On Some Mixed Polynomial Exponential Diophantine Equation: $\alpha^n + \beta^n + a(\alpha^s \pm \beta^s)^m + D = r(u^k + v^k + w^k)$

Original Research Article

Abstract

The study of Mixed polynomial diophantine equation is still an active area of research. May be because of the fact that diophantine equation have direct application in real life. Many studies seems to have focused on Ramanujan Nagell equation $x^2 + D = AB^n$ where x, n, A and B are variables and D is a fixed integer. Let $a, \alpha, \beta, r, u, v, w$ and D be any integers and suppose that n, m, s and k are non-negative exponent. In this paper, the diophantine equation $\alpha^n + \beta^n + a(\alpha^s \pm \beta^s)^m + D = r(u^k + v^k + w^k)$ is developed and investigated for integer solution and its various polynomial identities. Moreover, the study formulates some conjectures for the title equation
Keywords: Diophantine Equation, Mixed Polynomial.

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

1 Introduction

The study of integer decomposition into sums of powers are classical and has been a subject of considerable attention in recent past. Perhaps, may be because of the fact that, the study of integer decomposition has a direct application in the field of cryptography. Most researchers seems to have devoted their attention on Ramanujan Nagell Equation and sums of powers. For recent work on polynomial equations of sums of powers the reader may survey [9,10,11,12,13,14, 15,16,17,18,19,20] and for detailed recap on Ramanujan Nagell Equation the reader may refer to[1, 2,3,4,5,6,7,8] . In most of this studies, the literature involving mixed polynomial and sums of powers is still hardily available. Moreover, documented results on diophantine equation $x^n + y^n + a(x^s \pm y^s)^m + D = r(u^k + v^k + w^k)$ proposed in this is not known. This study is therefore, set to introduce and develop the formula $x^n + y^n + a(x^s \pm y^s)^m + D = r(u^k + v^k + w^k)$.

2 Main Results

The following assumptions will apply in this research. All numbers will be treated as integers, and it will be assumed that β is greater than α.

Conjecture 2.1. *For any integer $\beta > \alpha$ and exponent $m, n, k \geq 2$ and $s \geq 1$, there exist integers a, u, v, w and r such that*

$$\alpha^n + \beta^n + a(\alpha^s \pm \beta^s)^m + D = r(u^k + v^k + w^k) \cdots (1)$$

where D is an integer

In the sequel, we begin by constructing some solution of conjecture 2.1. We prioritize, determination of the unknowns a, m, n, r, s, k and D for which the equation $\alpha^n + \beta^n + a(\alpha^s \pm \beta^s)^m + D = r(u^k + v^k + w^k)$ has solution. The following cases has been considered. That is, $(a, m, n, k, r, s, D) = (1, 4, 4, 2, 2, 1, 0), (a, m, n, k, r, s, D) = (1, 4, 4, 2, 2, 1, 2)$.

Theorem 2.2. *Consider equation (1) satisfying the condition $(a, m, n, k, r, s, D) = (1, 4, 4, 2, 2, 1, 0)$. Then the diophantine equation $\alpha^4 + \beta^4 + (\alpha + \beta)^4 = 2(u_1^2 + v_1^2 + w_1^2)$ has solution in integers if α and β are consecutive*

Proof. Suppose α and β are consecutive integers and Consider the equation $\alpha^4 + \beta^4 + (\alpha + \beta)^4 = 2(u_1^2 + v_1^2 + w_1^2)$. The L.H.S expressed as $\alpha^4 + \beta^4 + (\alpha + \beta)^4 = \alpha^4 + (\alpha + 1)^4 + (2\alpha + 1)^4$ simplifies to $18\alpha^4 + 36\alpha^3 + 30\alpha^2 + 12\alpha + 2$. Rewriting the equation $18\alpha^4 + 36\alpha^3 + 30\alpha^2 + 12\alpha + 2 = 2(u_1^2 + v_1^2 + w_1^2)$ and dividing both sides by 2 we get $9\alpha^4 + 18\alpha^3 + 15\alpha^2 + 6\alpha + 1 = u_1^2 + v_1^2 + w_1^2$. To determine the value of u_1, v_1 and w_1 assume $u_1 = a\alpha^2 + b\alpha + c, v_1 = d\alpha^2 + e\alpha + f$ and $w_1 = g\alpha^2 + h\alpha + i$. Thus, $u_1^2 + v_1^2 + w_1^2 = (a\alpha^2 + b\alpha + c)^2 + (d\alpha^2 + e\alpha + f)^2 + (g\alpha^2 + h\alpha + i)^2 = a^2\alpha^4 + 2ab\alpha^3 + (2ac + b^2)\alpha^2 + 2bc\alpha + c^2 + d^2\alpha^4 + 2de\alpha^3 + (2df + e^2)\alpha^2 + 2ef\alpha + f^2 + g^2\alpha^4 + 2gh\alpha^3 + (2gi + h^2)\alpha^2 + 2hi\alpha + i^2 = (a^2 + d^2 + g^2)\alpha^4 + (2ab + 2de + 2gh)\alpha^3 + (2ac + b^2 + 2df + e^2 + 2gi + h^2)\alpha^2 + (2bc + 2ef + 2hi)\alpha + (c^2 + f^2 + i^2) = 9\alpha^4 + 18\alpha^3 + 15\alpha^2 + 6\alpha + 1$. Matching the coefficient we have

$$\begin{cases} a^2 + d^2 + g^2 = 9 \cdots (i), \\ 2ab + 2de + 2gh = 18 \cdots (ii), \\ 2ac + b^2 + 2df + e^2 + 2gi + h^2 = 15 \cdots (iii), \\ 2bc + 2ef + 2hi = 6 \cdots (iv), \\ c^2 + f^2 + i^2 = 1 \cdots (v). \end{cases}$$

Clearly, the system has 9 variables and 5 equation, thus there is no viable method to solve the system. Hence, we result to method of inspection. To solve the system we find the possible integer values $(a, b, c, d, e, f, g, h, i)$ that satisfy the system. We shall use step by step approach to determine correctly the solution set. From equation (1), $a^2 + d^2 + g^2 = 9$. Assume $a = 1$, thus $d^2 + g^2 = 8$. We need to find integer values for which $d^2 + g^2 = 8$. The only positive integer values are $d = 2$ and $g = 2$. Hence, $a^2 + d^2 + g^2 = 1^2 + 2^2 + 2^2 = 9$. Substituting the solution set $(a, d, g) = (1, 2, 2)$ into equation (ii) we obtain $2b+4e+4h=18$. Dividing both sides by 2 we obtain $b+2e+2h=9$. Need to find the solution set (b,e,h) which satisfy $b+2e+2h=9$. Letting $b=1, e=1, h=3$ we have $1+2(1)+2(3)=9$. Thus, $(b,e,h)=(1,1,3)$ is a solution. Substituting the solution $(a,d,g,b,e,h)=(1,2,2,1,1,3)$ in equation (iii) we have $2c+4f+4i=4$. Assuming $c=0, f=0, i=1$ we get $2(0)+4(0)+4(1)=4$. Thus $(c,f,i)=(0,0,1)$ is a solution. Since all the solution set have been determined i.e $(a,d,g,b,e,h,c,f,i)=(1,2,2,1,1,3,0,0,1)$ we consider this solution into equation (iv) and (v). Considering equation (iv),$2bc+2ef+2hi=2(1)(0)+2(1)(0)+2(3)(1)=6$. Hence, equation (iv) is satisfied. Finally, considering equation (v) we have $c^2+f^2+i^2=0^2+0^2+1^2=1$ which satisfies the equation. Consequently, $u_1=\alpha^2+\alpha, v_1=2\alpha^2+\alpha$ and $w_1=2\alpha^2+3\alpha+1$. Since u_1, v_1 and w_1 are known the result easily follows. □

2.1 Applications

In this subsection, we provide some examples to argument our results in Theorem 2.1 for case (i).

α^4	β^4	$(\alpha+\beta)^4$	I	$u_1^2=(\alpha^2+\alpha)^2$	$v_1^2=(2\alpha^2+\alpha)^2$	$w_1^2=(2\alpha^2+3\alpha+1)^2$	
1	16	81	98	4	9	36	
16	81	625	722	36	100	225	
81	256	2401	2738	144	441	784	
256	625	6561	7442	400	1296	2025	
625	1296	14641	16562	900	3025	4356	
1296	2401	28561	32258	1764	6084	8281	
2401	4096	50625	57122	3136	11025	14400	

Table 1: $\alpha^4+\beta^4+(\alpha+\beta)^4=I=2(u^2+v^2+w^2)$

Theorem 2.3. *Consider equation* (1) *satisfying the condition* $(a,m,n,k,r,s,D)=(1,4,4,2,2,1,2)$. *Then the diophantine equation* $\alpha^4+\beta^4+(\alpha+\beta)^4+2=2(u^2+v^2+w^2)$ *has solution in integers if* α *and* β *are consecutive*

Proof. Suppose α and β are consecutive integers and Consider the equation $\alpha^4+\beta^4+(\alpha+\beta)^4+2=2(u^2+v^2+w^2)$. The L.H.S expressed as $\alpha^4+\beta^4+(\alpha+\beta)^4+2=\alpha^4+(\alpha+1)^4+(2\alpha+1)^4+2$ simplifies to $18\alpha^4+36\alpha^3+30\alpha^2+12\alpha+4$. Rewriting the equation $18\alpha^4+36\alpha^3+30\alpha^2+12\alpha+4=2(u^2+v^2+w^2)$ and dividing both sides by 2 we get $9\alpha^4+18\alpha^3+15\alpha^2+6\alpha+2=u^2+v^2+w^2$. To determine the value of u, v and w assume $u=a\alpha^2+b\alpha+c, v=d\alpha^2+e\alpha+f$ and $w=g\alpha^2+h\alpha+i$. Thus, $u^2+v^2+w^2=(a\alpha^2+b\alpha+c)^2+(d\alpha^2+e\alpha+f)^2+(g\alpha^2+h\alpha+i)^2=a^2\alpha^4+2ab\alpha^3+(2ac+b^2)\alpha^2+2bc\alpha+c^2+d^2\alpha^4+2de\alpha^3+(2df+e^2)\alpha^2+2ef\alpha+f^2+g^2\alpha^4+2gh\alpha^3+(2gi+h^2)\alpha^2+2hi\alpha+i^2=(a^2+d^2+g^2)\alpha^4+(2ab+2de+2gh)\alpha^3+(2ac+b^2+2df+e^2+2gi+h^2)\alpha^2+(2bc+2ef+2hi)\alpha+(c^2+f^2+i^2)=9\alpha^4+18\alpha^3+15\alpha^2+6\alpha+1$. Matching the coefficient we have

$$\begin{cases} a^2+d^2+g^2=9\cdots(i),\\ 2ab+2de+2gh=18\cdots(ii),\\ 2ac+b^2+2df+e^2+2gi+h^2=15\cdots(iii),\\ 2bc+2ef+2hi=6\cdots(iv),\\ c^2+f^2+i^2=2\cdots(v).\end{cases}$$

Clearly, the system has 9 variables and 5 equation, thus there is no viable method to solve the system. Hence, we result to method of inspection. To solve the system we find the possible integer values $(a, b, c, d, e, f, g, h, i)$ that satisfy the system. We shall use step by step approach to determine correctly the solution set. From equation (1), $a^2 + d^2 + g^2 = 9$. Assume $a = 1$, thus $d^2 + g^2 = 8$. We need to find integer values for which $d^2 + g^2 = 8$. The only positive integer values are $d = 2$ and $g = 2$. Hence, $a^2 + d^2 + g^2 = 1^2 + 2^2 + 2^2 = 9$. Substituting the solution set $(a, d, g) = (1, 2, 2)$ into equation (ii) we obtain $2b + 4e + 4h = 18$. Dividing both sides by 2 we obtain $b + 2e + 2h = 9$. Need to find the solution set (b, e, h) which satisfy $b + 2e + 2h = 9$. Letting $b = 1, e = 2, h = 2$ we have $1 + 2(2) + 2(2) = 9$. Thus, $(b, e, h) = (1, 2, 2)$ is a solution. Substituting the solution $(a, d, g, b, e, h) = (1, 2, 2, 1, 2, 2)$ in equation (iii) we have $2c + 4f + 4i = 6$. Assuming $c = 1, f = 0, i = 1$ we get $2(1) + 4(0) + 4(1) = 6$. Thus $(c, f, i) = (1, 0, 1)$ is a solution. Since all the solution set have been determined i.e $(a, d, g, b, e, h, c, f, i) = (1, 2, 2, 1, 1, 2, 1, 0, 1)$ we consider this solution into equation (iv) and (v). Considering equation (iv),$2bc + 2ef + 2hi = 2(1)(1) + 2(1)(0) + 2(2)(1) = 6$. Hence, equation (iv) is satisfied. Finally, considering equation (v) we have $c^2 + f^2 + i^2 = 1^2 + 0^2 + 1^2 = 2$ which satisfies the equation. Consequently, $u = \alpha^2 + \alpha + 1, v = 2\alpha^2 + 2\alpha$ and $v = 2\alpha^2 + 2\alpha + 1$. Since u, v and w are known the result easily follows. ☐

In this subsection, we provide some examples to argument our results in Theorem 2.3 for case (ii).

α^4	β^4	$(\alpha+\beta)^4+2$	I_1	$u^2 = (\alpha^2+\alpha+1)^2$	$v^2 = (\alpha^2+2\alpha)^2$	$w^2 = (2\alpha^2+2\alpha+1)^2$
1	16	83	100	9	16	25
16	81	627	724	49	144	169
81	256	2403	2740	169	576	625
256	625	6563	7444	441	1600	1681
625	1296	14643	16564	961	3600	7225
1296	2401	28563	32260	1849	7056	12769
2401	4096	50627	57124	3249	12544	21025

Table 2: $\alpha^4 + \beta^4 + (\alpha+\beta)^4 + 2 = I_1 = 2(u^2 + v^2 + w^2)$

Theorem 2.4. *Consider equation* (1) *satisfying the condition* $(a, m, n, k, r, s, D) = (1, 4, 4, 2, 2, 1, 0)$. *Then the diophantine equation* $\alpha^4 + \beta^4 + (\beta - \alpha)^4 = 2(u_2^2 + v_2^2 + w_2^2)$ *has solution in integers if* α *and* β *are consecutive*

Proof. Suppose α and β are consecutive integers and Consider the equation $\alpha^4 + \beta^4 + (\beta - \alpha)^4 = 2(u_2^2 + v_2^2 + w_2^2)$. The L.H.S expressed as $\alpha^4 + \beta^4 + (\beta - \alpha)^4 = \alpha^4 + (\alpha + 1)^4 + 1^4$ simplifies to $2\alpha^4 + 4\alpha^3 + 6\alpha^2 + 4\alpha + 2$. Rewriting the equation $2\alpha^4 + 4\alpha^3 + 6\alpha^2 + 4\alpha + 2 = 2(u_2^2 + v_2^2 + w_2^2)$ and dividing both sides by 2 we get $\alpha^4 + 2\alpha^3 + 3\alpha^2 + 2\alpha + 1 = u_2^2 + v_2^2 + w_2^2$. To determine the value of u_2, v_2 and w_2 assume $u_2 = a\alpha^2 + b\alpha + c, v_2 = d\alpha^2 + e\alpha + f$ and $w_2 = g\alpha^2 + h\alpha + i$. Thus, $u_2^2 + v_2^2 + w_2^2 = (a\alpha^2 + b\alpha + c)^2 + (d\alpha^2 + e\alpha + f)^2 + (g\alpha^2 + h\alpha + i)^2 = a^2\alpha^4 + 2ab\alpha^3 + (2ac + b^2)\alpha^2 + 2bc\alpha + c^2 + d^2\alpha^4 + 2de\alpha^3 + (2df + e^2)\alpha^2 + 2ef\alpha + f^2 + g^2\alpha^4 + 2gh\alpha^3 + (2gi + h^2)\alpha^2 + 2hi\alpha + i^2 = (a^2 + d^2 + g^2)\alpha^4 + (2ab + 2de + 2gh)\alpha^3 + (2ac + b^2 + 2df + e^2 + 2gi + h^2)\alpha^2 + (2bc + 2ef + 2hi)\alpha + (c^2 + f^2 + i^2) = \alpha^4 + 2\alpha^3 + 3\alpha^2 + 2\alpha + 1$. Matching the coefficient we have

$$\begin{cases} a^2 + d^2 + g^2 = 1 \cdots (i), \\ 2ab + 2de + 2gh = 2 \cdots (ii), \\ 2ac + b^2 + 2df + e^2 + 2gi + h^2 = 3 \cdots (iii), \\ 2bc + 2ef + 2hi = 2 \cdots (iv), \\ c^2 + f^2 + i^2 = 1 \cdots (v). \end{cases}$$

Clearly, the system has 9 variables and 5 equation, thus there is no viable method to solve the system. Hence, we result to method of inspection. To solve the system we find the possible integer values $(a, b, c, d, e, f, g, h, i)$ that satisfy the system. We shall use step by step approach to determine correctly the solution set. From equation (1), $a^2 + d^2 + g^2 = 1$. Assume $a = 0$, thus $d^2 + g^2 = 1$. We need to find integer values for which $d^2 + g^2 = 1$. The integer values are $d = 0$ and $g = 1$. Hence, $a^2 + d^2 + g^2 = 0^2 + 0^2 + 1^2 = 1$. Substituting the solution set $(a, d, g) = (0, 0, 1)$ into equation (ii) we obtain $2gh = 2$. Dividing both sides by 2 we obtain $gh = 1$. Clearly, $g = 1$ and $h = 1$. Thus, $(g, h) = (1, 1)$ is a solution. Substituting the solution $(a, d, g, b, h) = (0, 0, 1, 1, 1)$ in equation (iii) we have $b^2 + e^2 = 2$. Assuming $b = 1, e = 1$ we get $1^2 + 1^2 = 2$. Thus $(b, e) = (1, 1)$ is a solution. Substituting the solution set $(a, d, g, b, e, h) = (0, 0, 1, 1, 1, 1)$ into equation (iv) we have $c = 0$. Finally, considering equation (v) we have $c^2 + f^2 + i^2 = 0^2 + 1^2 + 0^2 = 1$ which satisfies the equation. Consequently, $u_2 = \alpha, v_2 = \alpha + 1$ and $w_2 = \alpha^2 + \alpha$. Since u_2, v_2 and w_2 are known the result easily follows. □

In this subsection, we provide some examples to argument our results in Theorem 2.4 for case (i).

α^4	β^4	$(\beta-\alpha)^4$	I_2	$u_2^2 = \alpha^2$	$v_2^2 = (\alpha+1)^2$	$w_2^2 = (\alpha^2+\alpha)^2$	
1	16	1	18	1	4	4	
16	81	1	98	4	9	36	
81	256	1	338	9	16	144	
256	625	1	882	16	25	400	
625	1296	1	1922	25	36	900	
1296	2401	1	3698	36	49	1764	
2401	4096	1	6498	49	64	3136	

Table 3: $\alpha^4 + \beta^4 + (\beta - \alpha)^4 = I_2 = 2(u_2^2 + v_2^2 + w_2^2)$

3 Conclusion

In this present study, the diophantine equation $\alpha^n + \beta^n + a(\alpha^s \pm \beta^s)^m + D = r(u^k + v^k + w^k)$ is introduced and partially analysed for distinct integer solution and examined for various identities. Admittedly, research in this area is still minimal and therefore future research may consider the same diophantine for detailed analysis.

References

[1] Abumuriefa .F.S, (2006). On the Diophantine Equation $x^2 + 5^{2k} = y^n$. *Demonstratio Mathematica*, 319(2): 285-289.

[2] Abu Muriefa.F.S,Luca. S., Tengely. Sz, (2009). On the Diophantine Equation $x^2+C=2y^n$ *Int.J. Number Theory* ,(5), 1117-1128.

[3] Arif.S., Abu Muriefa (1997). On the Diophantine Equation $x^2+2^k=y^n$, *Internat.J.Math.Sci* , (20):299-304.

[4] Arif.S., Abu Muriefa (1998). On the Diophantine Equation $x^2+3^k=y^n$, *Internat.J.Math.Sci* , (21):619-620.

[5] Arif.S., Abu Muriefa (2001). The Diophantine Equation $x^2+q^{2k+1}=y^n$, *J.Number Theory* ,95-100.

[6] Arif.S., Abu Muriefa (1998). The Diophantine Equation $x^2+q^{2k}=y^n$, *Arab.J.Math.Sci* , (226):53-62.

[7] Bugeaud.Y,(1997). On Diophantine Equation $x^2-p^m=\pm y^n$, *Acta Arith.* , (80):213-223.

[8] Bugeaud.Y, Abu Muriefa. F.S.,(2006). The Diophantine Equation $x^2+c=y^n$, *Revista Colombiana de Matematicas.* , (40),31-37.

[9] David A., (2016) A partition-theoretic proof of Fermat's two squares theorem.*Discrete Mathematics* ,339:4:1410–1411, DOI:10.1016/j.disc.2015.12.002.

[10] Lao H., Kinyanjui J.,Zachary K., (2024).On Sums of Squares Involving Integer Sequence: $\sum_{r=1}^{n} w_r^2+\frac{n}{3}d^2=3(\frac{nd^2}{3}+\sum_{r=1}^{\frac{n}{3}} w_{3r-1}^2)$.*Journal of Advances in Mathematics and Computer Science* , Volume 39, Issue 7, Page 1-6, 2024; Article no.JAMCS.118505 ,ISSN: 2456-9968

[11] Lao H., (2024). Analytical Solution of Some Higher Degree Equation Via Radicals.*Journal of Advances in Mathematics and Computer Science* ,volume 39(3):20-28,Article no.JAMCS.113540ISSN: 2456-9968, DOI: 10.9734/JAMCS/2024/v39i31872

[12] Lao H., (2022).Some Formulae For Integer Sums of Two Squares.*Journal of Advances in Mathematics and Computer Science* , 38(8): 47-52, Article no.JAMCS.101314,ISSN: 2456-9968.

[13] Lao H.,Zachary K., Kinyanjui J. (2023).Some Generalized Formula For Sums of Cube.*Journal of Advances in Mathematics and Computer Science* , 37(4): 53-57, Article no.JAMCS.87824,ISSN: 2456-9968.

[14] Mude, Lao Hussein. 2022. On The Diophantine Equation $ab(cd+1)+L=u^2+v^2$,*Asian Research Journal of Mathematics*, 18 (9):8-13. https://doi.org/10.9734/arjom/2022/v18i93040.

[15] Kimtai B., Lao, H., (2023). *On Generalized Sum of Six, Seven and Nine Cube*. Mundi,Vol 3(1),pp.135-142, Science Mundi ISSN:2788-5844 http://sciencemundi.net DOI: https://doi.org/10.51867/scimundi.3.1.14

[16] Mochimaru, Y. (2005). Solution of Sextic Equations. International Journal of Pure and Applied Mathematics, 23(4), 575-583. https://ijpam.eu/contents/2005-23-4/9/9.

[17] Najman .F., (2010). On The Diophantine Equation $x^4\pm y^4=iz^2$ in Gaussian Integers. *Amer. Math. Monthly*, 117(7), 637-641.

[18] Najman .F., (2011).Torsion of elliptic curves over quadratic cyclotomic fields *Math. J. Okayama Univ.*. , 53, 75-82.

[19] Par Y. , (2016).Waring-Golbach problem. Two squares and Higher Powers. *Journal de Theorie des Nombres* ,791-810.

[20] Ruffini, P. (1799). Teoria Generale delle Equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto [General Theory of equations, in which the algebraic solution of general equations of degree higher than four is proven impossible]. Book on Demand Ltd. ISBN: 978-5519056762