

Enhanced Adomian Decomposition Method for Accurate Numerical Solutions of PDE Systems

Abstract

This research addresses critical challenges in numerical solutions, which are vital for various engineering and physical fields. The Modified Adomian Decomposition Method (MADM) is proposed as a novel approach for solving linear and nonlinear partial differential equations (PDEs). MADM builds upon the Adomian Decomposition Method (ADM) by incorporating a new integral operator that significantly improves convergence rates and accuracy.

Numerical examples demonstrate the effectiveness of MADM in handling complex nonlinear PDEs. Compared to traditional ADM, MADM consistently achieves more accurate and rapidly converging solutions. This enhancement is attributed to the novel integral operator, which addresses the limitations of ADM for intricate nonlinear problems.

The paper outlines the application of MADM, its solution procedure, and its effectiveness through numerical examples. Comparisons with standard ADM solutions and exact solutions validate MADM's accuracy and superiority. The results suggest that MADM is a promising tool for expanding the applicability of Adomian methods in the field of solving PDEs.

keywords: Adomian decomposition method, system of PDEs, numerical methods, accuracy, convergence, approximate solutions.

1 Introduction

Partial Differential Equations (PDEs) are indispensable tools for modeling diverse scientific and engineering phenomena. Accurately solving these equations is vital for predicting system behavior, optimizing designs, and analyzing

experimental data [16, 17, 27]. Traditional methods, however, may not be suitable for all PDEs, especially nonlinear ones. This has led to the exploration of alternative approaches [27], such as the well-established Adomian Decomposition Method (ADM).

ADM addresses nonlinear PDEs by decomposing the solution into an infinite series [1, 25]. Researchers have continuously explored modifications and extensions to ADM [2, 3, 5, 7, 11, 12, 13], broadening its applicability to various types of differential equations [9, 14, 15, 20, 26]. Recent advancements, including the integration of integral transforms, have significantly enhanced ADM's capability to solve complex, higher-order, and fractional differential systems [18, 23, 19].

This research introduces a novel approach, the Modified Adomian Decomposition Method (MADM), which builds upon ADM. MADM incorporates a new integral operator specifically [10, 4] designed to improve convergence and accuracy for complex nonlinear problems. Numerical experiments demonstrate MADM's superior performance in obtaining approximate solutions compared to traditional methods. The potential applications of MADM in fields such as fluid dynamics and heat transfer highlight its significance as a valuable tool for computational mathematics.

The following sections provide a detailed explanation of ADM and MADM, including their application to specific nonlinear PDEs. Numerical examples are presented to showcase MADM's effectiveness in handling nonlinearities and converging towards approximate solutions. The conclusion emphasizes the potential benefits of MADM in solving complex nonlinear problems.

2 ADM and MADM for PDE system solutions

2.1 Analysis of ADM

The system of partial differential equations that will be studied in this section takes the general form of:

$$u_i^i(x, t) + \lambda_i(t)u_i(x, t) + N_i(u_1, \dots, u_m) = g_i(x), \quad i = 1, 2, \dots, m. \quad (2.1)$$

The initial conditions are given by

$$u_i(x, 0) = f_i(x).$$

The equation (2.1) is equivalent to

$$L_i u_i(x, t) + \lambda_i(t)u_i(x, t) + N_i(u_1, \dots, u_m) = g_i(x), \quad i = 1, 2, \dots, m. \quad (2.2)$$

where $L_i = \frac{d}{dt}$ represents partial differential operators, R_i and N_i represent linear and nonlinear operator respectively.

Applying the inverse operators $L_i^{-1}(\cdot) = \int_0^t (\cdot) dt$ to system (2.2) and applying beginning conditions, then

$$u_i(x, t) = f_i(x) + L_i^{-1}g_i(x) - L_i^{-1}\lambda_i(t)u_i(x, t) - L_i^{-1}N_i(u_1, \dots, u_m), \quad i = 1, 2, \dots, m. \quad (2.3)$$

The Adomian decomposition method approach presupposes that the unknown functions $u_i(x, t)$ can be written as an infinite series of the form

$$u_i(x, t) = \sum_{n=0}^{\infty} u_n^i(x, t), \quad i = 1, 2, \dots, m. \quad (2.4)$$

The nonlinear operator $N_i(u_1, \dots, u_m)$ is defined using Adomian polynomials.

$$N_i(u_1, \dots, u_m) = \sum_{n=0}^{\infty} A_n^i, \quad i = 1, 2, \dots, m, \quad (2.5)$$

where

$$A_n^i = \frac{1}{n!} \left[\frac{d^n}{d\mu^n} N_i \left(\sum_{n=0}^{\infty} u_{1n} \mu^n, \dots, \sum_{n=0}^{\infty} u_{mn} \mu^n \right) \right]_{\mu=0}, \quad n \geq 0,$$

is an Adomian polynomial that can be constructed for all forms of all nonlinearity.

By (2.4) and (2.5) in (2.3) yields:

$$\sum_{n=0}^{\infty} u_n^i(x, t) = f_i(x) + L_i^{-1} g_i(x) - L_i^{-1} \left[\lambda_i(t) \left(\sum_{n=0}^{\infty} u_n^1, \dots, \sum_{n=0}^{\infty} u_n^m \right) \right] - L_i^{-1} \left(\sum_{n=0}^{\infty} A_n^i \right). \quad (2.6)$$

Identifying the Zeroth component u_0^i for all terms. The recurrence relations can be used calculate the remaining components f_i and g_i , u_n^i , $n \geq 0$ can be determined by using the recurrence relations:

$$\begin{aligned} u_0^i &= f_i(x) + L_i^{-1} g_i(x), \\ u_{n+1}^i &= -L_i^{-1} \left[\lambda_i(t) \left(\sum_{n=0}^{\infty} u_n^1, \dots, \sum_{n=0}^{\infty} u_n^m \right) \right] - L_i^{-1} \left(\sum_{n=0}^{\infty} A_n^i \right), \quad i = 1, 2, \dots, m, \quad n \geq 0. \end{aligned}$$

2.2 Analysis of MADM

In order to solve a system (2.1) that matched the following conditions. We can rewrite the system (2.1) as

$$u_t^i(x, t) + \lambda_i(t) u_i(x, t) = g_i(x) - N_i(u_1, \dots, u_m), \quad i = 1, 2, \dots, m. \quad (2.7)$$

We define the direct operator and its corresponding inverse operator as follows

$$L_i = e^{-\int \lambda_i(t) dt} \frac{d}{dt} e^{\int \lambda_i(t) dt}, \quad L_i^{-1}(\cdot) = e^{-\int \lambda_i(t) dt} \int_0^t e^{\int \lambda_i(t) dt}(\cdot) dt$$

Therefore

$$L_i u_i(x, t) = g_i(x) - N_i(u_1, \dots, u_m), \quad i = 1, 2, \dots, m. \quad (2.8)$$

Taking L_i^{-1} of the two terms $u_t^i + \lambda_i(t) u_i$ in the equation (2.7), we get

$$\begin{aligned} L_i^{-1}(u_t^i(x, t) + \lambda_i(t) u_i(x, t)) &= e^{-\int \lambda_i(t) dt} \int_0^t e^{\int \lambda_i(t) dt} (u_i(x, t) + \lambda_i(t) u_i(x, t)) dt \\ &= u_i(x, t) - u_i(x, 0) \Phi(0) \Phi^{-1}(t), \end{aligned}$$

where $\Phi^{-1}(t) = e^{-\int \lambda_i(t) dt}$. Applying the inverse operators to the equation (2.8), we get

$$u_i(x, t) = u_i(x, 0)\Phi(0)e^{-\int \lambda_i(t) dt} + L_i^{-1}g_i(x) - L_i^{-1}(N_i(u_1, \dots, u_m)). \quad (2.9)$$

The recursive relation is identified by:

$$\begin{aligned} u_0^i &= f_i(x)\Phi(0)\Phi^{-1}(t) + L_i^{-1}g_i(x), \\ u_{n+1}^i &= -L_i^{-1}\left(\sum_{n=0}^{\infty} A_n^i\right) \quad i = 1, 2, \dots, m. \end{aligned}$$

3 Numerical Example

Example 3.1. Consider the system of nonlinear PDEs

$$\begin{aligned} u_t + v_x w_y - v_y w_x + u &= 0, \\ v_t + w_x u_y + w_y u_x - v &= 0, \\ w_t + u_x v_y + u_y v_x - w &= 0. \end{aligned} \quad (3.1)$$

With initial conditions

$$\begin{aligned} u(x, y, 0) &= e^{x+y}, \\ v(x, y, 0) &= e^{x-y}, \\ w(x, y, 0) &= e^{-x+y}. \end{aligned}$$

To solve the system by the proposed method, we will be using the following direct and inverse operators $L_1 = e^{-\int dt} \frac{d}{dt} e^{\int dt}$, $L_2 = e^{\int dt} \frac{d}{dt} e^{-\int dt}$, $L_3 = e^{\int dt} \frac{d}{dt} e^{-\int dt}$,

$L_1^{-1}(\cdot) = e^{-\int dt} \int_0^t e^{\int dt}(\cdot) dt$, $L_2^{-1}(\cdot) = e^{\int dt} \int_0^t e^{-\int dt}(\cdot) dt$, $L_3^{-1}(\cdot) = e^{\int dt} \int_0^t e^{-\int dt}(\cdot) dt$. So we can write the equation (3.1) in the form

$$\begin{aligned} L_1(u) &= -(v_x w_y - v_y w_x), \\ L_2(v) &= -(w_x u_y + w_y u_x), \\ L_3(w) &= -(u_x v_y + u_y v_x). \end{aligned} \quad (3.2)$$

Taking the operator L_1^{-1} , L_2^{-1} and L_3^{-1} for $u_t + u$, $v_t - v$, $w_t - w$ of the system (3.1) respectively, we obtain

$$\begin{aligned} L_1^{-1}(u_t + u) &= e^{-t} \int_0^t e^t (u_t + u) dt \\ &= u(x, y, t) - e^{x+y-t}, \\ L_2^{-1}(v_t - v) &= e^t \int_0^t e^{-t} (v_t - v) dt \\ &= v(x, y, t) - e^{x-y+t}, \\ L_3^{-1}(w_t - w) &= e^t \int_0^t e^{-t} (w_t - w) dt \\ &= w(x, y, t) - e^{-x+y+t}. \end{aligned} \quad (3.3)$$

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Taking the operator L_1^{-1} , L_2^{-1} and L_3^{-1} for equations of the system (3.2) respectively, we obtain

$$\begin{aligned} u(x, y, t) &= e^{x+y-t} - L_1^{-1}(v_x w_y - v_y w_x), \\ v(x, y, t) &= e^{x-y+t} - L_2^{-1}(w_x u_y + w_y u_x), \\ w(x, y, t) &= e^{-x+y+t} - L_3^{-1}(u_x v_y + u_y v_x). \end{aligned} \quad (3.4)$$

Using the formulae $u = \sum_{n=0}^{\infty} u_n$, $v = \sum_{n=0}^{\infty} v_n$ and $w = \sum_{n=0}^{\infty} w_n$ in equation (3.4), yields

$$\begin{aligned} \sum_{n=0}^{\infty} u_n &= e^{x+y-t} - L_1^{-1} \left(\sum_{n=0}^{\infty} A_n \right), \\ \sum_{n=0}^{\infty} v_n &= e^{x-y+t} - L_2^{-1} \left(\sum_{n=0}^{\infty} B_n \right), \\ \sum_{n=0}^{\infty} w_n &= e^{-x+y+t} - L_3^{-1} \left(\sum_{n=0}^{\infty} C_n \right), \end{aligned} \quad (3.5)$$

where A_n , B_n , C_n are the polynomials for the nonlinear expression $(v_x w_y - v_y w_x)$, $(w_x u_y + w_y u_x)$, $(u_x v_y + u_y v_x)$ respectively. The general solution of the system is as follows

$$\begin{aligned} u_0(x, y, t) &= e^{x+y-t}, \\ u_{k+1}(x, t) &= -L_1^{-1} A_k = 0, \quad k \geq 0, \\ v_0(x, y, t) &= e^{x-y+t}, \\ v_{k+1}(x, t) &= -L_2^{-1} B_k = 0, \quad k \geq 0, \\ w_0(x, y, t) &= e^{-x+y+t}, \\ w_{k+1}(x, y, t) &= -L_3^{-1} C_k = 0, \quad k \geq 0. \end{aligned} \quad (3.6)$$

The exact solution of the system is given by

$$(u, v, w) = (u_0(x, y, t), v_0(x, y, t), w_0(x, y, t)) = (e^{x+y-t}, e^{x-y+t}, e^{-x+y+t}).$$

That is closed solution [8, 16, 22].

Example 3.2. Consider system of linear partial differential equations of the following form:

$$\begin{aligned} u_t - v_x - u + v &= -2, \\ v_t - u_x - u + v &= -2. \end{aligned} \quad (3.7)$$

With initial conditions:

$$\begin{aligned} u(x, t) &= 1 + e^x, \\ w(x, t) &= -1 + e^x. \end{aligned}$$

The exact solution is as follows [14].

$$u(x, t) = 1 + e^{x+t}, \quad w(x, t) = -1 + e^{x+t},$$

Let $L_1 = e^{\int dt} \frac{d}{dt} e^{-\int dt}$, $L_2 = e^{-\int dt} \frac{d}{dt} e^{\int dt}$,
so $L_1^{-1}() = e^{\int dt} \int_0^t e^{-\int dt} () dt$, $L_2^{-1}() = e^{-\int dt} \int_0^t e^{\int dt} () dt$.

We can be written Eq.(3.7) as

$$\begin{aligned} L_1(u) &= -2 + v_x - v, \\ L_2(v) &= -2 + u_x + u. \end{aligned} \quad (3.8)$$

Taking the operator L_1^{-1}, L_2^{-1} for $u_t - u, v_t + v$ of the system (3.7) respectively, we obtain

$$\begin{aligned} L_1^{-1}(u_t - u) &= e^t \int_0^t e^{-t} (u_t - u) dt \\ &= u - (e^t + e^{x+t}), \\ L_2^{-1}(v_t + v) &= e^{-t} \int_0^t e^t (v_t + v) dt \\ &= v - (-e^{-t} + e^{x-t}). \end{aligned} \quad (3.9)$$

Taking the operator L_1^{-1}, L_2^{-1} for equations of the system (3.8) respectively, we obtain

$$\begin{aligned} u(x, t) &= e^t + e^{x+t} + L_1^{-1}(-2) + L_1^{-1}(v_x - v), \\ v(x, t) &= -e^{-t} + e^{x-t} + L_2^{-1}(-2) + L_2^{-1}(u_x + u). \end{aligned} \quad (3.10)$$

The general solution of the system is as follows

$$\begin{aligned} u_0(x, t) &= e^t + e^{x+t} + L_1^{-1}(-2) \\ u_{k+1}(x, t) &= L_1^{-1}(v_{k_x} - v_k), \quad k \geq 0, \\ v_0(x, t) &= -e^{-t} + e^{x-t} + L_2^{-1}(-2), \\ v_{k+1}(x, t) &= L_2^{-1}(u_{k_x} + u_k), \quad k \geq 0, \end{aligned} \quad (3.11)$$

The first three solutions iterations are as follows

$$\begin{aligned} u_0 &= e^{x+t} - e^t + 2, \\ v_0 &= e^{x-t} + e^{-t} - 2, \\ u_1 &= \frac{3}{2}e^t + \frac{1}{2}e^{-t} - 2, \\ v_1 &= -\frac{3}{2}e^{-t} - \frac{1}{2}e^t + 2, \\ u_2 &= \frac{1}{2}te^t - \frac{5}{4}e^t - \frac{3}{4}e^{-t} + 2, \\ v_2 &= \frac{1}{2}te^{-t} + \frac{5}{4}e^{-t} + \frac{3}{4}e^t - 2. \\ &\vdots \end{aligned} \quad (3.12)$$

Summing the above iterations yields

$$\begin{aligned} u(x, t) &= u_0 + u_1 + u_2 + \dots = e^{x+t} - \frac{1}{4}e^{-t} + \frac{1}{2}te^t - \frac{3}{4}e^t + 2 + \dots \\ v(x, t) &= v_0 + v_1 + v_2 + \dots = e^{x-t} + \frac{1}{4}e^t + \frac{1}{2}te^{-t} + \frac{3}{4}e^{-t} - 2 + \dots \end{aligned} \quad (3.13)$$

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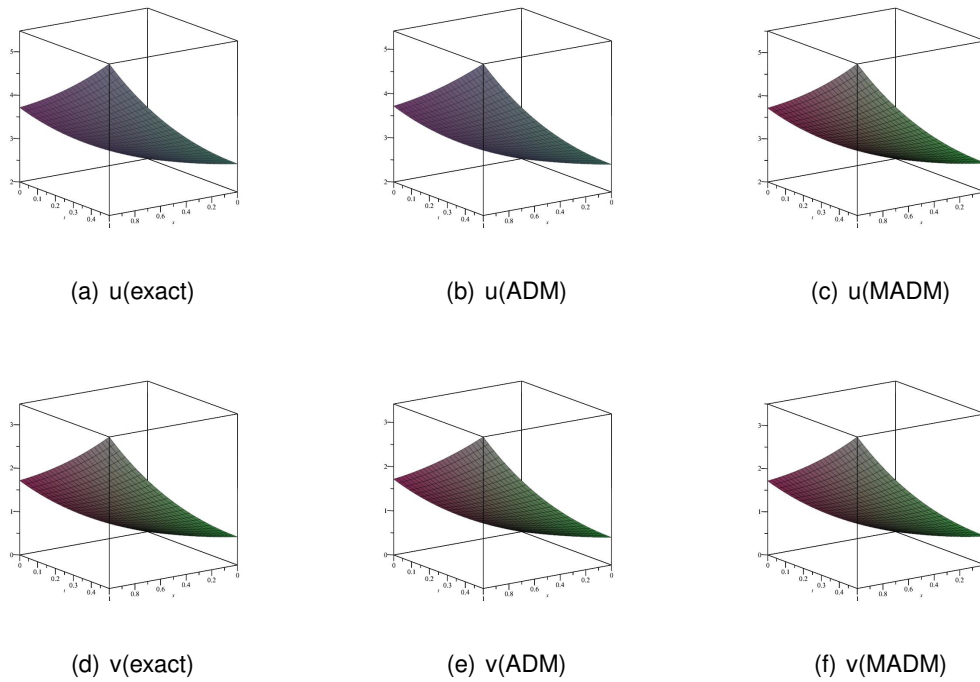


Figure 1: 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM for **Example(3.2)**

Which are the series of the approximate solution for $u(x,t)$ and $v(x,t)$ given using MADM. And The series of the approximate solution for $u(x,t)$ and $v(x,t)$ which are given using ADM as

$$\begin{aligned}
 u(x,t) &= u_0 + u_1 + u_2 + \dots = 1 + e^x + te^x + \frac{1}{2}t^2e^x + \dots \\
 v(x,t) &= v_0 + v_1 + v_2 + \dots = -1 + e^x + te^x + \frac{1}{2}t^2e^x + \dots
 \end{aligned}
 \tag{3.14}$$

Example 3.3. Consider the coupled system of nonlinear partial differential equations of the following form:

$$\begin{aligned} u_t + wu_x - \alpha u &= \beta, \\ w_t - uw_x + \alpha w &= \beta. \end{aligned} \quad (3.15)$$

With initial conditions:

$$\begin{aligned} u(x, t) &= e^{\beta x}, \\ w(x, t) &= e^{-\beta x}. \end{aligned}$$

The exact solution is as follows ([8],[16]).

$$\begin{aligned} u(x, t) &= e^{\beta x + \alpha t}, \\ w(x, t) &= e^{-\beta x - \alpha t}, \end{aligned}$$

where α and β are constants.

To utilize the proposed technique, we start with this system representation:

$$\begin{aligned} L_1 u &= \beta - wu_x, \\ L_2 w &= \beta + uw_x, \end{aligned} \quad (3.16)$$

where $L_1 = e^{\int \alpha dt} \frac{d}{dt} e^{-\int \alpha dt}$, $L_2 = e^{-\int \alpha dt} \frac{d}{dt} e^{\int \alpha dt}$,

so $L_1^{-1}() = e^{\int \alpha dt} \int_0^t e^{-\int \alpha dt} () dt$, $L_2^{-1}() = e^{-\int \alpha dt} \int_0^t e^{\int \alpha dt} () dt$

Using the inverse integration operator L_1^{-1}, L_2^{-1} for $u_t - \alpha u$, $w_t + \alpha w$ of the system (3.15) respectively, possible to get

$$\begin{aligned} L_1^{-1}(u_t - \alpha u) &= e^{\alpha t} \int_0^t e^{-\alpha t} (u_t - \alpha u) dt = u - e^{\beta x + \alpha t}, \\ L_2^{-1}(w_t + \alpha w) &= e^{-\alpha t} \int_0^t e^{\alpha t} (w_t + \alpha w) dt = w - e^{-\beta x - \alpha t}. \end{aligned} \quad (3.17)$$

Using the inverse integration operator L_1^{-1}, L_2^{-1} for the equations of the system (3.16) respectively, we get

$$\begin{aligned} u(x, t) &= e^{\beta x + \alpha t} + L_1^{-1}(\beta) - L_1^{-1}(wu_x), \\ w(x, t) &= e^{-\beta x - \alpha t} + L_2^{-1}(\beta) + L_2^{-1}(uw_x). \end{aligned} \quad (3.18)$$

The complete solution set of system as follows

$$\begin{aligned} u_0(x, t) &= e^{\beta x + \alpha t} + L_1^{-1}(\beta), \\ u_{k+1}(x, t) &= -L_1^{-1}A_k, \quad k \geq 0, \\ w_0(x, t) &= e^{-\beta x - \alpha t} + L_2^{-1}(\beta), \\ w_{k+1}(x, t) &= L_2^{-1}B_k, \quad k \geq 0. \end{aligned} \quad (3.19)$$

Therefore

$$\begin{aligned} u_0(x, t) = e^{\beta x + \alpha t} + L_1^{-1}(\beta) &= e^{\beta x + \alpha t} + e^{\alpha t} \int_0^t e^{-\alpha dt} (\beta) dt \\ &= e^{\beta x + \alpha t} + \frac{e^{\alpha t} \beta}{\alpha} - \frac{\beta}{\alpha}, \end{aligned}$$

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$$\begin{aligned} w_0(x, t) = e^{-\beta x - \alpha t} + L_2^{-1}(\beta) &= e^{-\beta x - \alpha t} + e^{-\alpha t} \int_0^t e^{\alpha dt} (\beta) dt \\ &= e^{-\beta x - \alpha t} - \frac{e^{-\alpha t} \beta}{\alpha} + \frac{\beta}{\alpha}. \end{aligned}$$

$$\begin{aligned} u_1(x, t) = -L_1^{-1} A_0 &= -e^{\alpha t} \int_0^t e^{-\alpha t} (w_0 u_{0x}) dt \\ &= -\frac{\beta^2}{\alpha} t e^{\beta x + \alpha t} + \frac{\beta^2}{\alpha^2} e^{\beta x + \alpha t} - \frac{\beta}{\alpha} e^{\alpha t} - \frac{\beta^2}{\alpha^2} e^{\beta x} + \frac{\beta}{\alpha}, \end{aligned}$$

$$\begin{aligned} w_1(x, t) = L_2^{-1} B_0 &= e^{-\alpha t} \int_0^t e^{\alpha t} (u_0 w_{0x}) dt \\ &= \frac{\beta^2}{\alpha} t e^{-\beta x - \alpha t} + \frac{\beta^2}{\alpha^2} e^{-\beta x - \alpha t} + \frac{\beta}{\alpha} e^{-\alpha t} - \frac{\beta^2}{\alpha^2} e^{-\beta x} - \frac{\beta}{\alpha}. \end{aligned}$$

⋮

The decomposition series solution for the system which we obtained by MADM are given by:

$$\begin{aligned} u(x, t) &= u_0 + u_1 + \dots \\ &= e^{\beta x + \alpha t} - \frac{\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{\beta^2}{\alpha} t e^{\beta x + \alpha t} + \frac{\beta^2}{\alpha} t e^{\beta x + \alpha t} + \frac{\beta^4}{2\alpha^2} t^2 e^{\beta x + \alpha t} + \dots \\ &= e^{\beta x + \alpha t} + \frac{\beta^4}{2\alpha^2} t^2 e^{\beta x + \alpha t} + \dots \end{aligned}$$

$$\begin{aligned} w(x, t) &= w_0 + w_1 + \dots \\ &= e^{-\beta x - \alpha t} + \frac{\beta}{\alpha} - \frac{\beta}{\alpha} + \frac{\beta^2}{\alpha} t e^{-\beta x - \alpha t} - \frac{\beta^2}{\alpha} t e^{-\beta x - \alpha t} + \frac{\beta^4}{2\alpha^2} t^2 e^{-\beta x - \alpha t} + \dots \\ &= e^{-\beta x - \alpha t} + \frac{\beta^4}{2\alpha^2} t^2 e^{-\beta x - \alpha t} + \dots \end{aligned}$$

The decomposition series solution for system (3.15) which we got via ADM are given by:

$$\begin{aligned} u(x, t) &= u_0 + u_1 + u_2 + \dots \\ &= e^{\beta x} + \alpha t e^{\beta x} + \alpha^2 \frac{t^2}{2} e^{\beta x} + \alpha^2 \beta \frac{t^3}{6} - \alpha \beta^2 \frac{t^3}{3} e^{\beta x} + \beta^3 \frac{t^3}{3} + \beta^4 \frac{t^4}{8} e^{\beta x} + \dots \end{aligned}$$

$$\begin{aligned} w(x, t) &= w_0 + w_1 + w_2 + \dots \\ &= e^{-\beta x} - \alpha t e^{-\beta x} + \alpha^2 \frac{t^2}{2} e^{-\beta x} + \alpha^2 \beta \frac{t^3}{6} + \alpha \beta^2 \frac{t^3}{3} e^{-\beta x} + \beta^3 \frac{t^3}{3} + \beta^4 \frac{t^4}{8} e^{-\beta x} + \dots \end{aligned}$$

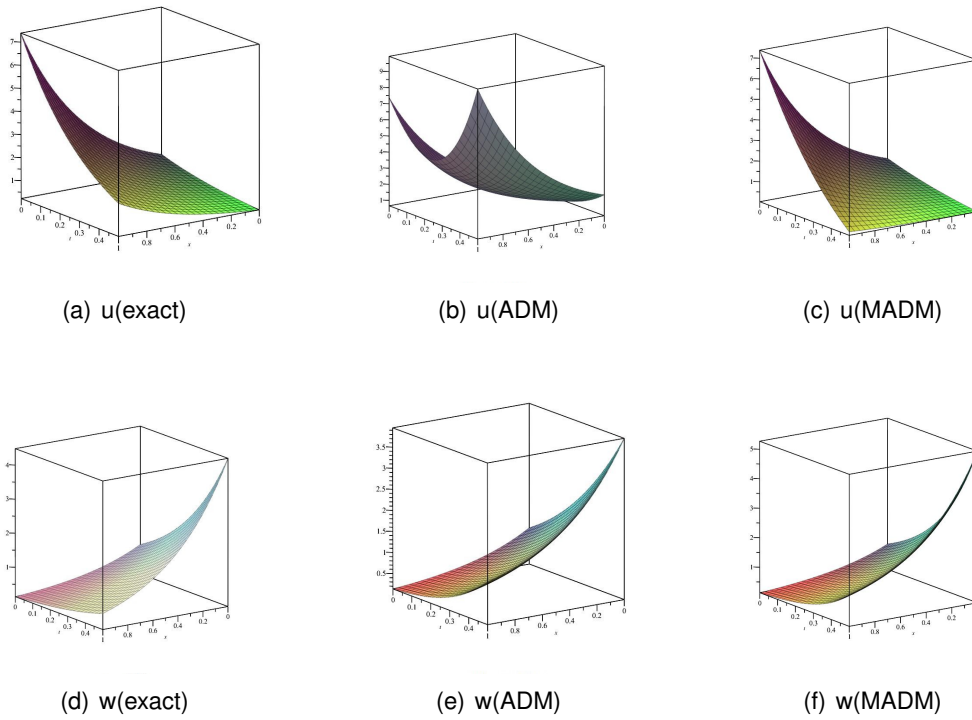


Figure 2: 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM at $\alpha = -3$, $\beta = 2$ for **Example(3.3)**

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Example 3.4. Consider the system of nonlinear PDEs

$$\begin{aligned} u_t + vu_x - 2u &= 1, \\ v_t - uv_x + 2v &= 1. \end{aligned} \quad (3.20)$$

With initial conditions

$$\begin{aligned} u(x, 0) &= e^x, \\ v(x, 0) &= e^{-x}. \end{aligned}$$

This example is a special case of example (3.2), with $\alpha = 2, \beta = 1$ and the exact solution is $u = e^{x+t}, v = e^{-x-t}$. The system can be expressed as follows

$$\begin{aligned} L_1 u &= 1 - vu_x, \\ L_2 v &= 1 + uv_x, \end{aligned} \quad (3.21)$$

where $L_1 = e^{\int 2dt} \frac{d}{dt} e^{-\int 2dt}$, $L_2 = e^{-\int 2dt} \frac{d}{dt} e^{\int 2dt}$,
 $L_1^{-1}(\cdot) = e^{\int 2dt} \int_0^t e^{-\int 2dt}(\cdot) dt$, $L_2^{-1}(\cdot) = e^{-\int 2dt} \int_0^t e^{\int 2dt}(\cdot) dt$.

Using L_1^{-1}, L_2^{-1} for $u_t - 2u, v_t + 2v$ of the system (3.20) respectively, we may obtain

$$\begin{aligned} L_1^{-1}(u_t - 2u) &= e^{2t} \int_0^t e^{-2t}(u_t - 2u) dt = u - e^{x+2t}, \\ L_2^{-1}(v_t + 2v) &= e^{-2t} \int_0^t e^{2t}(v_t + 2v) dt = v - e^{-x-2t}. \end{aligned} \quad (3.22)$$

Applying L_1^{-1}, L_2^{-1} for equations of the system (3.21) respectively, we may obtain

$$\begin{aligned} u(x, t) &= e^{x+2t} + L_1^{-1}(1) - L_1^{-1}vu_x, \\ v(x, t) &= e^{-x-2t} + L_2^{-1}(1) + L_2^{-1}uv_x. \end{aligned} \quad (3.23)$$

The general solution of the system is as follows

$$\begin{aligned} u_0(x, t) &= e^{x+2t} + L_1^{-1}(1), \\ u_{k+1}(x, t) &= -L_1^{-1}A_k, \quad k \geq 0, \\ v_0(x, t) &= e^{-x-2t} + L_2^{-1}(1), \\ v_{k+1}(x, t) &= +L_2^{-1}B_k, \quad k \geq 0. \end{aligned} \quad (3.24)$$

The first two solutions iterations are as follows

$$\begin{aligned} u_0 &= e^{x+2t} + \frac{e^{2t}}{2} - \frac{1}{2}, \\ u_1 &= -L_t^{-1}A_0 = -\frac{1}{2}te^{x+2t} + \frac{1}{4}e^{x+2t} - \frac{1}{2}e^{2t} - \frac{1}{4}e^x + \frac{1}{2}, \\ &\vdots \\ v_0 &= e^{-x-2t} - \frac{e^{-2t}}{2} + \frac{1}{2}, \\ v_1 &= L_t^{-1}B_0 = \frac{1}{2}te^{-x-2t} + \frac{1}{4}e^{-x-2t} + \frac{1}{2}e^{-2t} - \frac{1}{4}e^{-x} - \frac{1}{2}, \end{aligned}$$

⋮

Summing the above iterations yields

$$\begin{aligned} u_{app}(x, t) &= u_0 + u_1 + \dots = \frac{5}{4}e^{x+2t} + \frac{1}{2}te^{x+2t} - \frac{1}{4}e^x + \dots \\ v_{app}(x, t) &= v_0 + v_1 + \dots = \frac{5}{4}e^{-x-2t} + \frac{1}{2}te^{-x-2t} - \frac{1}{4}e^{-x} + \dots \end{aligned} \quad (3.25)$$

Which are the series of the approximate solution for $u(x, t)$ and $v(x, t)$ given using MADM. And

$$\begin{aligned} u &= \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \dots = e^x + 2te^x + 2t^2e^x + t^3 - 2\frac{t^3}{3}e^x + \frac{t^4}{8}e^x + \dots \\ v &= \sum_{n=0}^{\infty} v_n = v_0 + v_1 + v_2 + \dots = e^{-x} - 2te^{-x} + 2t^2e^{-x} + t^3 + 2\frac{t^3}{3}e^{-x} + \frac{t^4}{8}e^{-x} + \dots \end{aligned} \quad (3.26)$$

Which are the series of the approximate solution for $u(x, t)$, $v(x, t)$ that we produced via ADM.

3 NUMERICAL EXAMPLE

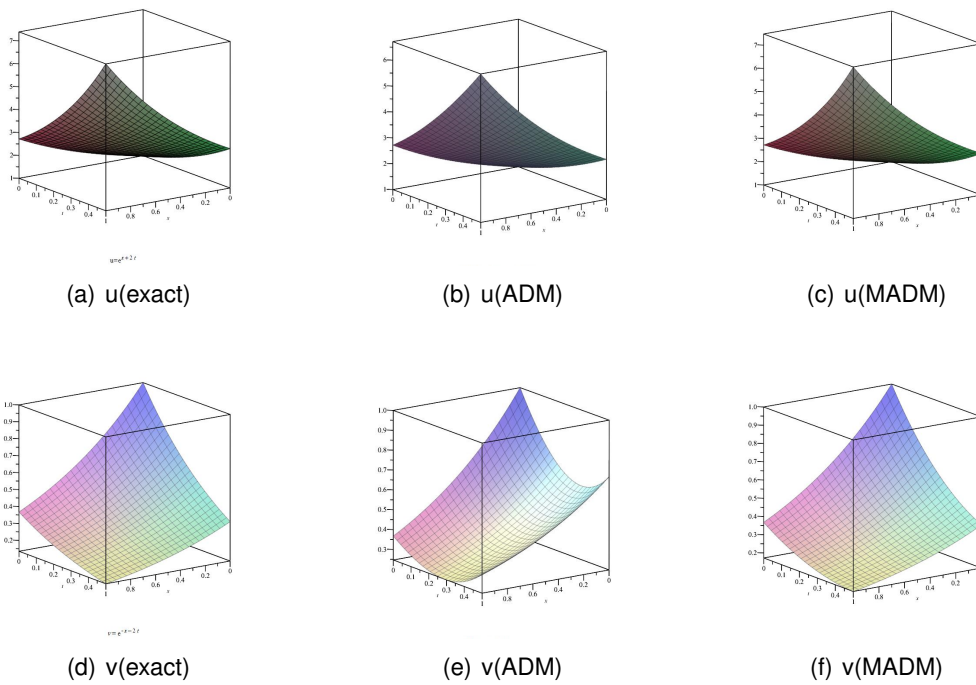


Figure 3: 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM for **Example(3.4)**

Example 3.5. Consider the system of nonlinear PDEs

$$\begin{aligned}\frac{\partial u}{\partial t} + \left(v \frac{\partial u}{\partial x}\right)^2 + u &= 1, \\ \frac{\partial v}{\partial t} + \left(u \frac{\partial v}{\partial x}\right)^2 - v &= 1.\end{aligned}\tag{3.27}$$

With initial conditions

$$\begin{aligned}u(x, 0) &= e^x, \\ v(x, 0) &= e^{-x}.\end{aligned}$$

The exact solution is given in as [6, 21]

$$\begin{aligned}u(x, t) &= e^{x-t}, \\ v(x, t) &= e^{-x+t}.\end{aligned}$$

For the modified method, the system is expressed as follows

$$\begin{aligned}L_1 u &= 1 - (vu_x)^2, \\ L_2 v &= 1 - (uv_x)^2,\end{aligned}\tag{3.28}$$

where $L_1 = e^{-\int dt} \frac{d}{dt} e^{\int dt}$, $L_2 = e^{\int dt} \frac{d}{dt} e^{-\int dt}$,
 $L_1^{-1}(\cdot) = e^{-\int dt} \int_0^t e^{\int dt}(\cdot) dt$, $L_2^{-1}(\cdot) = e^{\int dt} \int_0^t e^{-\int dt}(\cdot) dt$.

Using L_1^{-1} , L_2^{-1} for $u_t + u$, $v_t - v$, of the system (3.27) respectively, it produces the following

$$\begin{aligned}L_1^{-1}(u_t + u) &= e^{-t} \int_0^t e^t (u_t + u) dt = u - e^{x-t}, \\ L_2^{-1}(v_t - v) &= e^t \int_0^t e^{-t} (v_t - v) dt = u - e^{-x+t}.\end{aligned}\tag{3.29}$$

Employing L_1^{-1} , L_2^{-1} for equations of the system (3.28) respectively, we may obtain

$$\begin{aligned}u(x, t) &= e^{x-t} + L_1^{-1}(1) - L_1^{-1}(vu_x)^2 \\ v(x, t) &= e^{-x+t} + L_2^{-1}(1) - L_2^{-1}(uv_x)^2\end{aligned}\tag{3.30}$$

The general solution of system as follows

$$\begin{aligned}u_0(x, t) &= e^{x-t} + L_1^{-1}(1), \\ u_{k+1}(x, t) &= -L_1^{-1}A_k^2, \quad k \geq 0, \\ v_0 &= e^{-x+t} + L_2^{-1}(1), \\ v_{k+1}(x, t) &= -L_2^{-1}B_k^2, \quad k \geq 0.\end{aligned}\tag{3.31}$$

The first two terms of solutions iterations are as follows

3 NUMERICAL EXAMPLE

$$\begin{aligned}
 u_0(x, t) &= e^{x-t} + 1 - e^{-t} \\
 v_0(x, t) &= e^{-x+t} - 1 + e^t \\
 u_1(x, t) &= -L_1^{-1}A_0^2 = -e^{-t} \int_0^t e^t (v_0 u_{0_x})^2 dt \\
 &= 2te^{2x-t} + 2te^{x-t} + e^{2x-2t} + 2e^{x-t} + e^{-t} - e^{2x} - 2e^x - 1 \\
 v_1(x, t) &= -L_2^{-1}B_0^2 = -e^t \int_0^t e^{-t} (u_0 v_{0_x})^2 dt \\
 &= 2te^{-2x+t} - 2te^{-x+t} - e^{-2x+2t} + 2e^{-x+t} - e^t + e^{-2x} - 2e^{-x} + 1 \\
 &\vdots
 \end{aligned}$$

Summing the above iterations yields

$$\begin{aligned}
 u(x, t) &= u_0 + u_1 + \dots = 2te^{2x-t} + 2te^{x-t} + e^{2x-2t} + 3e^{x-t} - e^{2x} - 2e^x + \dots \\
 v(x, t) &= v_0 + v_1 + \dots = 2te^{-2x+t} - 2te^{-x+t} - e^{-2x+2t} + 2e^{-x+t} + e^{-2x} - 2e^{-x} + \dots
 \end{aligned}$$

Which the approximate solutions for $u(x,t)$, $v(x,t)$ of the system that we obtained by MADM are given.

And the approximate solutions for $u(x,t)$, $v(x,t)$ of the system that we obtained by ADM are given.

$$\begin{aligned}
 u(x, t) = u_0 + u_1 + u_2 + \dots &= e^x + t - \frac{e^{2x}t^3}{3} - e^x t^2 - \frac{t^2}{2} - t - te^x \\
 &- \frac{4}{81}t^9 e^{4x} - \frac{1}{18}(e^x + e^{3x})t^8 \\
 &- \frac{1}{7} \left[\frac{4}{3}(2e^{2x} + \frac{1}{2}e^{3x}) - \frac{1}{9}(e^{-x} + 5e^x)^2 \right] t^7 \\
 &- \frac{1}{6} \left[\frac{4}{3}e^{3x} + \frac{2}{3}(2 + \frac{1}{2}e^x)(e^{-x} + e^{5x}) \right] t^6 \\
 &- \frac{1}{5} \left[\frac{2}{3}(1 + 5e^{2x}) - (2 + \frac{1}{2}e^x)^2 \right] t^5 \\
 &- \frac{1}{4} \left[2e^x(2 + \frac{1}{2}e^x) - \frac{1}{3}e^{2x} \right] t^4 \\
 &- \frac{1}{3} \left[e^{2x} - e^x - \frac{1}{2} \right] t^3 + \frac{1}{2} [1 + e^x] t^2 + \dots
 \end{aligned}$$

3 NUMERICAL EXAMPLE

$$\begin{aligned}
 v(x, t) = v_0 + v_1 + v_2 + \dots &= e^{-x} + t - \frac{e^{-2x}t^3}{3} - e^{-x}t^2 + \frac{t^2}{2} - t + te^{-x} \\
 &- \frac{4}{81}t^9e^{-4x} - \frac{1}{18}(e^{-x} + e^{-3x})t^8 \\
 &- \frac{1}{7} \left[\frac{4}{3}(2e^{-2x} - \frac{1}{2}e^{-3x}) + \frac{1}{9}(e^x + 5e^{-x})^2 \right] t^7 \\
 &- \frac{1}{6} \left[\frac{4}{3}e^{-3x} + \frac{2}{3}(2 - \frac{1}{2}e^{-x})(e^x + e^{-5x}) \right] t^6 \\
 &- \frac{1}{5} \left[\frac{2}{3}(1 + 5e^{-2x}) + (2 - \frac{1}{2}e^{-x})^2 \right] t^5 \\
 &- \frac{1}{4} \left[2e^{-x}(2 - \frac{1}{2}e^{-x}) + \frac{1}{3}e^{-2x} \right] t^4 \\
 &- \frac{1}{3} \left[e^{-2x} + e^{-x} - \frac{1}{2} \right] t^3 - \frac{1}{2} [1 - e^x] t^2 + \dots
 \end{aligned}$$

Remark: By using the noise terms phenomenon in the above example

$$\begin{aligned}
 u_0(x, t) &= u_0\Phi(0)\Phi^{-1}(t) = e^{x-t}, \quad \Phi^{-1}(t) = e^{-\int dt}, \\
 v_0(x, t) &= v_0\Phi(0)\Phi^{-1}(t) = e^{-x+t}, \quad \Phi^{-1}(t) = e^{\int dt}, \\
 u_1(x, t) &= L_1^{-1}(1) - L_1^{-1}A_0^2 = 0, \\
 v_1(x, t) &= L_2^{-1}(1) - L_2^{-1}B_0^2 = 0,
 \end{aligned}$$

⋮

$$\begin{aligned}
 u_{k+1}(x, t) &= -L_1^{-1}A_k^2 = 0, \quad k \geq 1, \\
 v_{k+1}(x, t) &= -L_2^{-1}B_k^2 = 0, \quad k \geq 1.
 \end{aligned}$$

We have

$$(u, v) = (u_0(x, t), v_0(x, t)) = (e^{x-t}, e^{-x+t}),$$

which represents a closed solution. This also applies to Ex(3.3) and Ex(3.4)

3 NUMERICAL EXAMPLE

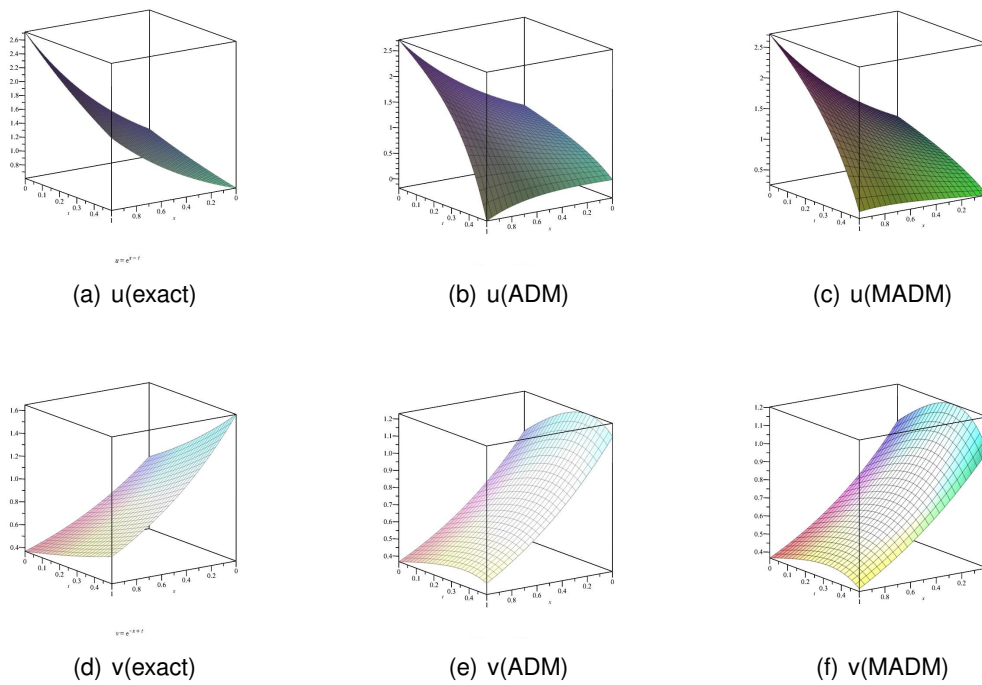


Figure 4: 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM for **Example(3.5)**

4 Conclusion

This research demonstrates that the Modified Adomian Decomposition Method (MADM) is a valuable tool for tackling complex nonlinear partial differential equations. While both MADM and the standard Adomian Decomposition Method (ADM) are effective for linear systems, MADM offers improved convergence and accuracy due to the incorporation of a new integral operator. The successful application of MADM in the presented numerical examples expands the capabilities of Adomian methods for solving a wider range of nonlinear PDEs.

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