Original Research Article

Interest Rate Risk Modelling Using Semi-Heavy Tail Distributions of Normal Variance-Mean Mixtures: Central Bank of Kenya Interest Rates.

Abstract

Derivative prices such as options and bond prices as well as swaps depend on the distributional assumptions of the underlying economic variables, normally, interest rates. The risk associated with changes in interest rates may worsen the value of the contract that depend on it since the values of these assets (derivative contracts) are affected directly by the fluctuations in interest rates. The distribution of interest rates, therefore, needs to be well understood to reduce the risks of losses associated with it. The Binomial Option pricing model assumes that interest rates are constant, with no returns, throughout the life of the option. Another common assumption of the underlying economic variables is that their returns are normally distributed with constant volatility. These assumptions have been used in pricing derivatives and currencies and has led to over-pricing and in some cases under-pricing. These assumptions have been considered inaccurate and misleading. This research uses mixture models exhibiting properties that appropriately capture the peakedness and skewness of interest rates as fundamental variables in pricing. The models of the Normal Variance-Mean Mixtures shows better performance than the normal distribution. The GARCH model is used under the assumption that 91-day Treasury Bills interest rates follow a Generalized Hyperbolic distribution while the Commercial Bank interest rates follows a Normal Inverse Gaussian distribution. A 99pc Value at Risk is then computed for the two models and calculates the minimum expected returns in the subsequent months. This research forms a foundation for the development of advanced pricing models that incorporates the fluctuations of interest rate in the pricing industry.

Keywords: Normal Variance-Mean Mixtures, Semi-Heavy Tail, Risk, GARCH, Pricing
2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

1 Introduction

Interest rates are one of the most common underlying variables in the pricing of financial derivatives such as options, equity, and bond valuation. They can simply be put as determinant of the prices of other derivatives. The most common rates according to the central bank of Kenya are the treasury bills' interest rates and the commercial bank's weighted average rates. A treasury bill is a financial instrument which is issued to lenders by the authority through the central bank (which is a fiscal agent) in order to raise project finances on a short-term basis. They mature after a period of 91, 182, and 364 days from the beginning of the contract and are sold at discounted prices to reflect the investor's return and redeemed at face (par) value. On the other hand, a weighted average rate is a group of data that analyze the calculated weighted average interest rate of the flow of transaction of the previous working day of the relevant month.
The risk, as measured by the volatility of interest rates and their Kurtosis, exhibit high frequency fluctuation over very short time intervals (seconds or minutes). The returns from interest rates exhibit a behavior of fat tails and peakedness with a skewed distribution

curve. This clearly invalidates the assumption that they follow a normal distribution with constant volatility. They also exhibit common statistical properties such volatility clustering and non-normality (1),(19) Moreover, the data, as in, (19) are fat tailed, and non-linear, with leptokurtic empirical distribution, meaning that they are more peaked with heavier tails than the normal distribution. By the property of semi heavy tails, a risk manager and derivative pricing manager need to use models relative to the characteristics of the data to bring out the right context and interpretation of the data in question. These models can capture the semi-heavy tails of the data and other statistical properties. Some studies have suggested the use of Gaussian assumption in modeling, pricing and forecasting financial derivatives which to date has been rendered obsolete. The use of the Black-Scholes Option pricing model (18), (41) and (4), the Random Walk model (46), and the Geometric Brownian Motion (49) have based their assumption on the fact that the distribution of interest rates is normal (52), while others have assumed a log-normal distribution. This assumption has been rendered obsolete and invalidated by (30) which confirms that these returns do not follow the normality assumption.

Despite the vast research on the financial derivative market and distribution of asset returns in the stock market and equity prices, very little is known about the risk associated with interest rates, as a financial variable, and its effect on pricing derivatives and currency options using the Black-Scholes options pricing model, GBM, and RW financial time series data forecasting models, as underlying variables. These pricing models(Brownian Motion (Geometric Brownian Motion) (49), n-period Binomial Model and the Cox-Ross-Rubinstein model and the famous Black-Scholes Option pricing model (18)) assume that the underlying variables such as interest rates are normally distributed with a constant volatility which in practice is not true. Besides, the normal model allows for a chance that the interest rates can be negative which is also highly unlikely. As much as the use of models with normality assumptions has been rampant and considered appropriate, the use of new more flexible, and analytically tractable models has been developed but has since not been applied in modeling interest rates risk. Therefore, modeling the CBK interest rates using semi-heavy tail distributions developed by (48) and used by (6), (25), (10), (29) and (13) in risk management has so far not been used in modeling interest rate risk. This study provides a better interest rates model that can be used in place of a normal distribution for the underlying interest rates as used in derivative pricing models as well as hedging and pricing volatility dependent claims such as Variance and Volatility Swaps. This will improve accuracy of these models therefore reducing risks of mis-pricing.

2 Review of Previous Literature

The Geometric Brownian Motion was first developed by Robert Brown to observe the irregular pollen grains from plants as they move on the surface of a liquid. He admitted that he had no scientific explanation for his observed phenomena (17). This was later studied by Albert Einstein in 1905 (3) who proposed the existence of Brownian Motion which was later proven by Norbert Wiener in 1923 (24). The standard Wiener process can be used to model asset returns but its main weakness is that its mean is zero, meaning that the growth rate of return is zero which, because of inflation (45), is not true in practice. With the modern application of Brownian Motion to model financial markets, the GBM was applied by Black and Scholes (27), Merton (50), and Paul Samuelson among others. Later in 1973,

Black and Scholes derived an option valuation model (27) which was extended by Merton (50). They however assumed that stock returns follow a Brownian Motion process.
Different scholars developed these models with their analysis beginning with the observed market price of financial instruments or a benchmark instrument, which is fairly priced. These models assume a random process with a drift term (mean) and volatility (variance) of interest rates. The Ho-Lee model was in 1986 by scientists Ho and Lee (34) to derive a model for movement of the arbitrage free interest rate and apply this model to interest rates. However, the model assumes that the market is friction-less, has discrete points in time, and is complete (a market with negligible transaction costs and perfect information). In this model, there is no mean reversion (an assumption that interest rates eventually move back to their average), and volatility is independent of the level of the short rate. In short, the Ho-Lee model is normal. The Hull-White model was later introduced by John Hull and Allan White in 1990 (35) to price interest rate derivative securities by allowing for mean reversion. This model (HW) (35) is a generalization of the Ho-Lee model (34) and its purpose was to overcome the weakness of (35) but still, it was a normal model. Kalotay-Williams-Fabozzi model (KWF thereafter) was later introduced by Andrew Kalotay, George Williams, and Frank J. Fabozzi in 1993 (38) to model bond rates and embedded options, it did not allow for mean reversion but allowed for changes in short rates to be modeled by natural logarithms of returns to attain stationarity. The Black-Karasinski model (BK thereafter) was introduced by Fisher Black and Piotr Karasinski in 1991 (11) to price bonds and options when short rates are lognormal. The BK was a logarithmic extension of the KWF. Black-Derman-Toy was earlier introduced by Fischer Black, Emmanuel Derman, and William Toy in 1990 (28) as a one-factor model for interest rates and Treasury bond options. Later, Heath-Jarrow-Morton (HJM herein) model was introduced in 1997 (22) and derived by Jeffrey Andrew (5) as a general continuous-time multi-factor model to price bonds and the term structure of interest rates. In these models, term structure dynamics are described using fundamental economic variables assumed to affect interest rates. The Vasicek was introduced in 1977 (54), the Cox, Ingersoll, and Ross in 1985 (20), Brennan and Schwartz (16), and Longstaff and Schwartz model (39) used economic variables affecting interest rates and used to price the term structure of interest rates.
In summary, these models have common limitations. They make an assumption that the interest rates follow a normal distribution which in practice is not true. The Vasicek model and the Hull and White models allows for a negative interest rates which in reality is not very likely. They also assume that the volatility as shown by the variance of interest rates is constant, which is also not true in practice. In derivative pricing, the most commonly used methods of options pricing are the famous continuous-time Black-Scholes Option pricing model (12), (51) and (19), the discrete time Binomial model (21), the Cox-Ross-Rubinstein model (21), (44) and the Stochastic models such as the Heston, SABR and ARCH models. These models bear a common assumption that the volatility and the interest rates are known and constant, which in practice is nearly impossible. The true implied volatility (volatility measured using the historical information) is not constant over time. Since the volatility is a key component in option pricing using these models, ignoring its variability nature by assuming that it is constant and known poses high risks in pricing process. There is need to incorporate models that have the capabilities to model the volatility skew and the tails in to the pricing models. The development of financial models has since taken a turn when other models were introduced in the literature. Berndorff-Nielsen in 1977 introduced hyperbolic distributions (48) as a model for wind blown sand and sizes

of grains. The distribution used was derived from the fact that its geometric returns form a hyperbola and hence a hyperbolic distribution. The distribution was later discussed by other authors, particularly in Physics to model turbulence. A special case such as NIG was introduced by BN in 1997 (47) and found to have long tails in both directions. Because of its attractive features and analytical tractability, it has attracted attention of authors in the investment market (55). Its tail behavior is seen as semi-heavy to mean that it is heavier than Gaussian but lighter than Pareto and non-Gaussian stable laws. Other special cases also have similar properties.

3 Research Methodology

3.1 Data Used in the Research

Secondary data for 91-day Treasury Bills interest rates and Commercial Bank Weighted average interest rates (Base lending rates) from the Central Bank of Kenya from 1991 to 2021 were used. The log was used to eliminate the unit root behavior intrinsic to I_t and therefore achieving stationarity. Maximum likelihood method was used to estimate the parameters.

3.2 The Modified Bessel Function of the Third Kind.

Represented as follows (index ν and order ω denoted by $K_\nu(\omega)$)

$$2K_\nu(\omega) = \int_0^\infty x^{\nu-1} e^{-\frac{\omega}{2}(x+\frac{1}{x})} dx\omega, x > 0$$

as proposed by (48) and properties analyzed by (2)

3.3 Generalized form of the Mixing distributions and Properties

The pdf's of the Generalized form of the mixing distribution (see (31), (9) and (36)) is obtained by substituting $\delta\gamma$ into the equation 3.1. This is then simplified to obtain :

$$g(z) = \left(\frac{\gamma}{\delta}\right)^\lambda \frac{z^{\lambda-1}}{2K_\lambda(\delta\gamma)} e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} \quad (3.1)$$

where δ, $\gamma > 0$ with the following properties:
the n-th moment is;

$$E[Z^n] = \left(\frac{\delta}{\gamma}\right)^n \frac{K_{\lambda+n}(\delta\gamma)}{K_\lambda(\delta\gamma)}$$

and the variance:

$$Var[Z] = \left(\frac{\delta}{\gamma}\right)^2 \Big[\frac{K_{\lambda+2}(\delta\gamma)}{K_{\lambda}(\delta\gamma)} - \Big(\frac{K_{\lambda+1}(\delta\gamma)}{K_{\lambda}(\delta\gamma)}\Big)^2\Big]$$

For more properties and proofs, see (7).

3.3.1 Construction of RIG Mixing Distribution

Suppose $\lambda = \frac{1}{2}$, and substituting in equation 3.1 above, we obtain;

$$\begin{aligned} g(z) = & \left(\frac{\gamma}{\delta}\right)^{\frac{1}{2}} \frac{z^{-\frac{1}{2}}}{2k_{\frac{1}{2}}(\delta\gamma)} e^{-\frac{1}{2}\left(\frac{\delta^2}{z} + \gamma^2 z\right)} \\ = & \frac{\gamma^{\frac{1}{2}}}{\delta^{\frac{1}{2}}} \left(\frac{z^{-\frac{1}{2}}}{2k_{\frac{1}{2}}(\delta\gamma)}\right) e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} \\ = & \frac{\gamma^{\frac{1}{2}}}{\delta^{\frac{1}{2}}} \frac{z^{-\frac{1}{2}}}{\sqrt{\frac{2\pi}{\delta\gamma}} e^{-\delta\gamma}} e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} \\ = & \frac{\gamma^{\frac{1}{2}}}{\delta^{\frac{1}{2}}} \frac{z^{-\frac{1}{2}}}{\sqrt{\frac{2\pi}{\delta\gamma}}} e^{\delta\gamma} e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)} \\ = & \frac{\gamma}{\sqrt{2\pi}} z^{-\frac{1}{2}} e^{\delta\gamma} e^{-\frac{1}{2}\left(\frac{\delta^2}{z} + \gamma^2 z\right)} \end{aligned} \tag{3.2}$$

The expectation, variance, Skewness and Kurtosis are provided. The proofs are simplified through the n-th moment generator as outlined in 3.1

$$
\begin{aligned}
E(Z^n) &= \left(\frac{\delta}{\gamma}\right)^n \frac{K_{\lambda+n}(\delta\gamma)}{K_\lambda \delta\gamma} \\
&= \frac{(\delta\gamma+1)}{\gamma^2} \\
E(Z^2) &= \left(\frac{\gamma}{\delta}\right)^2 \frac{k_{\frac{5}{2}}(\delta\gamma)}{k_{\frac{1}{2}}(\delta\gamma)} \\
&= \frac{\delta^2}{\gamma^2} * \frac{\delta^2\gamma^2+3\delta\gamma+3}{\delta^2\gamma^2} \\
&= \frac{\delta^2\gamma^2+3\delta\gamma+3}{\gamma^4} \\
Var(Z) &= \frac{\delta^2\gamma^2+3\delta\gamma+3}{\gamma^4} - \left[\frac{\delta^2\gamma^2+2\delta\gamma+1}{\gamma^4}\right] \\
&= \frac{\delta\gamma+2}{\gamma^4} \\
Sk(Z) &= \frac{\mu_3(Z)}{(Var(Z))^{1.5}} \\
&= \frac{3\delta\gamma+8}{\gamma^6}\left(\frac{\gamma^4}{\delta\gamma-1}\right)^{1.5} \\
&= \frac{3\delta\gamma+8}{(\delta\gamma-1)^{1.5}}
\end{aligned}
$$

Suppose $\lambda = -\frac{1}{2}$
Replacing into the equation 3.1, and going through the same process, we obtain the mixing distribution for the Normal Inverse Gaussian distribution) as follows;

$$
g(z) = \frac{\delta}{\sqrt{2\pi}} z^{-\frac{3}{2}} e^{(\delta\gamma)} e^{-\frac{1}{2}\left(\frac{\delta^2}{z}+\gamma^2 z\right)}
$$

Properties
The first, second, and third moments are studied here:

$$
\begin{aligned}
E(Z^n) &= \left(\frac{\delta}{\gamma}\right)^n \frac{K_{\lambda+n}(\delta\gamma)}{K_\lambda \delta\gamma} \\
&= \left(\frac{\delta}{\gamma}\right) \frac{K_{\frac{1}{2}}(\delta\gamma)}{K_{-\frac{1}{2}}\delta\gamma} \\
&= \left(\frac{\delta}{\gamma}\right) \\
E(Z^2) &= \left(\frac{\gamma}{\delta}\right)^2 \frac{k_{\frac{3}{2}}(\delta\gamma)}{k_{-\frac{1}{2}}(\delta\gamma)} \\
&= \frac{\delta^2}{\gamma^2} \frac{\delta\gamma + 1}{\delta\gamma} \\
&= \frac{\delta(\delta\gamma + 1)}{\gamma^3} \\
VAR(Z) &= \frac{\delta(\delta\gamma + 1)}{\gamma^3} - \frac{\delta^2}{\gamma^2} \\
&= \frac{\delta}{\gamma^3}
\end{aligned}
$$

3.3.2 The Normal Variance-Mean Mixtures

When $\lambda = -\frac{1}{2}$ and 3.3, as the mixing distribution and $X|Z \sim N(\mu + \beta z, z)$ as the conditional distribution, therefore, by using direct integration we obtain the pdf of the Normal Inverse Gaussian distribution as follows;

$$
\begin{aligned}
f(x) &= \int_0^\infty \frac{1}{\sqrt{2\pi z}} e^{-\frac{1}{2}\left[\frac{((x-\mu)+\beta z)^2}{z}\right]} * \frac{\delta}{\sqrt{2\pi}} z^{-\frac{3}{2}} e^{(\delta\gamma)} e^{-\frac{1}{2}} \left[\frac{\delta^2}{z} + \gamma^2 z\right] dz \\
&= \frac{\delta}{2\pi} e^{(\delta\gamma)} \int_0^\infty z^{-2} e^{-\frac{1}{2}\left[\frac{((x-\mu)+\beta z)^2}{z} + \frac{\delta^2}{z} + \gamma^2 z\right]} dz \\
&= \frac{\delta}{2\pi} e^{(\delta\gamma)} \int_0^\infty z^{-2} e^{-\frac{1}{2}\left[\frac{(x-\mu)^2}{z} + 2(x-\mu)\beta + \beta^2 z + \frac{\delta^2}{z} + \gamma^2 z\right]} dz \\
&= \frac{\delta}{2\pi} e^{(\delta\gamma)} e^{(x-\mu)\beta} \int_0^\infty z^{-2} e^{-\frac{(\beta^2+\gamma^2)}{2}\left[z + \frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)z}\right]} dz
\end{aligned}
$$

The aim is to simplify the exponential part so we use change of variable technique by letting $z = \sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}} t \Rightarrow dz = \sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}} dt$ Note also that; $(\beta^2 + \gamma^2) = \alpha^2$, therefore

$$
\begin{aligned}
f(x) &= \int_0^\infty \left[\sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}t\right]^{-2} e^{-\frac{(\beta^2+\gamma^2)}{2}\left[\left(\sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}t\right)+\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)(\sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}t)}\right]} \sqrt{\frac{(x-\mu)^2+\delta^2}{(\beta^2+\gamma^2)}}dt \\
&= \left[\sqrt{\frac{(x-\mu)^2+\delta^2}{(\alpha^2)}}\right]^{-1} \int_0^\infty t^{-2} e^{-\frac{\alpha^2}{2}*\sqrt{\frac{(x-\mu)^2+\delta^2}{(\alpha^2)}}(t+\frac{1}{t})} dt \\
&= \frac{\delta}{2\pi} e^{(\delta\gamma)} e^{(x-\mu)\beta} \frac{\alpha}{\sqrt{(x-\mu)^2+\delta^2}} 2K_1\left[\sqrt{\alpha^2\Big((x-\mu)^2+\delta^2\Big)}\right]
\end{aligned}
$$

with four parameters, $\alpha, \beta, \delta, \mu$ We also looked at other mixtures such as the Reciprocal Inverse Gaussia distribution, The Variance-Gamma, the Asymmetric Student-t and the Hyperbolic distribution and fitted to the data.

Properties

We can easily obtain the properties of the NIG distribution as follows;

$$
\begin{aligned}
E(X) &= \mu + \beta E(Z) \\
&= \mu + \beta\Big(\frac{\delta}{\gamma}\Big) \qquad (3.3) \\
Var(X) &= E(Z) + \beta^2 var(Z) \\
&= \frac{\alpha^2\delta}{\gamma^3} \\
Sk(X) &= 3\beta Var(Z) + \beta^3\mu_3(z) \qquad (3.4) \\
&= \frac{3\beta\Big(\frac{\delta}{\gamma^3}\Big) + \beta^3\Big(\frac{3\delta}{\delta^5}\Big)}{\Big(\frac{\delta}{\gamma^3}\Big)^{1.5}} \\
&= \frac{3\beta}{\alpha(\delta\gamma)^{0.5}} \qquad (3.5) \\
Kurt(X) &= 3\Big(\frac{1+4\beta^2/\alpha^2}{\gamma\delta}\Big) \qquad (3.6)
\end{aligned}
$$

Through similar approach for $\lambda = \frac{1}{2}$, we obtain the pdf of the Reciprocal Inverse Gaussian Distribution as follows;

$$
f(x) = \frac{\gamma}{\sqrt{2\pi}} e^{(\delta\gamma)} 2K_0\Big(\sqrt{\delta^2+\alpha(x-\mu)^2}\Big) e^{\beta(x-\mu)} \qquad (3.7)
$$

The Generalized Hyperbolic Distribution

The pdf of the hyperbolic distribution is(For proof and properties, see (43));

$$
f_{hyp}(x;\alpha,\beta,\delta,\mu) = \frac{\sqrt{(\alpha^2-\beta^2)}}{2\alpha\beta K_1(\delta\sqrt{(\alpha^2-\beta^2)})} e^{[-\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)]} \qquad (3.8)
$$

$\alpha > 0$, $\delta > 0$, $\mu \in \mathbb{R}$, $\beta \in$ (-α, α)

3.4 The G.A.R.C.H model

Introduced by (15) to allows for conditional variance to depend on the previous lag unlike the ARCH model. We will use this model as a forecasting tool for the interest rates.

3.4.1 Definition

Let η_t represent an iid sequence η distributed. ϵ_t is a G.A.R.C.H(p,q) process in the sequence η_t iff; $\epsilon_t = \sigma_t \eta_t$ we can sufficiently say that;

$$R_t{}^2 = \omega + \sum_{i=1}^{q} \alpha_i \sigma_{t-i}^2 \eta_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{3.9}$$

where ω, α_i, and β_j are constants and positive values. η is the distribution of the error term of the hyperbolic family.
For more properties and proof, see (26) and (15)

3.4.2 Model Selection Criteria

The models will be evaluated based on the Akaike Information Criterion approximated as follows;

$$AIC = -2log(H) + 2n \tag{3.10}$$

where H is the likelihood and n the number parameters in the model.
In some cases, we shall use Goodness-of-fit tests.

4 Analysis of Results and Discussion

4.1 Descriptive statistics

From the table, it is sufficient to conclude that the underlying interest rates used in pricing of bonds and stocks do not follow the normal distribution as assumed. This is shown by Skewness not equal to zero and Kurtosis greater than 3 as in the normal distribution.

4.2 Dickey-Fuller test for Mean Reversion

This method tests the stationarity a property of mean reversed data.In this test, the hypothesis is as follows,
H0: the data is non-stationary

Table 1: Summary Statistics

Rates	Mean	Skewness	Kurtosis
T.R	-0.00110384	-0.31146	8.5935
Bank	-0.00353	0.137206	18.9425

H1: the data is stationary.
The p-value is less that the alpha value. We therefore reject the (H0) and conclude that there is mean reversion in the data.

Table 2: Dickey-Fuller test for Commercial Bank and Treasury Bills Interest Rates

Rates	Dickey-Fuller	p-value
T.B	-8.2525	0.01
C.B	-12.543	0.01

4.3 Histograms of the Interest Rates Data

Fitting the normal distribution does not capture the peakedness well as seen in the histograms.

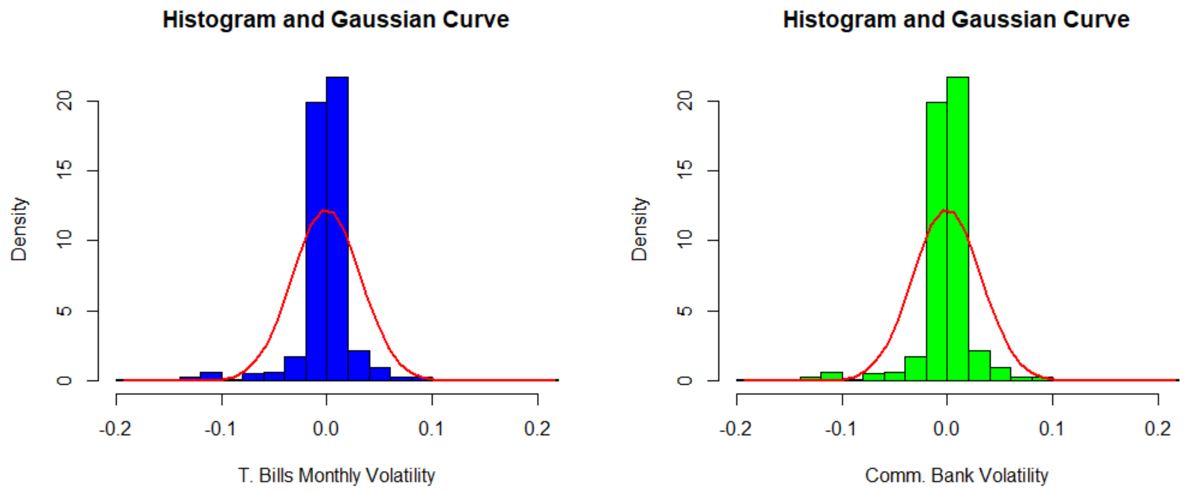

fig 1 &2: Histogram and Gaussian Curve in accordance with T. bills Monthly variable and Comm. Bank Volatility

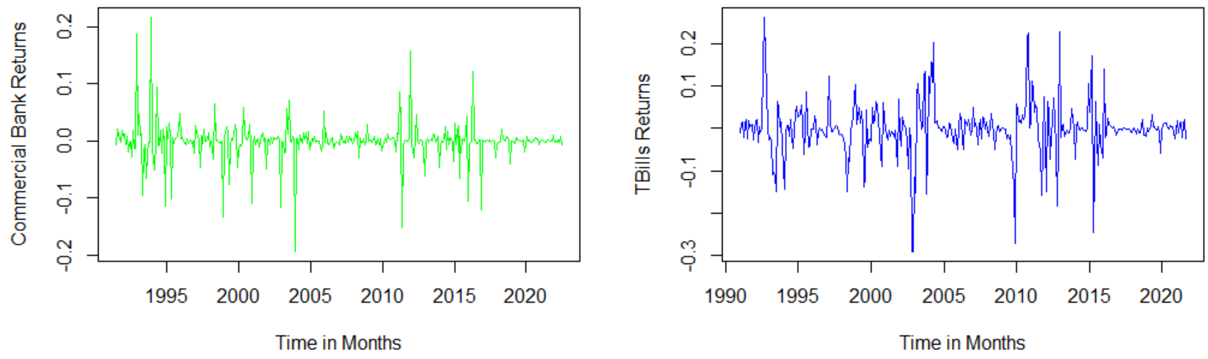

fig 3 & 4: Year based Comparison of T. bills return and Comm. Bank Returns

4.4 Parameter Estimates

In this section, we shall estimate the parameters of some of the distributions studied above using maximum likelihood method.
From the information in tables 3 and 4, it is shown that the Asymmetric NIG fits the Commercial bank interest rates better than the 91-day Treasury Bills interest rates.On the other hand, Asymmetric ghyp fits the 91-day data. This is due to the fact that it has the lowest AIC and the highest Log-Likelihood.

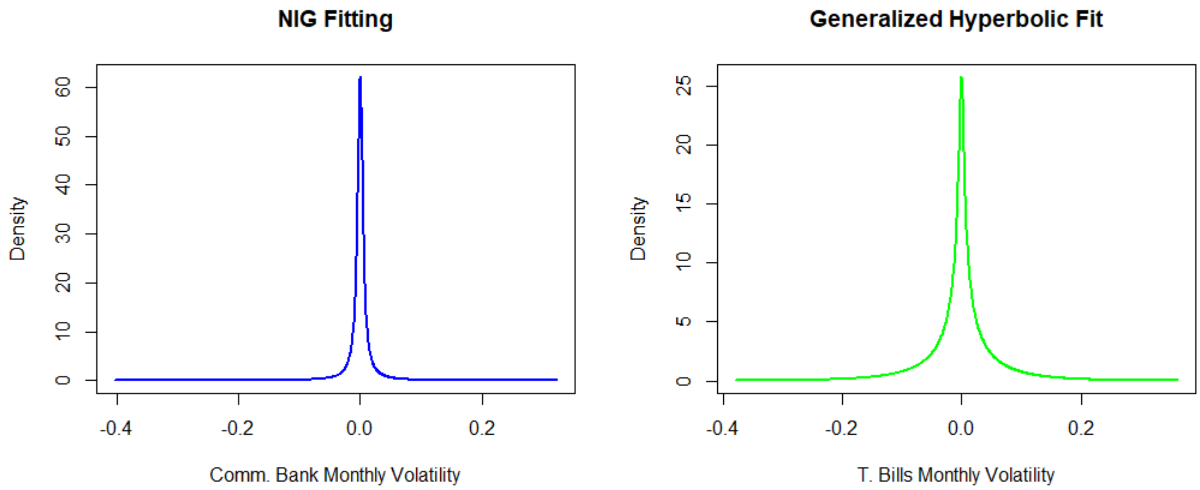

fig 5&6: Comparison of Asymmetric NIG and GHY Parametric Fits for Interest Rates: Analyzing Fit Quality for Commercial Banks and 91-Day Treasury Bills

4.5 ARCH L-M Test

This is a Chi-squared hypothesis test for the presence of ARCH effect using Lagrange Multiplier (L-M). H0:there is no ARCH effect while H1: there is ARCH effect.Since P ¡ 0.05,

Table 3: Parameter Estimates of the Asymmetric Normal Inverse Gaussian Distribution

Parameters	**91-day Treasury Bills**	**Commercial Bank**
α	0.0684618635	0.0173190064
μ	-0.0003035624	0.0004143653
δ	0.0719747059	0.0394937758
β	-0.0007691234	-0.0007874793
Log-Lik	617.5179	1034.177
AIC	-1227.036	-2060.353

Table 4: Parameter Estimates of the Asymmetric G.H Distribution

Parameters	**91-day Treasury Bills**	**Commercial Bank**
λ	0.1680857588	-0.3313386045
α	0.0435677071	0.0164281275
μ	-0.0001157169	0.0002918099
σ	0.0637759052	0.0042322110
γ	-0.0009652545	0.0042322110
Log-Lik	625.9611	1033.994
AIC	-1241.922	-2057.988

we reject the H0 and conclude that there is ARCH effect in the data. This means that there is no constant volatility and hence, GARCH (2,0) and ARMA (1,2) for the Treasury Bills interest rates and GARCH (1,1) with no mean model for the Commercial Bank interest rates selected because of their least values of the AIC will be used in forecasting.

Table 5: ARCH L-M test

	Chi-squared	p-value
91-day Treasury Bill	73.68	6.521e-11
Commercial Bank	68.151	7.092e-10

4.5.1 The Forecasted Values

The forecasts are as shown in the table below:

Table 6: 6- Month ghyp Forecasts

0-roll	Returns, T0 = Sept 2021	VaR(0.99)
T+1	0.03251	0.0867395
T+2	0.03738	0.099733
T+3	0.04159	0.110966
T+4	0.04543	0.1212112
T+5	0.04896	0.1306296
T+6	0.05225	0.139408

Table 7: 6- Month nig Forecasts

0-roll	Returns, T0 = Jul 2022	VaR(0.99)
T+1	0.01285	0.0347193
T+2	0.01282	0.0346383
T+3	0.01279	0.0345572
T+4	0.01275	0.0344491
T+5	0.01272	0.0343681
T+6	0.01269	0.034287

5 CONCLUSIONS

The objective was to use flexible distributions to understand these fluctuations in the returns of interest rates and contrary to the normality assumption as seen in the literature review, we note that these distribution provide a better fit than the normal distribution. In addition to that, the data displays special properties that require the use of GARCH models with special error terms assumption in forecasting the volatility. It is also found that this models provide good forecast and are better under normal assumption. From the Goodness-of Fit test and the AIC, we note that these models provide a good fit. We therefore recommend that;

a More distributions in this class explored in details and applied to the same data in order to determine the best performing distribution. This is because, this research did not explore all the distributions but four of the main ones. It also did not go into details in the properties except for the moments. More studies should also be done on proof of the properties of the tails.

b Other distributions outside the hyperbolic class should also be studied in details and subjected to the same conditions and data. Also, other heavy tail distributions and other approaches to model interest rate risk should be studied and result compared to ensure the best conditions for modeling interest rates.

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