The Modified Adomian Decomposition Method for Numerical Solutions for PDE Systems with Improved Accuracy and Convergence

Abstract

This study explores the potential of the Modified Adomian Decomposition Method (MADM) for solving linear and nonlinear partial differential equations (PDEs). MADM addresses limitations faced by traditional methods and builds upon the Adomian Decomposition Method (ADM) by incorporating a new integral operator. This operator aims to enhance convergence and accuracy for intricate nonlinear problems. The paper outlines the application of MADM, its solution procedure, and its effectiveness through numerical examples for both linear and nonlinear PDE systems. Comparisons are made with standard ADM solutions and exact solutions to assess MADM's accuracy. The results suggest MADM as a promising tool for handling complex nonlinear PDEs and expanding the applicability of Adomian methods in this field.

1 Introduction

This research investigates the application of the Modified Adomian Decomposition Method (MADM) for solving partial differential equations (PDEs). We focus particularly on its effectiveness in handling nonlinear problems. Partial differential equations (PDEs) are fundamental tools for modeling diverse scientific and engineering phenomena. Solving them is crucial for predicting system behavior, designing optimized systems, and analyzing experimental data ([16] ,[17]). Traditional methods may not always be applicable for solving PDEs, especially nonlinear ones [12] . This necessitates alternative approaches like the well-established Adomian Decomposition Method (ADM) [1]. ADM tackles nonlinear problems by decomposing the solution into an infinite series. Several studies have explored modified methods for solving differential equations ([2], [3], [9], [11] ,[15]). ADM itself has expanded the range of solvable problems for systems of ordinary and partial differential equations ([5], [7], [13]). This research delves into MADM, which builds upon ADM. MADM incorporates a new integral operator specifically designed to improve convergence ([10], [4]) and accuracy for complex nonlinear problems. We outline the solution procedure and present numerical examples to demonstrate MADM's effectiveness in obtaining approximate solutions. Additionally, we compare the approximate solutions obtained using MADM with the approximate solutions obtained using the standard Adomian Decomposition Method (ADM) and the exact solutions. We present the solutions in graphical forms as well. The following sections provide a detailed explanation of ADM and MADM, including their application to specific nonlinear PDEs. The examples showcase how MADM handles nonlinearities and converges towards an approximate solution. The conclusion emphasizes the potential benefits of MADM for solving complex nonlinear PDEs, and its contribution to this field.

2 ADM and MADM for PDE system solutions

2.1 Analysis of ADM

The system of partial differential equations that follows is expressed as

$$\mathbf{u^i_t(x,t) + R_i u_i(x,t) + N_i(u_1,\dots,u_m) = g_i(x)}, \quad i=1,2,\dots,m. \tag{2.1}$$

The initial conditions are given by

$$\mathbf{u_i(x,0) = f_i(x)}.$$

The equation (1) is equivalent to

$$\mathbf{L_t u_i(x,t) + R_i u_i(x,t) + N_i(u_1,\dots,u_m) = g_i(x)}, \quad i=1,2,\dots,m. \tag{2.2}$$

where $\mathbf{L_t} = \frac{\mathbf{d}}{\mathbf{dt}}$ represents partial differential operators, $\mathbf{R_i}$ and $\mathbf{N_i}$ represent linear and nonlinear operator respectively.
Applying the inverse operators $\mathbf{L_t^{-1}}() = \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{dt}()$ to system (2) and applying beginning conditions, then

$$\mathbf{u_i(x,t) = f_i(x) + L_t^{-1} g_i(x) - L_t^{-1} R_i u_i(x,t) - L_t^{-1} N_i(u_1,\dots,u_m)}, \quad i=1,2,\dots m. \tag{2.3}$$

The Adomian decomposition method approach presupposes that the unknown functions $u_i(x,t)$ can be written as an infinite series of the form

$$\mathbf{u_i(x,t)} = \sum_{\mathbf{n=0}}^{\infty} \mathbf{u^i_n(x,t)}, \quad i=1,2,\dots,m. \tag{2.4}$$

The nonlinear operator $\mathbf{N_i(u_1, \cdots, u_m)}$ is defined using Adomian polynomials.

$$\mathbf{N_i(u_1, \ldots, u_m) = \sum_{n=0}^{\infty} A_n^i}, \quad i = 1, 2, \ldots, m, \tag{2.5}$$

where

$$\mathbf{A_n^i = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N_i\left(\sum_{n=0}^{\infty} u_{1n}\lambda^n, \ldots, \sum_{n=0}^{\infty} u_{mn}\lambda^n\right)\right]_{\lambda=0}}, \quad n = 0, 1, 2, \cdots$$

is an Adomian polynomial that can be constructed for all forms of all nonlinearity.
Substituting Eqs.(4) and (5) into Eq.(3) yields:

$$\mathbf{\sum_{n=0}^{\infty} u_n^i(x,t) = f_i(x) + L_t^{-1} g_i(x) - L_t^{-1}\left[R_i\left(\sum_{n=0}^{\infty} u_n^1, \ldots, \sum_{n=0}^{\infty} u_n^m\right)\right] - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n^i\right)}. \tag{2.6}$$

Identifying the Zeroth component $\mathbf{u_o^i}$ for all terms. The recurrence relations can be used calculate the remaining components $\mathbf{f_i}$ and $\mathbf{g_i}$, $\mathbf{u_n^i}$, $\mathbf{n \geq 0}$ can be determined by using the recurrence relations:

$$\begin{aligned} \mathbf{u_0^i} &= \mathbf{f_i(x) + L_t^{-1} g_i(x)}, \\ \mathbf{u_{n+1}^i} &= \mathbf{-L_t^{-1}\left[R_i\left(\sum_{n=0}^{\infty} u_n^1, \ldots, \sum_{n=0}^{\infty} u_n^m\right)\right] - L_t^{-1}\left(\sum_{n=0}^{\infty} A_n^i\right)}, \quad i = 1, 2, \ldots, m, \quad n \geq 0. \end{aligned}$$

2.2 Analysis of MADM

In order to solve a system (1) that matched the following conditions. The new modified Adomian approach was employed in solving the system of nonlinear partial differential equations in the case of fulfilling the following condition $\mathbf{\lambda_i(t) u_i(x,t) + M_i u_i(x,t) = R_i u_i(x,t)}$, i=1,2,. . . ,m. We can rewrite the system (1) as

$$\mathbf{u_t^i(x,t) + \lambda_i(t) u_i(x,t) = g_i(x) - M_i u_i(x,t) - N_i(u_1, \ldots, u_m)}, \quad i = 1, 2, \ldots, m. \tag{2.7}$$

We define the direct operator and its corresponding inverse operator as follows
$\mathbf{L_i = e^{-\int \lambda_i(t)\, dt} \frac{d}{dt} e^{\int \lambda_i(t)\, dt}}$, $\mathbf{L_i^{-1}() = e^{-\int \lambda_{i(t)}\, dt} \int_0^t e^{\int \lambda_i(t)\, dt}()\, dt}$
Therefore

$$\mathbf{L_i u_i(x,t) = g_i(x) - M_i u_i(x,t) - N_i(u_1, \ldots, u_m)}, \quad i = 1, 2, \ldots, m. \tag{2.8}$$

Taking $\mathbf{L_i^{-1}}$ of the two terms u_t^i, $\lambda_i(t) u_i$ in the equation (7), we get

$$\begin{aligned} L_i^{-1}((\mathbf{u_t^i(x,t) + \lambda_i(t) u_i(x,t)})) &= \mathbf{e^{-\int \lambda_i(t)\, dt} \int_0^t e^{\int \lambda_{i(t)}\, dt}(u_i(x,t) + \lambda_i(t) u_i(x,t))\, dt} \\ &= \mathbf{u_i(x,t) - u_i(x,0)\Phi(0)\Phi^{-1}(t)} \end{aligned}$$

Where $\mathbf{\Phi^{-1}(t) = e^{-\int \lambda_i(t)dt}}$. Applying the inverse operators to the equation (8),we get

$$\mathbf{u_i(x,t) = u_i(x,0)\Phi(0)e^{-\int \lambda_i(t)dt} + L_i^{-1}g_i(x) - L_i^{-1}(M_i u_i(x,t) + N_i(u_1, \dots, u_m))}. \tag{2.9}$$

The recursive relation is identified by:

$$\begin{aligned} \mathbf{u_0^i} &= \mathbf{f_i(x)\Phi(0)\Phi^{-1}(t) + L_i^{-1}g_i(x)}, \\ \mathbf{u_{n+1}^i} &= \mathbf{-L_i^{-1}}\left[\mathbf{M_i}\left(\sum_{\mathbf{n=0}}^{\infty}\mathbf{u_n^1}, \dots, \sum_{\mathbf{n=0}}^{\infty}\mathbf{u_n^m}\right)\right] - \mathbf{L_i^{-1}}\left(\sum_{\mathbf{n=0}}^{\infty}\mathbf{A_n^i}\right) \quad i = 1, 2 \dots, m, \end{aligned}$$

3 Numerical Example

Example 3.1. *Consider the system of nonlinear PDEs*

$$\begin{aligned} u_t + v_x w_y - v_y w_x + u &= 0, \\ v_t + w_x u_y + w_y u_x - v &= 0, \\ w_t + u_x v_y + u_y v_x - w &= 0. \end{aligned} \tag{3.1}$$

With initial conditions

$$\begin{aligned} u(x,y,0) &= e^{x+y}, \\ v(x,y,0) &= e^{x-y}, \\ w(x,y,0) &= e^{-x+y}. \end{aligned}$$

To solve the system by the proposed method, we well be using the following direct and inverse operators: $L_1 = e^{-\int dt}\frac{d}{dt}e^{\int dt}$*,* $L_2 = e^{\int dt}\frac{d}{dt}e^{-\int dt}$*,* $L_3 = e^{\int dt}\frac{d}{dt}e^{-\int dt}$*,*
$L_1^{-1}() = e^{-\int dt}\int_0^t e^{\int dt}()dt$*,* $L_2^{-1}() = e^{\int dt}\int_0^t e^{-\int dt}()dt$*,* $L_3^{-1}() = e^{\int dt}\int_0^t e^{-\int dt}()\,dt$*.*
So we can write the equation (10),in the form

$$\begin{aligned} L_1(u) &= -(v_x w_y - v_y w_x), \\ L_2(v) &= -(w_x u_y + w_y u_x), \\ L_3(w) &= -(u_x v_y + u_y v_x). \end{aligned} \tag{3.2}$$

Taking the operator L_1^{-1}, L_2^{-1} *and* L_3^{-1} *for* $u_t + u$*,* $v_t - v$*,* $w_t - w$ *of the system (10)*
respectively, we obtain

$$\begin{aligned} L_1^{-1}(u_t + u) &= e^{-t}\int_0^t e^t(u_t + u)dt \\ &= u(x,y,t) - e^{x+y-t}, \\ L_2^{-1}(v_t - v) &= e^t\int_0^t e^{-t}(v_t - v)dt \\ &= v(x,y,t) - e^{x-y+t}, \\ L_3^{-1}(w_t - w) &= e^t\int_0^t e^{-t}(w_t - w)dt \\ &= w(x,y,t) - e^{-x+y+t}. \end{aligned} \tag{3.3}$$

Taking the operator L_1^{-1}, L_2^{-1} and L_3^{-1} for equations of the system (11) respectively, we obtain

$$\begin{aligned} u(x,y,t) &= e^{x+y-t} - L_1^{-1}(v_x w_y - v_y w_x), \\ v(x,y,t) &= e^{x-y+t} - L_2^{-1}(w_x u_y + w_y u_x), \\ w(x,y,t) &= e^{-x+y+t} - L_3^{-1}(u_x v_y + u_y v_x). \end{aligned} \tag{3.4}$$

Using the formulae $u = \sum_{n=0}^{\infty} u_n$, $v = \sum_{n=0}^{\infty} v_n$ and $w = \sum_{n=0}^{\infty} w_n$ in equation (13), yields

$$\begin{aligned} \sum_{n=0}^{\infty} u_n &= e^{x+y-t} - L_1^{-1}\left(\sum_{n=0}^{\infty} A_n\right), \\ \sum_{n=0}^{\infty} v_n &= e^{x-y+t} - L_2^{-1}\left(\sum_{n=0}^{\infty} B_n\right), \\ \sum_{n=0}^{\infty} w_n &= e^{-x+y+t} - L_3^{-1}\left(\sum_{n=0}^{\infty} C_n\right), \end{aligned} \tag{3.5}$$

where A_n, B_n, C_n are the polynomials for the nonlinear expression $(v_x w_y - v_y w_x)$, $(w_x u_y + w_y u_x)$, $(u_x v_y + u_y v_x)$ respectively.
The general solution of the system is as follows

$$\begin{aligned} u_0(x,y,t) &= e^{x+y-t}, \\ u_{k+1}(x,t) &= -L_1^{-1} A_k = 0, \quad k \geq 0, \\ v_0(x,y,t) &= e^{x-y+t}, \\ v_{k+1}(x,t) &= -L_2^{-1} B_k = 0, \quad k \geq 0, \\ w_0(x,y,t) &= e^{-x+y+t}, \\ w_{k+1}(x,y,t) &= -L_3^{-1} C_k = 0, \quad k \geq 0. \end{aligned} \tag{3.6}$$

The exact solution of the system is given by $(u, v, w) = u_0(x,y,t), v_0(x,y,t), w_0(x,y,t) = (e^{x+y-t}, e^{x-y+t}, e^{-x+y+t})$. That is closed solution ([8],[16]).

Example 3.2. *Consider system of linear partial differential equations of the following form:*

$$\begin{aligned} u_t - v_x - u + v &= -2, \\ v_t - u_x - u + v &= -2. \end{aligned} \tag{3.7}$$

With initial conditions:

$$\begin{aligned} u(x,t) &= 1 + e^x, \\ w(x,t) &= -1 + e^x. \end{aligned}$$

The exact solution is as follows [14].

$$\begin{aligned} u(x,t) &= 1 + e^{x+t}, \\ w(x,t) &= -1 + e^{x+t}, \end{aligned} \tag{3.8}$$

Let $L_1 = e^{\int dt}\frac{d}{dt}e^{-\int dt}$*,* $L_2 = e^{-\int dt}\frac{d}{dt}e^{\int dt}$*,*
so $L_1^{-1}() = e^{\int dt}\int_0^t e^{-\int dt}()dt$*,* $L_2^{-1}() = e^{-\int dt}\int_0^t e^{\int dt}()dt.$
We can be written Eg. (15) as

$$\begin{aligned} L_1(u) &= -2 + v_x - v, \\ L_2(v) &= -2 + u_x + u. \end{aligned} \tag{3.9}$$

Taking the operator L_1^{-1}, L_2^{-1} *for* $u_t - u$*,* $v_t + v$ *of the system (16) respectively, we obtain*

$$\begin{aligned} L_1^{-1}(u_t - u) = &\quad e^t \int_0^t e^{-t}(u_t - u)dt \\ = &\quad u - (e^t + e^{x+t}), \\ L_2^{-1}(v_t + v) = &\quad e^{-t} \int_0^t e^{t}(v_t + v)dt \\ = &\quad v - (-e^{-t} + e^{x-t}). \end{aligned} \tag{3.10}$$

Taking the operator L_1^{-1}*,* L_2^{-1} *for equations of the system (18) respectively, we obtain*

$$\begin{aligned} u(x,t) &= e^t + e^{x+t} + L_1^{-1}(-2) + L_1^{-1}(v_x - v), \\ v(x,t) &= -e^{-t} + e^{x-t} + L_2^{-1}(-2) + L_2^{-1}(u_x + u). \end{aligned} \tag{3.11}$$

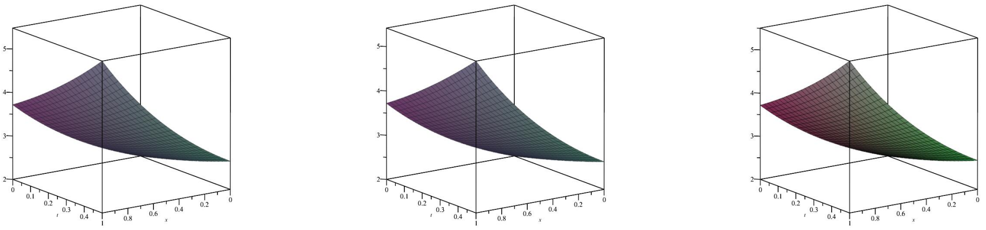
The general solution of the system is as follows

$$\begin{aligned} u_0(x,t) &= e^t + e^{x+t} + L_1^{-1}(-2) \\ u_{k+1}(x,t) &= L_1^{-1}(v_{k_x} - v_k), \quad k \geq 0, \\ v_0(x,t) &= -e^{-t} + e^{x-t} + L_2^{-1}(-2), \\ v_{k+1}(x,t) &= L_2^{-1}(u_{k_x} + u_k), \quad k \geq 0, \end{aligned} \tag{3.12}$$

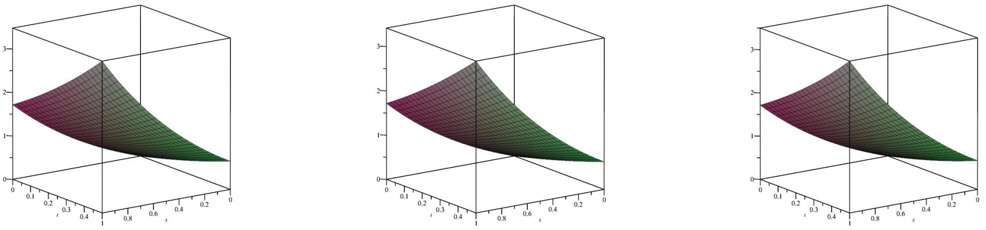
The first three solutions iterations are as follows

$$\begin{aligned} u_0 &= e^{x+t} - e^t + 2, \\ v_0 &= e^{x-t} + e^{-t} - 2, \\ u_1 &= \frac{3}{2}e^t + \frac{1}{2}e^{-t} - 2, \\ v_1 &= -\frac{3}{2}e^{-t} - \frac{1}{2}e^t + 2, \\ u_2 &= \frac{1}{2}te^t - \frac{5}{4}e^t - \frac{3}{4}e^{-t} + 2, \\ v_2 &= \frac{1}{2}te^{-t} + \frac{5}{4}e^{-t} + \frac{3}{4}e^t - 2. \end{aligned} \tag{3.13}$$

$$\vdots$$



(a) u(exact) (b) u(ADM) (c) u(MADM)



(d) v(exact) (e) v(ADM) (f) v(MADM)

Figure 1: 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM for **Example(3.2)**

Summing the above iterations yields

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + \cdots = e^{x+t} - \frac{1}{4}e^{-t} + \frac{1}{2}te^{t} - \frac{3}{4}e^{t} + 2 + \cdots \\ v(x,t) &= v_0 + v_1 + v_2 + \cdots = e^{x+t} + \frac{1}{4}e^{t} + \frac{1}{2}te^{-t} + \frac{3}{4}e^{-t} - 2 + \cdots \end{aligned} \tag{3.14}$$

Which are the series of the approximate solution for $u(x,t)$*and* $v(x,t)$ *given using MADM. And*
The series of the approximate solution for $u(x,t)$*and* $v(x,t)$ *which are given using ADM as*

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + \cdots = 1 + e^{x} + te^{x} + \frac{1}{2}t^2e^{x} + \cdots \\ v(x,t) &= v_0 + v_1 + v_2 + \cdots = -1 + e^{x} + te^{x} + \frac{1}{2}t^2e^{x} + \cdots \end{aligned} \tag{3.15}$$

Example 3.3. *Consider the coupled system of nonlinear partial differential equations of the following form:*

$$\begin{aligned} u_t + wu_x - \alpha u &= \beta, \\ w_t - uw_x + \alpha w &= \beta. \end{aligned} \tag{3.16}$$

With initial conditions:

$$\begin{aligned} u(x,t) &= e^{\beta x}, \\ w(x,t) &= e^{-\beta x}. \end{aligned}$$

The exact solution is as follows ([8],[16]).

$$\begin{aligned} u(x,t) &= e^{\beta x + \alpha t}, \\ w(x,t) &= e^{-\beta x - \alpha t}, \end{aligned} \tag{3.17}$$

where α and β are constants.
To utilize the proposed technique, we start with this system representation:

$$\begin{aligned} L_1 u &= \beta - wu_x, \\ L_2 w &= \beta + uw_x, \end{aligned} \tag{3.18}$$

where $L_1 = e^{\int \alpha dt} \frac{d}{dt} e^{\int -\alpha dt}$, $L_2 = e^{-\int \alpha \ dt} \frac{d}{dt} e^{\int \alpha dt}$,
so $L_1^{-1()=} e^{\int \alpha \ dt} \int_0^t e^{\int -\alpha \ dt} ()dt$, $L_2^{-1}() = e^{-\int \alpha \ dt} \int_0^t e^{\int \alpha \ dt} \ dt$
Using the inverse integration operator L_1^{-1}, L_2^{-1} for $u_t - \alpha u$, $w_t + \alpha w$ of the system (25) respectively, possible to get

$$\begin{aligned} L_1^{-1}(u_t - \alpha u) &= e^{\alpha t} \int_0^t e^{-\alpha t}(u_t - \alpha u) \ dt = u - e^{\beta x + \alpha t}, \\ L_2^{-1}(w_t + \alpha w) &= e^{-\alpha t} \int_0^t e^{\alpha t}(w_t + \alpha w) \ dt = w - e^{-\beta x - \alpha t}. \end{aligned} \tag{3.19}$$

Using the inverse integration operator L_1^{-1}, L_2^{-1} for the equations of the system (27) respectively, we get

$$\begin{aligned} u(x,t) &= e^{\beta x + \alpha t} + L_1^{-1}(\beta) - L_1^{-1}(wu_x), \\ w(x,t) &= e^{-\beta x - \alpha t} + L_2^{-1}(\beta) + L_2^{-1}(uw_x). \end{aligned} \tag{3.20}$$

The complete solution set of system as follows

$$\begin{aligned} u_0(x,t) &= e^{\beta x + \alpha t} + L_1^{-1}(\beta), \\ u_{k+1}(x,t) &= -L_1^{-1} A_k, \quad k \geq 0, \\ w_0(x,t) &= e^{-\beta x - \alpha t} + L_2^{-1}(\beta), \\ w_{k+1}(x,t) &= L_2^{-1} B_k, \quad k \geq 0. \end{aligned} \tag{3.21}$$

Therefore

$$\begin{aligned} u_0(x,t) = e^{\beta x + \alpha t} + L_1^{-1}(\beta) &= e^{\beta x + \alpha t} + e^{\alpha dt} \int_0^t e^{-\alpha dt}(\beta) dt \\ &= e^{\beta x + \alpha t} + \frac{e^{\alpha t}\beta}{\alpha} - \frac{\beta}{\alpha}, \end{aligned}$$

$$\begin{aligned} w_0(x,t) = e^{-\beta x-\alpha t} + L_2^{-1}(\beta) &= e^{-\beta x-\alpha t} + e^{-\alpha dt}\int_0^t e^{\alpha dt}(\beta)dt \\ &= e^{-\beta x-\alpha t} - \frac{e^{-\alpha t}\beta}{\alpha} + \frac{\beta}{\alpha}. \end{aligned}$$

$$\begin{aligned} u_1(x,t) = -L_1^{-1}A_0 &= -e^{\alpha t}\int_0^t e^{-\alpha t}(w_0 u_{0x})dt \\ &= -\frac{\beta^2}{\alpha}te^{\beta x+\alpha t} + \frac{\beta^2}{\alpha^2}e^{\beta x+\alpha t} - \frac{\beta}{\alpha}e^{\alpha t} - \frac{\beta^2}{\alpha^2}e^{\beta x} + \frac{\beta}{\alpha}, \end{aligned}$$

$$\begin{aligned} w_1(x,t) = L_2^{-1}B_0 &= e^{-\alpha t}\int_0^t e^{\alpha t}(u_0 w_{0x})dt \\ &= \frac{\beta^2}{\alpha}te^{-\beta x-\alpha t} + \frac{\beta^2}{\alpha^2}e^{-\beta x-\alpha t} + \frac{\beta}{\alpha}e^{-\alpha t} - \frac{\beta^2}{\alpha^2}e^{-\beta x} - \frac{\beta}{\alpha}. \end{aligned}$$

$$\vdots$$

The decomposition series solution for the system which we obtained by MADM are given by:

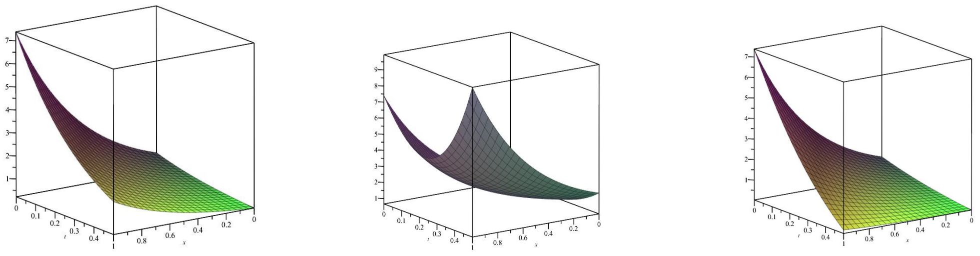
$$\begin{aligned} u(x,t) &= u_0 + u_1 + \cdots \\ &= e^{\beta x+\alpha t} - \frac{\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{\beta^2}{\alpha}te^{\beta x+\alpha t} + \frac{\beta^2}{\alpha}te^{\beta x+\alpha t} + \frac{\beta^4}{2\alpha^2}t^2e^{\beta x+\alpha t} + \cdots \\ &= e^{\beta x+\alpha t} + \frac{\beta^4}{2\alpha^2}t^2e^{\beta x+\alpha t} + \cdots \end{aligned}$$

$$\begin{aligned} w(x,t) &= w_0 + w_1 + \cdots \\ &= e^{-\beta x-\alpha t} + \frac{\beta}{\alpha} - \frac{\beta}{\alpha} + \frac{\beta^2}{\alpha}te^{-\beta x-\alpha t} - \frac{\beta^2}{\alpha}te^{-\beta x-\alpha t} + \frac{\beta^4}{2\alpha^2}t^2e^{-\beta x-\alpha t} + \cdots \\ &= e^{-\beta x-\alpha t} + \frac{\beta^4}{2\alpha^2}t^2e^{-\beta x-\alpha t} + \cdots \end{aligned}$$

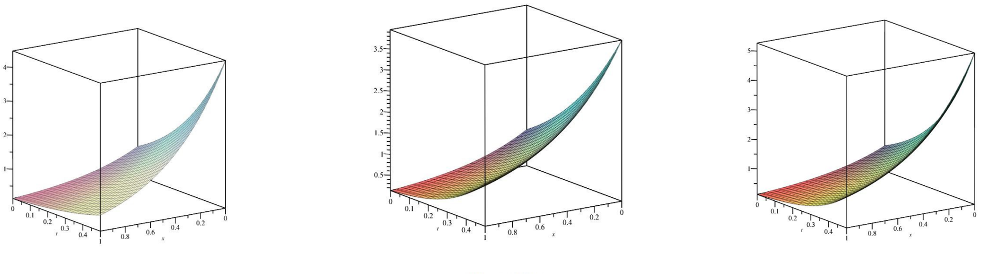
The decomposition series solution for system (25)which we got via ADM are given by:

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + \cdots \\ &= e^{\beta x} + \alpha te^{\beta x} + \alpha^2\frac{t^2}{2}e^{\beta x} + \alpha^2\beta\frac{t^3}{6} - \alpha\beta^2\frac{t^3}{3}e^{\beta x} + \beta^3\frac{t^3}{3} + \beta^4\frac{t^4}{8}e^{\beta x} + \cdots \end{aligned}$$

$$\begin{aligned} w(x,t) &= w_0 + w_1 + w_2 + \cdots \\ &= e^{-\beta x} - \alpha te^{-\beta x} + \alpha^2\frac{t^2}{2}e^{-\beta x} + \alpha^2\beta\frac{t^3}{6} + \alpha\beta^2\frac{t^3}{3}e^{-\beta x} + \beta^3\frac{t^3}{3} + \beta^4\frac{t^4}{8}e^{-\beta x} + \cdots \end{aligned}$$



(a) u(exact) (b) u(ADM) (c) u(MADM)



(d) w(exact) (e) w(ADM) (f) w(MADM)

Figure 2: 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM at α =-3, β =2 for **Example(3.3)**

Example 3.4. *Consider the system of nonlinear PDEs*

$$\begin{aligned} u_t + vu_x - 2u &= 1, \\ v_t - uv_x + 2v &= 1. \end{aligned} \tag{3.22}$$

With initial conditions

$$\begin{aligned} u(x,0) &= e^{x}, \\ v(x,0) &= e^{-x}. \end{aligned}$$

This example is a special case of example 2, with $\alpha = 2, \beta = 1$ *and the exact solution is* $u = e^{x+t}, v = e^{-x-t}$. *The system can be expressed as follows*

$$\begin{aligned} L_1 u &= 1 - vu_x, \\ L_2 v &= 1 + uv_x, \end{aligned} \tag{3.23}$$

where $L_1 = e^{\int 2dt}\frac{d}{dt}e^{-\int 2dt}$, $L_2 = e^{-\int 2dt}\frac{d}{dt}e^{\int 2dt}$,
$L_1^{-1}() = e^{\int 2dt}\int_0^t e^{-\int 2dt}()\ dt$, $L_2^{-1}() = e^{-\int 2dt}\int_0^t e^{\int 2\ dt}()dt$.
Using L_1^{-1}, L_2^{-1} *for* $u_t - 2u$, $v_t + 2v$ *of the system (31) respectively, we may obtain*

$$\begin{aligned} L_1^{-1}(u_t - 2u) &= e^{2t}\int_0^t e^{-2t}(u_t - 2u)\ dt = u - e^{x+2t}, \\ L_2^{-1}(v_t + 2v) &= e^{-2t}\int_0^t e^{2t}(v_t + 2v)\ dt = v - e^{-x-2t}. \end{aligned} \tag{3.24}$$

Applying L_1^{-1}, L_2^{-1} *for equations of the system (32) respectively, we may obtain*

$$\begin{aligned} u(x,t) &= e^{x+2t} + L_1^{-1}(1) - L_1^{-1}vu_x, \\ v(x,t) &= e^{-x-2t} + L_2^{-1}(1) + L_2^{-1}uv_x. \end{aligned} \tag{3.25}$$

The general solution of the system is as follows

$$\begin{aligned} u_0(x,t) &= e^{x+2t} + L_1^{-1}(1), \\ u_{k+1}(x,t) &= -L_1^{-1}A_k, \quad k \geq 0, \\ v_0(x,t) &= e^{-x-2t} + L_2^{-1}(1), \\ v_{k+1}(x,t) &= +L_2^{-1}B_k, \quad k \geq 0. \end{aligned} \tag{3.26}$$

The first two solutions iterations are as follows

$$\begin{aligned} u_0 &= e^{x+2t} + \frac{e^{2t}}{2} - \frac{1}{2}, \\ u_1 &= -L_t^{-1}A_0 = -\frac{1}{2}te^{x+2t} + \frac{1}{4}e^{x+2t} - \frac{1}{2}e^{2t} - \frac{1}{4}e^{x} + \frac{1}{2}, \end{aligned}$$

$$\vdots$$

$$
\begin{aligned}
v_0 &= e^{-x-2t} - \frac{e^{-2t}}{2} + \frac{1}{2}, \\
v_1 &= L_t^{-1}B_0 = \frac{1}{2}te^{-x-2t} + \frac{1}{4}e^{-x-2t} + \frac{1}{2}e^{-2t} - \frac{1}{4}e^{-x} - \frac{1}{2},
\end{aligned}
$$

$$\vdots$$

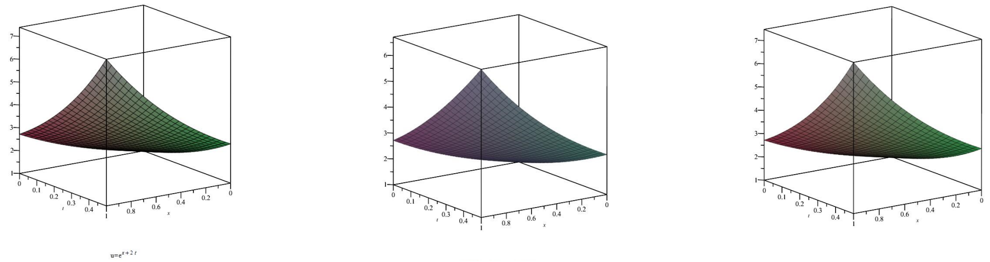
Summing the above iterations yields

$$
\begin{aligned}
u_{app}(x,t) = u_0 + u_1 + \cdots = \frac{5}{4}e^{x+2t} + \frac{1}{2}te^{x+2t} - \frac{1}{4}e^{x} + \cdots \\
v_{app}(x,t) = v_0 + v_1 + \cdots = \frac{5}{4}e^{-x-2t} + \frac{1}{2}te^{-x-2t} - \frac{1}{4}e^{-x} + \cdots
\end{aligned}
\tag{3.27}
$$

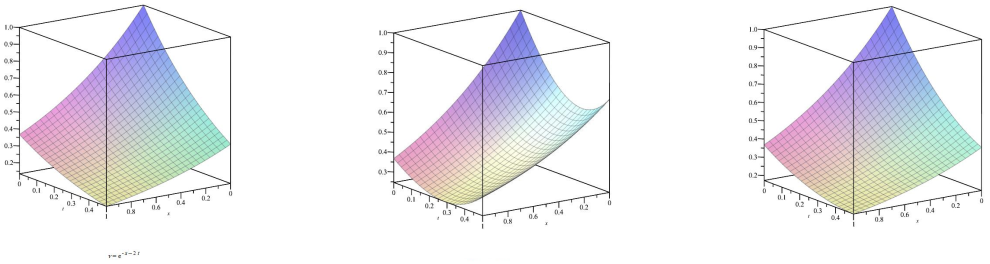
Which are the series of the approximate solution for $u(x,t)$*and* $v(x,t)$ *given using MADM. And*

$$
\begin{aligned}
u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots = e^{x} + 2te^{x} + 2t^2e^{x} + t^3 - 2\frac{t^3}{3}e^{x} + \frac{t^4}{8}e^{x} + \cdots \\
v = \sum_{n=0}^{\infty} v_n = v_0 + v_1 + v_2 + \cdots = e^{-x} - 2te^{-x} + 2t^2e^{-x} + t^3 + 2\frac{t^3}{3}e^{-x} + \frac{t^4}{8}e^{-x} + \cdots
\end{aligned}
\tag{3.28}
$$

Which are the series of the approximate solution for u(x,t), v(x,t) that we produced via ADM.



(a) u(exact) (b) u(ADM) (c) u(MADM)



(d) v(exact) (e) v(ADM) (f) v(MADM)

Figure 3: 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM for **Example(3.4)**

Example 3.5. *Consider the system of nonlinear PDEs*

$$\begin{aligned}\frac{\partial u}{\partial t}+\left(v\frac{\partial u}{\partial x}\right)^2+u=1,\\ \frac{\partial v}{\partial t}+\left(u\frac{\partial v}{\partial x}\right)^2-v=1.\end{aligned} \tag{3.29}$$

With initial conditions

$$\begin{aligned}u(x,0)=e^{x},\\ v(x,0)=e^{-x}.\end{aligned}$$

The exact solution is given in as [6]

$$\begin{aligned}u(x,t)=e^{x-t},\\ v(x,t)=e^{-x+t}.\end{aligned} \tag{3.30}$$

For the modified method, the system is expressed as follows

$$\begin{aligned}L_1u=1-(vu_x)^2,\\ L_2v=1-(uv_x)^2,\end{aligned} \tag{3.31}$$

where $L_1=e^{-\int\, dt}\frac{d}{dt}e^{\int\, dt}$, $L_2=e^{\int\, dt}\frac{d}{dt}e^{-\int\, dt}$,
$L_1^{-1}()=e^{-\int\, dt}\int_0^t e^{\int\, dt}()\,dt$, $L_2^{-1}()=e^{\int\, dt}\int_0^t e^{-\int\, dt}()\,dt$.
Using L_1^{-1},L_2^{-1} *for* u_t+u, v_t-v, *of the system (38) respectively, it produces the following*

$$\begin{aligned}L_1^{-1}(u_t+u)=e^{-t}\int_0^t e^{t}(u_t+u)\,dt=u-e^{x-t},\\ L_2^{-1}(v_t-v)=e^{t}\int_0^t e^{-t}(v_t-v)\,dt=u-e^{-x+t}.\end{aligned} \tag{3.32}$$

Employing L_1^{-1},L_2^{-1} *for equations of the system (40) respectively, we may obtain*

$$\begin{aligned}u(x,t)=e^{x-t}+L_1^{-1}(1)-L_1^{-1}(vu_x)^2\\ v(x,t)=e^{-x+t}+L_2^{-1}(1)-L_2^{-1}(uv_x)^2\end{aligned} \tag{3.33}$$

The general solution of system as follows

$$\begin{aligned}u_0(x,t)&=e^{x-t}+L_1^{-1}(1),\\ u_{k+1}(x,t)&=-L_1^{-1}A_k^2,\quad k\ge 0,\\ v_0&=e^{-x+t}+L_2^{-1}(1),\\ v_{k+1}(x,t)&=-L_2^{-1}B_k^2,\quad k\ge 0.\end{aligned} \tag{3.34}$$

The first two terms of solutions iterations are as follows

$$\begin{aligned}
u_0(x,t) &= e^{x-t}+1-e^{-t}\\
v_0(x,t) &= e^{-x+t}-1+e^{t}\\
u_1(x,t) &= -L_1^{-1}A_0^2 = -e^{-t}\int_0^t e^{t}(v_0 u_{0_x})^2\,dt\\
&= 2te^{2x-t}+2te^{x-t}+e^{2x-2t}+2e^{x-t}+e^{-t}-e^{2x}-2e^{x}-1\\
v_1(x,t) &= -L_2^{-1}B_0^2 = -e^{t}\int_0^t e^{-t}(u_0 v_{0_x})^2\,dt\\
&= 2te^{-2x+t}-2te^{-x+t}-e^{-2x+2t}+2e^{-x+t}-e^{t}+e^{-2x}-2e^{-x}+1
\end{aligned}$$

$$\vdots$$

Summing the above iterations yields

$$\begin{aligned}
u(x,t) &= u_0+u_1+\cdots = 2te^{2x-t}+2te^{x-t}+e^{2x-2t}+3e^{x-t}-e^{2x}-2e^{x}+\cdots\\
v(x,t) &= v_0+v_1+\cdots = 2te^{-2x+t}-2te^{-x+t}-e^{-2x+2t}+2e^{-x+t}+e^{-2x}-2e^{-x}+\cdots
\end{aligned}$$

Which the approximate solutions for u(x,t), v(x,t) of the system that we obtained by MADM are given.
And the approximate solutions for u(x,t), v(x,t) of the system that we obtained by ADM are given.

$$\begin{aligned}
u(x,t)=u_0+u_1+u_2+\cdots &= e^{x}+t-\frac{e^{2x}t^3}{3}-e^{x}t^2-\frac{t^2}{2}-t-te^{x}\\
&- \frac{4}{81}t^9e^{4x}-\frac{1}{18}(e^{x}+e^{3x})t^8\\
&- \frac{1}{7}\left[\frac{4}{3}(2e^{2x}+\frac{1}{2}e^{3x})-\frac{1}{9}(e^{-x}+5e^{x})^2\right]t^7\\
&- \frac{1}{6}\left[\frac{4}{3}e^{3x}+\frac{2}{3}(2+\frac{1}{2}e^{x})(e^{-x}+e^{5x})\right]t^6\\
&- \frac{1}{5}\left[\frac{2}{3}(1+5e^{2x})-(2+\frac{1}{2}e^{x})^2\right]t^5\\
&- \frac{1}{4}\left[2e^{x}(2+\frac{1}{2}e^{x})-\frac{1}{3}e^{2x}\right]t^4\\
&- \frac{1}{3}\left[e^{2x}-e^{x}-\frac{1}{2}\right]t^3+\frac{1}{2}\left[1+e^{x}\right]t^2+\cdots
\end{aligned}$$

$$\begin{aligned}
v(x,t) = v_0 + v_1 + v_2 + \cdots &= e^{-x} + t - \frac{e^{-2x}t^3}{3} - e^{-x}t^2 + \frac{t^2}{2} - t + te^{-x} \\
&- \frac{4}{81}t^9 e^{-4x} - \frac{1}{18}(e^{-x} + e^{-3x})t^8 \\
&- \frac{1}{7}\left[\frac{4}{3}(2e^{-2x} - \frac{1}{2}e^{-3x}) + \frac{1}{9}(e^{x} + 5e^{-x})^2\right] t^7 \\
&- \frac{1}{6}\left[\frac{4}{3}e^{-3x} + \frac{2}{3}(2 - \frac{1}{2}e^{-x})(e^{x} + e^{-5x})\right] t^6 \\
&- \frac{1}{5}\left[\frac{2}{3}(1 + 5e^{-2x}) + (2 - \frac{1}{2}e^{-x})^2\right] t^5 \\
&- \frac{1}{4}\left[2e^{-x}(2 - \frac{1}{2}e^{-x}) + \frac{1}{3}e^{-2x}\right] t^4 \\
&- \frac{1}{3}\left[e^{-2x} + e^{-x} - \frac{1}{2}\right] t^3 - \frac{1}{2}\left[1 - e^{x}\right] t^2 + \cdots
\end{aligned}$$

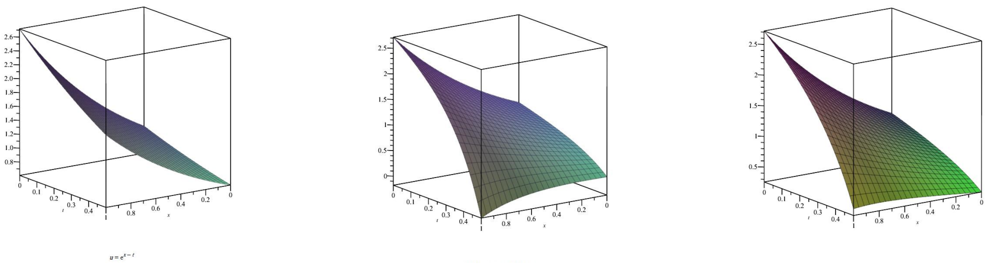
Remark: By using the noise terms phenomenon in the above example

$$\begin{aligned}
u_0(x,t) &= u_0\Phi(0)\Phi^{-1}(t) = e^{x-t}, \quad \Phi^{-1}(t) = e^{-\int dt}, \\
v_0(x,t) &= v_0\Phi(0)\Phi^{-1}(t) = e^{-x+t}, \quad \Phi^{-1}(t) = e^{\int dt}, \\
u_1(x,t) &= L_1^{-1}(1) - L_1^{-1}A_0^2 = 0, \\
v_1(x,t) &= L_2^{-1}(1) - L_2^{-1}B_0^2 = 0,
\end{aligned}$$

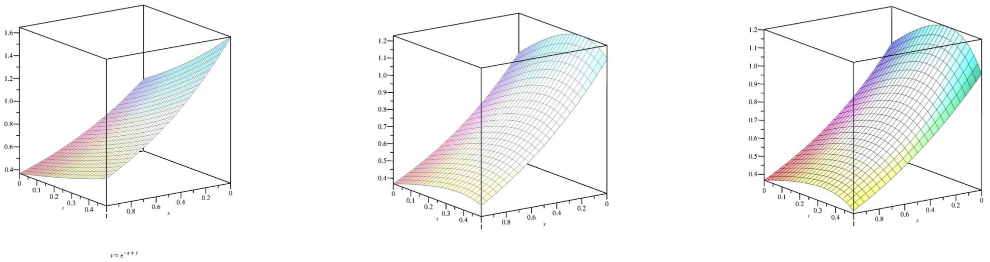
$$\vdots$$

$$\begin{aligned}
u_{k+1}(x,t \quad) &= \quad -L_1^{-1}A_k^2 = 0, \quad k \geq 1, \\
v_{k+1}(x,t) \quad &= \quad -L_2^{-1}B_k^2 = 0, \quad k \geq 1.
\end{aligned}$$

We have
$(u,v) = (u_0(x,t), v_0(x,t)) = (e^{x-t}, e^{-x+t})$,
which represents a closed solution. This also applies to Ex(3.3)and Ex(3.4)



(a) u(exact) (b) u(ADM) (c) u(MADM)



(d) v(exact) (e) v(ADM) (f) v(MADM)

Figure 4: 3D plots of the exact solution and 3-approximate solutions of the ADM and MADM for **Example(3.5)**

4 Conclusion

This research demonstrates that the Modified Adomian Decomposition Method (MADM) is a valuable tool for tackling complex nonlinear partial differential equations. While both MADM and the standard Adomian Decomposition Method (ADM) are effective for linear systems, MADM offers improved convergence and accuracy due to the incorporation of a new integral operator. The successful application of MADM in the presented numerical examples expands the capabilities of Adomian methods for solving a wider range of nonlinear PDEs.

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