Review Article

Stability Analysis of the Chaotic Reverse Butterfly-Shaped Dynamical System Using Hurwitz Polynomials

Abstract

The stability of a system of differential equations describes how it responds to significantly small perturbation. This qualitative behavior of system of differential equation can be studied using Lyapunov or Hurwitz polynomials. The later reduces the problem of stability to an algebraic problem, providing necessary and sufficient criteria for which the system is stable. In this paper, the stability analysis of the chaotic reverse butterfly-shaped dynamical system is presented using Hurwitz polynomials. Results are verified with numerical simulations in MAPLE software.

Keywords: Differential equations, Dynamical system, Hurwitz polynomial, Stability, Routh-Hurwitz criterion, Bifurcation, chaos, reverse butterfly-shaped system

2010 Mathematics Subject Classification: 34D10; 34H10; 34H20

1 Introduction

In the study of systems of differential equations, stability analysis aims at establishing necessary and sufficient conditions for which trajectories close to the system's initial condition remain so at all future times or tends to stationary solutions. Stability theory is concern with the qualitative behavior of a solution of dynamical systems to a significantly small perturbation of the initial value of the solution [(4)]. This notion of proximity of variation in the initial conditions of systems began in the early days of the study of mechanics. Initially studied by famous physicists and mathematics such as Lagrange and Dirichlet. Aleksandr Mikhailovich Lyapunov, a Russian mathematician is known to have laid the foundation of stability theory in his Ph.D. dissertation titled 'The General Problem of Motion Stability' in 1892 [(10; 5)]. Lyapunov established two methods to analyze the stability of a system of ordinary differential equations, one of which is the based on the use Hurwitz type polynomials.

Chaotic systems are sensitive to initial conditions. Henri Poincare had a first glimpse of the complexity chaos in his quest to find a solution to Newton's three body problem during a competition in honor of King Oscar II [(1)]. Edward Lorenz, a meteorologist and professor at MIT reintroduced the theory of chaos in 1960 during his research to simulate and predict the weather. Lorenz developed a deterministic model which he would simulate, yet he could not predict the outcome. He observed a great variation in his climatological results with a slight variation (decimal difference) in the initial condition which he had entered mistakenly. Lorenz' experience awakened research into stability, chaos, bifurcation among others [(6; 8; 3)]. The reverse butterfly-shaped dynamical system is a chaotic system similar to the Lorenz system but with different topological structure which was first propsed by [(9)].

To determine the stability of the nonlinear expressed in the form $\dot{x} = Ax$ it is relevant to observe the nature of the roots of the characteristic polynomial associated with corresponding eigenvalues of A. Thus, the problem of stability analysis becomes an algebraic problem. A system is considered asymptotically stable if all the roots of the characteristics polynomial have negative real parts. Thus, lie in the left half of the complex plane. Such a characteristic polynomial is called a Hurwitz polynomial.

This paper presents the stability analyses of the chaotic reverse butterfly-shaped system using the standard Routh-Hurwitz stability criteria. The rest of the paper is structured as follows; section two outlines the Routh-Hurwitz stability criteria. Section three presents stability analysis of the reverse butterfly-shaped system. Finally, we perform simulation of numerical results using Maple Software in section four.

2 Hurwitz polynomial

Let

$$\dot{x} = Ax \tag{2.1}$$

be the linearization of the nonlinear system

$$\dot{x} = f(x) \tag{2.2}$$

where x is a vector and A, a square matrix. The stability of the equilibrium points of the nonlinear system (**??**) can be determined from the nature of eigenvalues of the associated matrix A of (2.1). If all the eigenvalues of A are negative, then we conclude that (2.2) is asssymptotically stable. Thus, the problem determining stability of the system is finding necessary and sufficient conditions for which all the roots of the characteristic polynomial lie in the left half of the complex plane. Edward Routh and Adolf Hurwitz developed an algorithm which solves this problem providing explicit conditions.

2.1 Routh – Hurwitz criterion

Suppose that

$$p(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n \tag{2.3}$$

is the characteristic polynomial corresponding to the matrix A of the linear system (2.1). We construct a matrix H from the coefficients of the characteristic polynomial (2.3) as follows;

$$H = \begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 \\ a_0 & a_2 & a_4 & a_6 & \cdots & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 \\ 0 & a_2 & a_4 & a_6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix} \tag{2.4}$$

where we Write the coefficients of the polynomial (2.3) with odd positions beginning with a_1 on the first row of H. Coefficients of the polynomial (2.3) with even positions fill the second row beginning with a_0. Subsequent row entries are formed as follows;

$$h_{ij} = \begin{cases} a_{2j-i}, & 0 < 2j - i \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

With the exception of the last element on the leading diagonal of H, which has the coefficient a_n - the last coefficient of the polynomial (2.3), all other elements of the last column of H are null. The matrix H is called Hurwitz matrix.

Theorem 2.1. *The polynomial (2.3), with its positive leading coefficient* (a_0) *is a Hurwitz polynomial if and only if all the diagonal principal minors of the Hurwitz matrix are positive [(11)].*

The principal diagonal minors of the matrix H are;

$$\Delta_1 = |a_1|, \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \quad \Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

$$\Delta_n = a_n \cdot \Delta_{n-1}.$$

Also, from the polynomial (2.3), the Routh array is constructed as follows;

$$\begin{array}{c|cccc} \lambda^n & a_0 & a_2 & a_4 & \cdots \\ \lambda^{n-1} & a_1 & a_3 & a_5 & \cdots \\ \lambda^{n-2} & b_0 & b_1 & b_2 & \cdots \\ \lambda^{n-3} & c_0 & c_1 & c_2 & \cdots \\ \lambda^{n-4} & d_0 & d_1 & d_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{2.5}$$

Again, notice that the first row of the array (2.5) begins with a_0 which is the first coefficient of the polynomial (2.3), followed by coefficients with even positions. Coefficients a_1 of the polynomial (2.3) (3) begins the second row of the array (2.5) followed by the next odd position coefficient a_3, and so on. Subsequent array entries are obtained as follows;

$$\begin{aligned} b_0 &= a_2 - \tfrac{a_0}{a_1} a_3, \quad b_1 = a_4 - \tfrac{a_2}{a_3} a_5, \quad \cdots \\ c_0 &= a_3 - \tfrac{a_1}{b_0} b_3, \quad c_1 = a_5 - \tfrac{a_3}{b_1} b_2, \quad \cdots \\ d_0 &= b_1 - \tfrac{b_0}{c_0} c_1, \quad d_1 = b_2 - \tfrac{b_1}{c_1} c_2, \quad \cdots \end{aligned} \tag{2.6}$$

The number roots of the polynomial (2.3) in the right half plane of the complex plane is equal to the number of sign variations of the first column in Routh's array (2.5). Moreover, the characteristic polynomial (2.3) Hurwitz if and only if when performing Routh's array (2.5) all values in the first column are nonzero of the same sign [(7)].

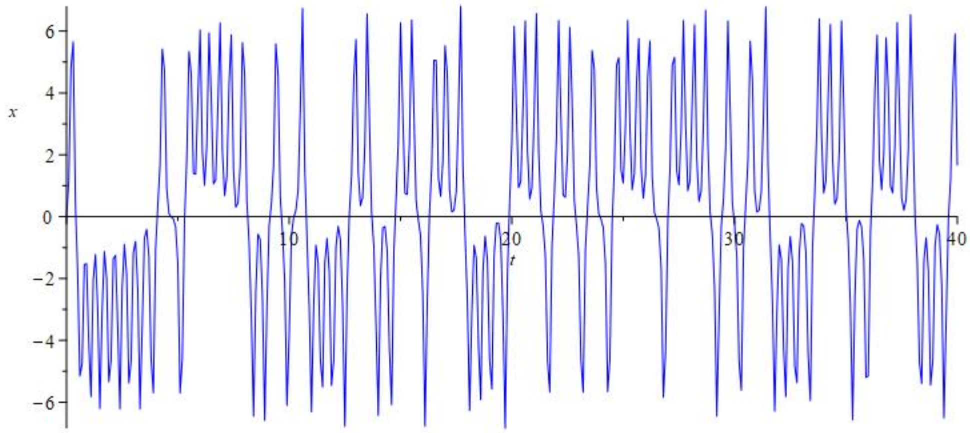

Figure 1: x timeseries solutions of (3.7)

3 Reverse Butterfly-Shaped System

The reverse bbutterfly-shaped ssystem is given by

$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= bx + kxz \\ \dot{z} &= -cz - hxy \end{aligned} \tag{3.1}$$

where x, y, z are the state variables and a, b, c, h, k are positive parameters. Having jux two nonlinearities xy and xz, the deterministic system exhibits complicated dynamics such as random yet without any stochastic input as shown in figure (1).

To demonstrate the system's sensitivity to initial condition which a fundamental characteristic of chaotic system, we construct a time series plot for the flow of x for initial conditions $x(0) = y(0) = 1, z(t) = 0$ captured in blue color and $x(0) = 1.001, y(0) = 1, z(0) = 1$ also captured in green color. The figure (2) shows a significant difference in the solution with just a 0.1 percentage change in the initial conditions of the variable x. The solutions are same for a short while after which it enters into the state of unpredictability. At $t = 10$, the two solutions are on different sides as seen in figure (2).

3.1 Lyapunov Exponents

Lyapunov exponents quantify a system's sensitivity to initial conditions.

Definition 3.1. Let x_0 be an initial condition and a nearby point $x_0 + \delta_0$. Let δ_n be the separation of orbit from x_0 and orbit δ_n. If $|\delta_n| \approx |\delta_0|e^{n\lambda}$, then λ is called Lyapunov Exponent. $\lambda > 0$ is a signature of chaos.

Using [(2)] algorithm for computing Lyapunov Exponents we obtain the Lyapunov exponents $LE1 = 1.2674$, $LE2 = -0.0089$ and $LE3 = -13.3502$. LE1 positive implies that the reverse butterfly shaped system is chaotic. Thus, close orbits grow exponentially, separating from each other. The following table shows data for the figure (3) obtained from Matlab.

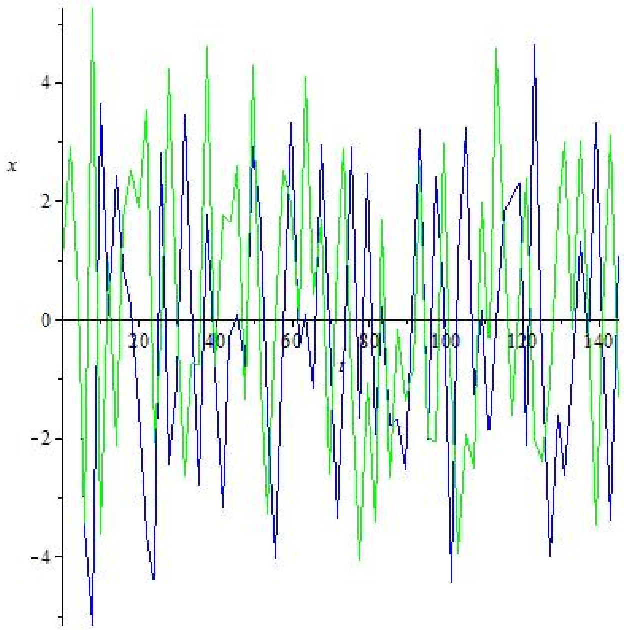

Figure 2: Sensitivity to initial conditions

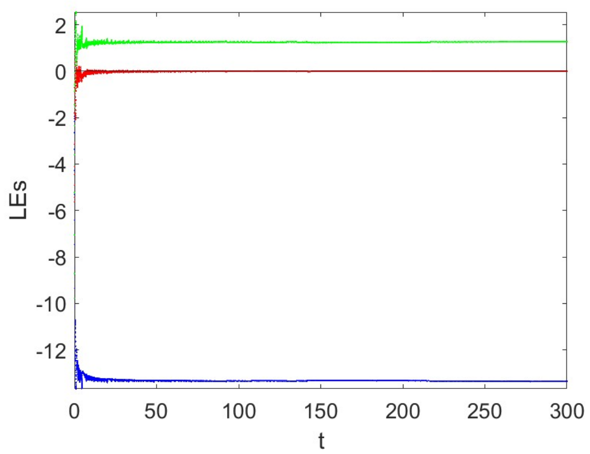

Figure 3: Lyapunov exponents for $t \in [0, 300]$

Time	LE1	LE2	LE3
20.00	1.0581	-0.1213	-13.0163
40.00	1.2286	-0.0466	-13.2582
60.00	1.2148	-0.0432	-13.2553
80.00	1.2332	-0.0233	-13.2941
100.00	1.2504	-0.0206	-13.3136
120.00	1.2327	-0.0139	-13.3050
140.00	1.2260	-0.0203	-13.2938
160.00	1.2199	-0.0194	-13.2887
180.00	1.2182	-0.0086	-13.2985
200.00	1.2241	-0.0147	-13.2997
220.00	1.2547	-0.0049	-13.3425
240.00	1.2571	-0.0055	-13.3447
260.00	1.2100	-0.0084	-13.2934
280.00	1.2141	-0.0048	-13.3011
300.00	1.2144	-0.0048	-13.3012

Table 1: Lyapunov exponents of reverse butterfly-shaped system.

3.2 Equilibrium Points]

The system (3.7) can be rewritten in the vector form

$$\dot{X} = F$$

where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } F = \begin{pmatrix} a(y-x) \\ bx + kxz \\ -cz - hxy \end{pmatrix}.$$

The stationary points are given by $F = 0$. Solving this system shows that the stationary points are

1. origin, $o(0, 0, 0)$.
2. non-trivial symmetric twin equilibria $E^+ = \left(\sqrt{\frac{bc}{kh}}, \sqrt{\frac{bc}{kh}}, -\frac{b}{k}\right)$ and $E^- = \left(-\sqrt{\frac{bc}{kh}}, -\sqrt{\frac{bc}{kh}}, -\frac{b}{k}\right)$.

3.3 Stability of the Origin

The Jacobian matrix of the system (3.7) is

$$J(x, y, z) = \begin{bmatrix} -a & a & 0 \\ kz + b & 0 & kx \\ -hy & -hx & -c \end{bmatrix} \tag{3.2}$$

Evaluating the Jacobian matrix at the origin equilibrium point gives

$$J(0, 0, 0) = \begin{bmatrix} -a & a & 0 \\ 0 & 0 & 0 \\ 0 & 0x & -c \end{bmatrix} \tag{3.3}$$

and the characteristic polynomial

$$P(\lambda) = \lambda^3 + (a+c)\lambda^2 + (ac-ab)\lambda - abc = 0 \tag{3.4}$$

The Hurwitz matrix associated with the polynomial (**??**) for $a_0 = 1, a_1 = (a+c), a_2 = a(c-b)$ and $a_3 = -abc$ is given by

$$H = \begin{bmatrix} (a+c) & -abc \\ 1 & a(c-b) \end{bmatrix} \tag{3.5}$$

From section .. the characteristic polynomial (3.4) is Hurwitz if and only if

$$\Delta_1 = |a+c| > 0$$

$$\Delta_2 = \begin{vmatrix} (a+c) & -abc \\ 1 & a(c-b) \end{vmatrix} > 0$$

Now, a, c and postive, hence Δ_1 is positive. $Delta_2 > 0$ if the following condition holds;

$$-a^2b + a^2c + ac^2 > 0 \tag{3.6}$$

The terms a^2c and ac^2 are positive for all positive values of a and c. Howevers, $-a^2b$ is only positive when $b < 0$ which can not be be b is nonnegative. Therefore the characteristic polynomial (3.4) is not Hurwitz.
Also, from the linearized system;

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= bx \\ \dot{z} &= -cz \end{aligned} \tag{3.7}$$

It is immediately observed that, the system's y and z equations break out with solutions $y(t) = e^{bt}$ and $z(t) = e^{-ct}$. This reveals that $y(t)$ grows exponential fast from the origin whereas $z(t)$ approaches the origin exponentially which indicates that the system is unstable. Furthermore, judging from the eigenvalues of the charactertic polynomial (3.4), which are;

$$\begin{aligned} \lambda_1 &= -c \\ \lambda_2 &= -\frac{1}{2}a + \frac{1}{2}\sqrt{a^2+4ab} \\ \lambda_3 &= -\frac{1}{2}a - \frac{1}{2}\sqrt{a^2+4ab} \end{aligned} \tag{3.8}$$

From the eigenvalues (3.8), $\lambda_1, \lambda_3 < 0$. λ_2 is also negative because the square root is an increasing function in $[0, \infty$. We conclude that the origin is a saddle.

3.4 Stability of $E^{\pm}$

Evaluating the Jacobian matrix (3.2) at the equilibria $E^{\pm}$ yields the common characteristic polynomial

$$P^{E^{\pm}}_{a,b,c}(\lambda) = \lambda^3 + (a+c)\lambda^2 + (ac+bc)\lambda + 2abc \tag{3.9}$$

Following from section...the matrix H is obtain as

$$H = \begin{bmatrix} (a+c) & 2abc \\ 1 & c(a+b) \end{bmatrix} \tag{3.10}$$

for $a_0, a_1 = (a+c), a_(2) = a(a+b)$ and $a_3 = 2abc$. The characteristic polynomial (3.9) is Hurwitz if and only if

$$\Delta_1 = |a+c| > 0$$

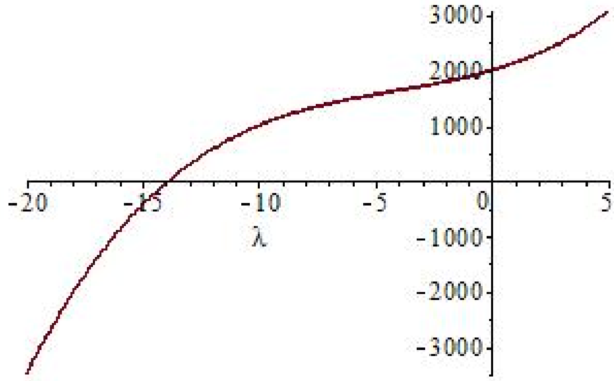

Figure 4: Graph of $P^{E^{\pm}}_{a,b,c}(\lambda)$

$$\Delta_2 = \begin{vmatrix} a+c) & 2abc \\ 1 & c(a+b) \end{vmatrix} > 0$$

. We construct the associated Routh arrary as follows;

$$\begin{array}{ccc} \lambda^3 & 1 & c(a+b) \\ \lambda^2 & a+c & 2abc \\ \lambda^1 & b_0 & b_1 \\ \lambda^0 & c_0 & c_1 \end{array}$$

where;

$$b_0 = \frac{c(a+b)(a+c) - 2bc}{a+c}$$

and

$$c_0 = 2abc > 0$$

b_1 and c_1 are zero.
From Routh Hurwitz criterion, the number of roots of the characteristic polynomial with positive real parts is equal to the number of changes in sign of the first column of the Routh array. c_0 is positive, ttherefore, the only necessary and sufficient for $E^{\pm}$ to be stable is $b_0 > 0$. Thus;

$$\frac{c(a+b)(a+c) - 2bc}{a+c} > 0$$

implying that

$$c > \frac{a(b-a)}{a+b}. \tag{3.11}$$

At (3.11), the characteristic polynomial (2.3) has a negative real eqigevalue and a complex conjugate with positive real parts as shown in figure (4). Next, we investigate the possibility of $P^{E^{\pm}}_{a,b,c}(\lambda)$ assuming purely imaginary roots. To do this, we assume $\lambda = i\omega$ is an eigenvalue and substitute into $P^{E^{\pm}}_{a,b,c}(\lambda)$. We get

$$(i\omega)^3 + (a+c)(i\omega)^2 + (ac+bc)i\omega + 2abc = 0 \tag{3.12}$$

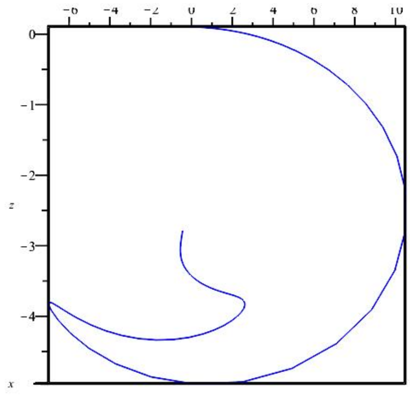

Figure 5: Origin unstable at $c = 1$, $t = 0..1$

Equating the real and imaginary parts to zero gives

$$c = \frac{2ab}{a+b} - a \tag{3.13}$$

Thus at (3.13), the system has purely imaginary roots. For the systems parameters, this occurs at $c^* = 6.0$ where a Hopf bifurcation occurs.

4 Numerical Results

In this section, we verify the stability results obtained from the previous section with simulations performed in MAPLE software. We c use as a control parameter for our simulations.

4.1 Stability of the Origin $O(0, 0, 0)$ and $E^{\pm}$

We obtained that the equilibrium point $O(0, 0, 0)$ is unstable for all values of c. We show this in figures (5) and (6) Trajectory started close to the origin $x(0) = y(0) = z(0) = 0.1$ travel away from it indicating that the origin not attracting. However, as time increases these trajectories from the unstable manifold of the origin are attracted onto the stable manifolds of the equilibria $E^{\pm}$ as shown in figures (7) and (8). At $c = 2.4$, the equilibria $E^{\pm}$ are chaotic with strange attractors. This is shown in figure (9). At $c = 6.0$, the system 3.7) experiences a Hopf bifurcation. The characteristic polynomial (2.3) has a real eigenvalue and a pair of complex conjugates with zero real part. Figure (10) illustrates the structural change at $c = 6.0$. At $c > 6.0$, the system (3.7) is stable with a negative real eigenvalue, and a pai of complex conjugates, also with same negative real parts. This is shown in figure (11) with $c = 7.0$ for instance.

5 Conclusion

Hurwitz polynomials reduces the problem of stability determination of system of equations to an algebra problem. These algebraic stability criteria from Hurwitz polynomials provides necessary and sufficient condition for which the system's stability can be achieved. Hurwitz polynomials have

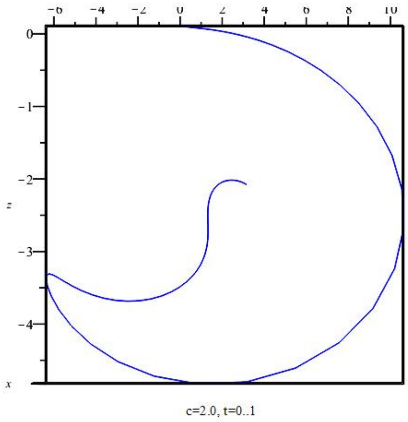

Figure 6: Origin unstable at $c = 2$, $t = 0..1$

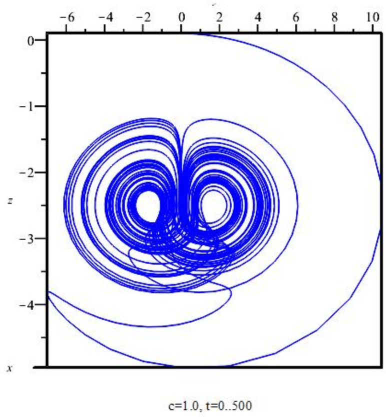

Figure 7: Stablity of $E^{\pm}$ at $c = 1$, $t = 0..1$

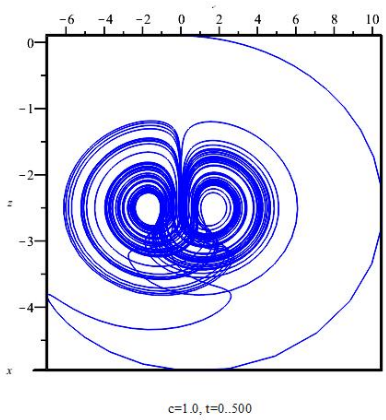

Figure 8: Stability of $E^{\pm}$ at $c = 1$, $t = 0..500$

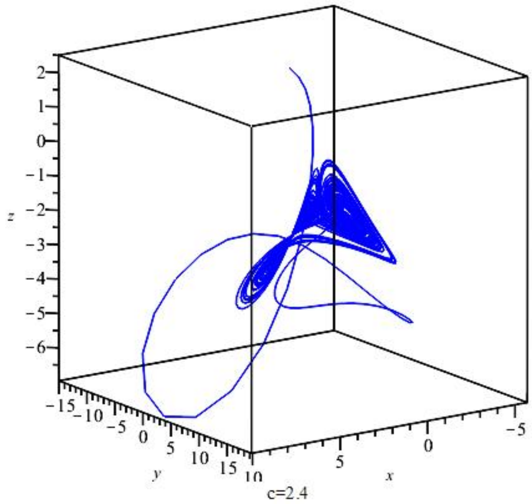

Figure 9: System is Chaotic at $c = 2.4$

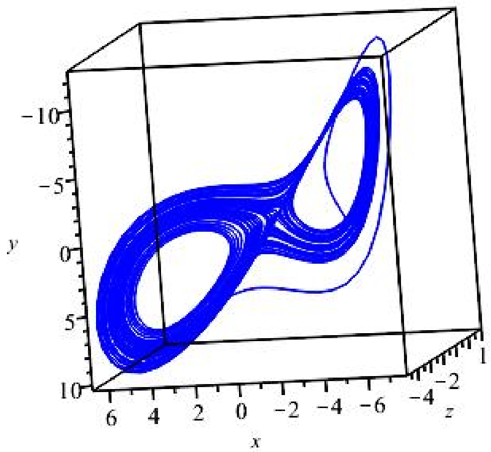

Figure 10: Bifurcation at $c = 6.0$

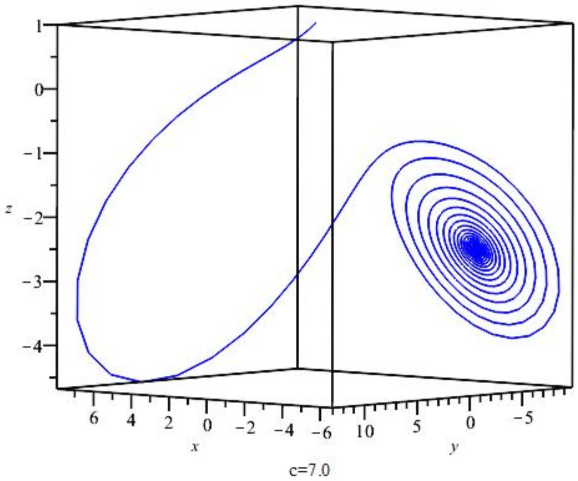

Figure 11: Stability at $c = 7.0$

been used to perform the stability analysis of the chaotic reverse butterfly-shaped dynamical system. Stability results were verified with simulation in MAPLE software.

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