Kth-DIMESIONAL DIFFERENTIAL TRANSFORM METHOD FOR THE GENERAL SOLUTION OF INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

ABSTRACT

This work studies the effect of the kth-order differential transform method to obtain the general solution of initial value problems of ordinary differential equations. The given equation was transformed using the methodologies of the conventional differential transform; the transformed equation was expanded with Taylor series; the resulting terms of the series and the order of the differential equation determines the order of terms in the transformed differential equation to be used. This approach also provides a closed-form solution; therefore, it is very powerful and effective in finding numerical solutions of first and second order ordinary differential equations. Some numerical examples were used to ascertain the usability of the method. The Method was found to be consistent and accurate; consequently, it is recommended for use for solutions and research purpose.

Keywords: Differential Equation, Ordinary Differential Equation, Power Series, Taylor Series, Initial Value Problem, Differential Transform

1.0 INTRODUCTION

1.1 Differential Equation (DE)

A Differential Equation (DE) is any equation which contains derivatives, either ordinary derivatives or partial derivatives. They are fundamental to the study of science, engineering, physics, etc. Many physical problems are capable of description in terms of a single first order ordinary differential equation, while other more complicated problems involve coupled first order differential equations that after the elimination of all but one of the independent variables, can be replaced by a single higher order equation for the remaining dependent variable. Thus, first order ordinary differential equations can be considered as the building blocks in the study of higher order equations, and their properties are particularly important and easy to obtain when the equations are linear. The study and properties of the simple class of equations called constant coefficient equations is very important, as it forms the foundation of the study of higher order constant coefficient equations that will be developed later in this work.

1.2 Ordinary Differential Equation (ODE)

An ordinary differential equation (ODE) is an equation that relates a function $y(x)$ to some of its derivatives

$$y^{(r)}(x) = \frac{d^r y}{dx^r} \qquad (1.0)$$

It is usual to call x the in-dependent variable and y the dependent variable, and to write the most general ordinary differential equation as:

$$F(x, y, y^{(1)}, y^{(2)}, \dots, y^{(n)}) = 0 \qquad (1.1)$$

Example of ODE

$$\frac{dy}{dx} + 2xy = 2e^{-x^2} \qquad (1.2)$$

1.2 Differential Transform (DT)

The Differential Transform is a numerical method for solving differential equations. The concept of the differential transform was first proposed by Zhou (1986), and its main application therein is to solve

both linear and nonlinear initial value problems in electric circuit analysis. The differential transformation method is one of the semi-analytical methods used for solving ordinary and partial differential equations in the forms of polynomials as approximations of the exact solutions.
In this work, we shall use a DT as follows:

$$p' = f(x,p), \qquad p(x_0) = p_0 \tag{1.3}$$

$$p'' = f(x,p,p'), \qquad p(x_0) = p_0, p'(x_0) = p_1 \tag{1.4}$$

Where the one-dimensional k^{th} differential transform of the $p(x)$ defined as $P(k)$ is given as:

$$P(k) = \frac{1}{k!}[\frac{d^k f(x)}{dx^k}]\ x = x_\circ \tag{1.5}$$

Zhou J. k. (1986) applied the DT of the above kind for solving differential equation with an initial value from electrical circuits, his methods was found to be accurate and consistent, hence producing further research in this area.

Edeki et al (2014) investigated the use of differential transform for solving second order linear ordinary differential equations which works for both homogenous and non-homogenous cases, there method was found to be accurate, and consistent.
The initial methods were effected on linear DEs but later on, Kenmogne (2015) examined the generalization of differential transform for solving non-linear differential equations, without having to go through linearization, discretization or perturbation of a differential problem. In same manner, Khudair et al (2016) applied differential transform to solve second order random differential equations their approach was unique and generally acceptable.
Mirzaee (2011) noted that the solutions of linear and non-linear systems of ordinary differential equations can be obtained by differential transform approach. In the same line, Muhammad et al (2013) also investigated a class of stiff systems by differential transform method to find the numerical solution of some selected problems with the output aligning with the exact solution.

However, Mohmoud S. and Gubara M. (2016) examined the reduced differential transform which is also as efficient as the differential transform but Mohammad et al (2011) argued that differential transform method of solutions are only valid for small values of the independent variable and he resulted into using the multi-step differential transform method on a flow of a second-grade fluid over a stretching or shrinking sheet. He concluded that the multi-step differential transform method is an accurate approximate solutions for systems of differential equations.

Some other notable authors who worked on Numerical approach to solutions of DEs are as follows (Kayode S. J. and Abejide K.S (2019), 2023).

Other literatures reviewed in this work are as follows (Vedat (2007), Saurabh et al (2014), Ayaz (2004))

2.0 DERIVATION OF METHOD

In this work, there is an improvement on the conventional Differential Transform for some selected first and second order differential equations. In this work, lower case letters were used to represent the original functions and the upper-case letters to represent the transformed functions. Where $y(x)$ is used to express the approximate analytical solution as a power series.

2.1 Statement of Problem

Consider $$y' = f(x,y), \qquad y(x_0) = y_0 \qquad (2.0)$$

$$y'' = f(x,y,y'), \qquad y(x_0) = y_0, y'(x_0) = y_1 \qquad (2.1)$$

Where the one-dimensional k^{th} differential transform of the $y(x)$ defined as $Y(k)$ is given as:

$$Y(k) = \frac{1}{k!}[\frac{d^k f(x)}{dx^k}]\ x = x_\circ \qquad (2.2)$$

Equation (2.2) is the transformed function of $y(x)$

The differential inverse transform of $Y(k)$ is a Taylor series expansion of the function $y(x)$ about $x = x_\circ = 0$, defined as:

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) \qquad (2.3)$$

Combining equation (2.2) and (2.3) yields:

$$y(x) = \sum_{k=0}^{\infty}[\frac{d^k f(x)}{dx^k}]\frac{x^k}{k!} \qquad (2.4)$$

The basis function for this work is Taylor series.

3.0 ANALYSIS OF METHOD

In the method derived above it is sufficient to note that one-dimensional differential transform to solve first and second order ode with initial value problems are given below.

From Taylor's series we can have the following transforms

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots \qquad (3.0)$$

$$y(x) = 1 + x\frac{dy}{dx} + \frac{x^2}{2!}\frac{d^2y}{dx^2} + \cdots + \frac{x^k}{k!}\frac{d^ky}{dx^k} + \frac{x^{k+1}}{(k+1)!}\frac{d^{k+1}y}{dx^{k+1}} + \frac{x^{k+2}}{(k+2)!}\frac{d^{k+2}y}{dx^{k+2}} + \cdots \; x = 0 \qquad (3.1)$$

The order of the differential equation determines the relevant term in the series above.

First order ODE takes on (K+1) term of the series

Second order ODE takes on (K+2) term of the series

The iterative term of the DT equation generates a power series of the exact solution with the use of MATLAB programming language.

3.1 Numerical Examples

Consider the first order ordinary differential equation below:

$$2\frac{dy}{dx} - y = 4\sin 3x \qquad (3.2)$$

With initial conditions, $y(0) = 1$

Applying differential transform method with the initial conditions of the differential transformation given as: $Y(0) = 1$ to equation (3.3) and using the above operations.

From;

$$y(x) = 1 + x\frac{dy}{dx} + \frac{x^2}{2!}\frac{d^2y}{dx^2} + \cdots + \frac{x^k}{k!}\frac{d^ky}{dx^k} + \frac{x^{k+1}}{(k+1)!}\frac{d^{k+1}y}{dx^{k+1}} + \frac{x^{k+2}}{(k+2)!}\frac{d^{k+2}y}{dx^{k+2}} + \cdots \; x = 0 \qquad (3.3)$$

From the given condition, the first order O.D.E, we use the (k+1) term as follows

$$Y(k+1) = \frac{x^{k+1}}{(k+1)!}\frac{d^{k+1}y}{dx^{k+1}} \; x = 0 \qquad (3.4)$$

$$Y(k+1) = Y(k)\frac{x}{(k+1)}\frac{dy}{dx} \; x = 0 \qquad (3.5)$$

From equation (3.2):

$$\frac{dy}{dx} = \frac{y}{2} + 2\sin 3x$$

We now have the transformed equation as:

$$Y(k+1) = \frac{1}{(k+1)}\left[\frac{Y(k)}{2} + 2\left(\frac{3^k}{k!}\sin\left(\frac{k\pi}{2}\right)\right)\right] \quad (3.6)$$

Iteration:

$$\left.\begin{array}{ll} k = 0, & Y(1) = 0.5 \\ k = 1, & Y(2) = 3.125 \\ k = 2, & Y(3) = 0.520833333 \\ k = 3, & Y(4) = -2.184895833 \\ k = 4, & Y(5) = -0.218489583 \\ k = 5, & Y(6) = 0.6567925347 \end{array}\right\} \quad (3.7)$$

Then we have the solution of the differential equation as a power series

$$y(x) = x^0Y(0) + x^1Y(1) + x^2Y(2) + x^3Y(3) + x^4Y(4) + \cdots \quad (3.7)$$

$$y(x) = 1 + 0.5x + 3.125x^2 + 0.520833333x^3 - \cdots \quad (3.8)$$

Where the Exact solution is:

$$y(x) = -\frac{24}{37}\cos(3x) - \frac{4}{37}\sin(3x) + \frac{61}{37}e^{\frac{x}{2}} \quad (3.9)$$

Table 1: Shows the approximate analytic solution, the exact solution and the error when n = 50

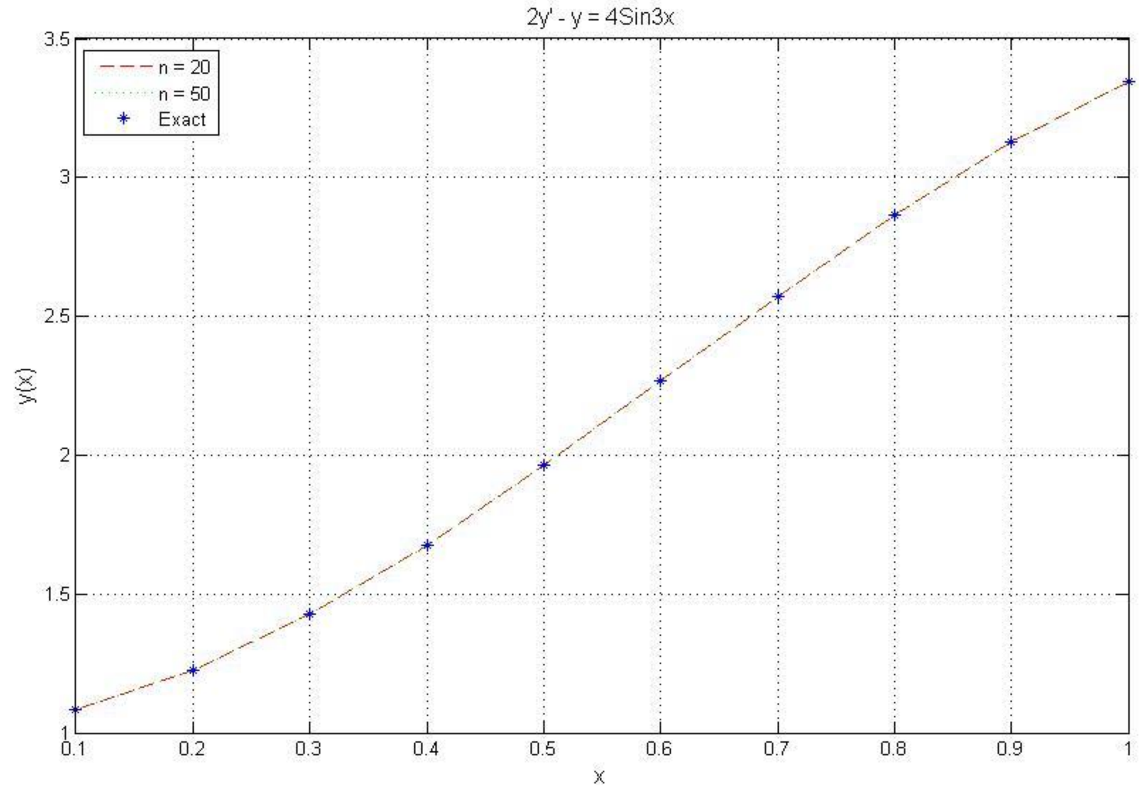
$Y_{Exact}(x) = -\frac{24}{37}\cos(3x) - \frac{4}{37}\sin(3x) + \frac{61}{37}e^{\frac{x}{2}}$

$y_{DT}(x) = 1 + 0.5x + 3.125x^2 + 0.520833333x^3 - \cdots$

x	$Exact$	$D.T, n = 50$	$Absolute\ Error$
0	1	1	0
0.1	1.081550819277772	1.081550819277772	0.000000000000000
0.2	1.225643279762219	1.225643279762219	0.000000000000000
0.3	1.427565998092416	1.427565998092416	0.000000000000000
0.4	1.677860156661116	1.677860156661116	0.000000000000000
0.5	1.963185881932734	1.963185881932733	0.000000000000000
0.6	2.267536405709663	2.267536405709663	0.000000000000000
0.7	2.573691608324048	2.573691608324047	0.000000000000000
0.8	2.864781162348724	2.864781162348724	0.000000000000000

0.9	3.125817708172282	3.125817708172282	0.000000000000000
1.0	3.345063172942677	3.345063172942676	0.000000000000001

Figure 1: Representing the exact solution and the numerical solution



DT on Second Order Ordinary Differential Equations

Consider this linear differential equation:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = e^{3x} \qquad (3.10)$$

With initial conditions, $y(0) = 0; y'(0) = 2$

Applying differential transform method with the initial conditions of the differential transformation given as: $Y(0) = 0; Y(1) = 2$ to equation (3.10) and using the above operations

From;

$$y(x) = 1 + x\frac{dy}{dx} + \frac{x^2}{2!}\frac{d^2y}{dx^2} + \cdots + \frac{x^k}{k!}\frac{d^ky}{dx^k} + \frac{x^{k+1}}{(k+1)!}\frac{d^{k+1}y}{dx^{k+1}} + \frac{x^{k+2}}{(k+2)!}\frac{d^{k+2}y}{dx^{k+2}} + \cdots \; x = 0 \qquad (3.11)$$

Since, its 2nd order O.D.E, we use the (k+2) term

$$Y(k+2) = \frac{x^{k+2}}{(k+2)!}\frac{d^{k+2}y}{dx^{k+2}}\,x = 0 \quad (3.12)$$

$$Y(k+2) = Y(k)\frac{x^2}{(k+1)(k+2)}\frac{d^2y}{dx^2}\,x = 0 \quad (3.13)$$

From equation (3.10):

$$\frac{d^2y}{dx^2} = e^{3x} + 6\frac{dy}{dx} - 8y \quad (3.14)$$

We now have the transformed equation as:

$$Y(k+2) = \frac{1}{(k+1)(k+2)}\left[\frac{3^k}{k!} + 6(k+1)Y(k+1) - 8Y(k)\right] \quad (3.15)$$

Iteration:

$$\left.\begin{aligned} k = 0, &\quad Y(2) = \frac{13}{2} \\ k = 1, &\quad Y(3) = \frac{65}{6} \\ k = 2, &\quad Y(4) = \frac{295}{24} \\ k = 3, &\quad Y(5) = \frac{1277}{120} \\ k = 4, &\quad Y(6) = \frac{5383}{720} \end{aligned}\right\} \quad (3.16)$$

Then we have the solution of the differential equation as a power series

$$y(x) = x^0Y(0) + x^1Y(1) + x^2Y(2) + x^3Y(3) + \cdots \quad (3.17)$$

$$y(x) = 2x + \frac{13}{2!}x^2 + \frac{65}{3!}x^3 + \frac{295}{4!}x^4 + \frac{1277}{5!}x^5 + \cdots \quad (3.18)$$

Where the Exact solution is:

$$y(x) = \frac{3}{2}e^{4x} - \frac{1}{2}e^{2x} - e^{3x} \quad (3.19)$$

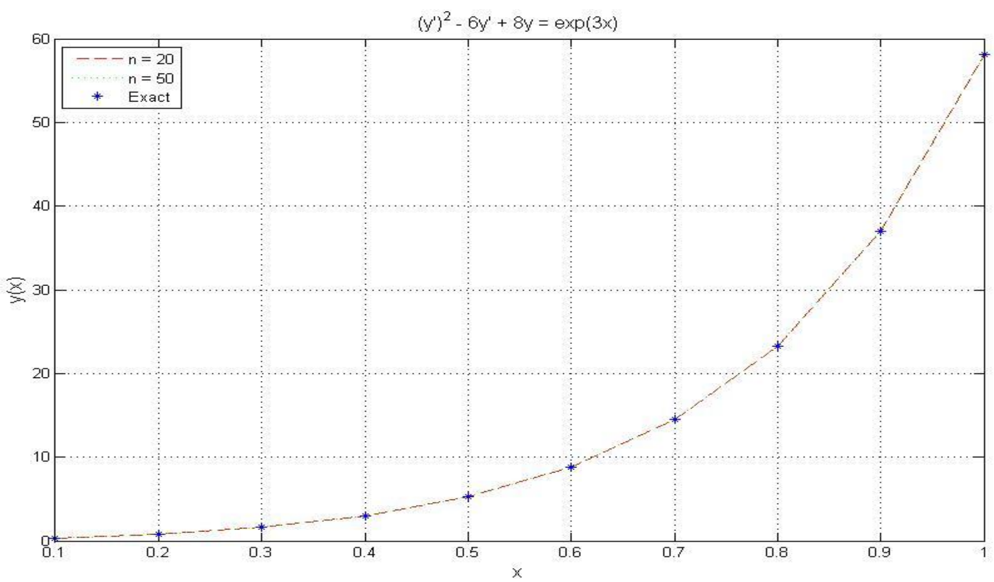
Table 2: Shows the approximate analytic solution, the exact solution and the error when n = 50.

$y_{DT}(x) = 2x + \frac{13}{2!}x^2 + \frac{65}{3!}x^3 + \frac{295}{4!}x^4 + \frac{1277}{5!}x^5 + \cdots$

$y_{Exact}(x) = \frac{3}{2}e^{4x} - \frac{1}{2}e^{2x} - e^{3x}$

x	$Exact$	$D.T, n = 50$	$Absolute\ Error$
0	0	0	0
0.1	0.277176859805818	0.277176859805817	0.000000000000001
0.2	0.770280243527558	0.770280243527558	0.000000000000000
0.3	1.609512872752617	1.609512872752618	0.000000000000001
0.4	2.996661249609891	2.996661249609891	0.000000000000068
0.5	5.242754163828388	5.242754163828320	0.000000000003176
0.6	8.825058645181183	8.825058645178007	0.000000000082442
0.7	14.473200260655588	14.473200260573146	0.000000000000068
0.8	23.299102702824865	23.299102701435007	0.000000001389857
0.9	36.992796208437667	36.992796191597400	0.000000016840268
1.0	58.117160077063367	58.117159919807044	0.000000157256324

Figure 2: Representing the exact solution and the numerical solution



4.0 RESULT OF DISCUSSION

In this study, approximate analytical solution was obtained using the differential transform for solving first and second order ODEs. The Differential Transform approach produces unique advantages over other numerical techniques as it does not involve linearization, discretization or perturbation of a given problem; hence it has no effect of computational round off error. This approach also provides a closed-form solution; therefore, it is very powerful and effective in finding both analytical and numerical solutions of first and second order ordinary differential equations. The method gives rapidly converging series solutions.

Based on the results obtained in this work, from the figures and tables we can see that for the method gives an accurate result for the linear equations and for the non-linear equations, it gives an approximate analytical solution. Thus, the accuracy of the obtained solutions can be improved by taking more terms in the series.

5.0 CONCLUSION

This study reveals that with the use of Differential Transform, lesser and easier computations is effective and convenient for obtaining the exact and approximate analytic solution. It's simple in applicability as it does not require linearization, discretization or perturbation like other numerical and approximate methods. It shows that the approach is reliable, powerful and a promising method for linear and non-linear equations. The accuracy of the obtained solution can be improved by taking more terms in the solution. In many cases, the series solution obtained with differential transform method can be written in exact closed form.

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