35), (36) and (37), an iterative method such as the Newton-Raphson method is adopted for finding its solution.

9. Application to real data set

In this section, we illustrate the flexibility and applicability of the BES distribution with two real data sets. The data comprises of the sum of skin folds in 202 athletes collected at Australian Institute of sports and were used by Weisberg (2005):

28.0, 98.0, 89.0, 68.9, 69.9, 109.0, 52.3, 52.8, 46.7, 82.7, 42.3, 109.1, 96.8, 98.3, 103.6, 110.2, 98.1, 57.0, 43.1, 71.1, 29.7, 96.3, 102.8, 80.3, 122.1, 71.3, 200.8, 80.6, 65.3, 78.0, 65.9, 38.9, 56.5, 104.6, 74.9, 90.4, 54.6, 131.9, 68.3, 52.0, 40.8, 34.3, 44.8, 105.7, 126.4, 83.0, 106.9, 88.2, 33.8, 47.6, 42.7, 41.5, 34.6, 30.9, 100.7, 80.3, 91.0, 156.6, 95.4, 43.5, 61.9, 35.2, 50.9, 31.8, 44.0, 56.8, 75.2, 76.2, 101.1, 47.5, 46.2, 38.2, 49.2, 49.6, 34.5, 37.5, 75.9, 87.2, 52.6, 126.4, 55.6, 73.9, 43.5, 61.8, 88.9, 31.0, 37.6, 52.8, 97.9, 111.1, 114.0, 62.9, 36.8, 56.8, 46.5, 48.3, 32.6, 31.7, 47.8, 75.1, 110.7, 70.0, 52.5, 67.0, 41.6, 34.8, 61.8, 31.5, 36.6, 76.0, 65.1, 74.7, 77.0, 62.6, 41.1, 58.9, 60.2, 43.0, 32.6, 48.0, 61.2, 171.1, 113.5, 148.9, 49.9, 59.4, 44.5, 48.1, 61.1, 31.0, 41.9, 75.6, 76.8, 99.8, 80.1, 57.9, 48.4, 41.8, 44.5, 43.8, 33.7, 30.9, 43.3, 117.8, 80.3, 156.6, 109.6, 50.0, 33.7, 54.0, 54.2, 30.3, 52.8, 49.5, 90.2, 109.5, 115.9, 98.5, 54.6, 50.9, 44.7, 41.8, 38.0, 43.2, 70.0, 97.2, 123.6, 181.7, 136.3, 42.3, 40.5, 64.9, 34.1, 55.7, 113.5, 75.7, 99.9, 91.2, 71.6, 103.6, 46.1, 51.2, 43.8, 30.5, 37.5, 96.9, 57.7, 125.9, 49.0, 143.5, 102.8, 46.3, 54.4, 58.3, 34.0, 112.5, 49.3, 67.2, 56.5, 47.6, 60.4, 34.9

The goodness of fit of the new lifetime distribution would be assessed by means of comparing its fitting performance with those of

(1) Pareto Distribution (PD), **(n.d)**

$$g\left(x\right)=\frac{\alpha x^{\alpha}}{x^{\alpha+1}},\ x>0,\alpha>0 \tag{40}$$

and

$$G\left(x\right)=1-x^{\alpha} \tag{41}$$

(2) Lindley distribution (LD), (Ghitney, *et al.*, 2008)

$$g\left(x\right)=\frac{\theta^{2}}{\theta+1}\left(1+x\right)e^{-\theta x},x>0,\theta>0 \tag{42}$$

and

$$G\left(x\right)=1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x},x>0,\theta>0 \tag{43}$$

Comparison of the fitted models would be based on the following goodness of fit measures:

the Akaike information criterion (AIC) due to Akaike (1992), given by

$$AIC=-2l+2k, \tag{44}$$

the Bayesian information criterion (BIC) due to Schwarz (1978), given by

$$BIC=k\ln\left(n\right)-2l, \tag{45}$$

where k is the number of parameters in the KW-Suja distribution l is the maximized value of the log-likelihood function of the KW-Suja distribution, $\hat{F}\left(x_i\right)$ is the value of the cdf of the

KW-Suja distribution and n is the sample size. The smaller the criterion statistics the better the model.

Table 1: Maximum likelihood estimates of parameters of the KW-Suja distribution, the standard error of estimates, model selection criteria and goodness-of-fit.

Models	Estimates	SE	ℓ	AIC	BIC	KS	P-value
KW-SD	$\alpha = 11.9566$ $\beta = 0.1585$ $\theta = 0.2430$	0.0769 0.0116 0.0025	948.957	1903.914	1913.839	0.0661	0.3261
SD	$\theta = 0.2430$	0.0123	962.3421	1932.919	1943.211	0.0532	0.4213
LD	$\theta = 0.2859$	0.0014	1001.743	2005.486	2008.795	0.2154	0.0508
PD	$\alpha = 0.0580$	0.0020	965.7686	1933.537	1936.846	0.0918	0.0621

10. Conclusion

This paper introduces a new lifetime distribution, called the Kumaraswamy Suja distribution, which generalizes the Suja distribution. We have provided explicit mathematical expressions for some of its basic statistical properties such as the probability density function, cumulative density function, rth crude and central moments, variance, coefficient of variation, skewness, kurtosis, and quantile function and some reliability characteristics like the reliability, hazard rate, cumulative hazard and reverse hazard functions. Rényi entropy was discussed. Also, the distributions of rth, first and largest order statistics were introduced. Estimation of the model parameters was approached through the method of maximum likelihood estimates. The flexibility and applicability of the new lifetime distribution was illustrated with a real data and the results obtained revealed that the Kumaraswamy Suja distribution provides the best fit among all the compared related distributions. The Kumaraswamy Suja distribution is recommended for modelling unimodal or bimodal continuous lifetime data with a non-decreasing shape and bathtub shaped hazard rate function and hope that it would receive significant applications in the future.

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