THE $\mathcal{JK}$-METHOD OF INTERPOLATION FOR FINITE FAMILIES OF BANACH SPACES

Abstract

Let $\mathcal{I}$ be an operator ideal of those considered here. We present a method of interpolation for finite families such that if $\overline{A}$ and $\overline{B}$ are (n+1)-tuples, T is an interpolation operator and $\langle A\rangle, \langle B\rangle$ are the interpolation spaces obtained by this method then, $T : \langle A\rangle \to \langle B\rangle$ is in $\mathcal{I}$ if and only if the operator from the intersection $\mathcal{J}(\overline{A})$ into the sum $\mathcal{S}(\overline{B})$ is in $\mathcal{I}$.

Key words and phrases: real method of interpolation, operator ideals.

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1 Introduction

In this introductory section we will sketch in a light form, that is, brief and informally, what we shall do in this paper.

Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be two interpolation pairs of Banach spaces and let $T : \overline{A} \to \overline{B}$ be an interpolation operator; the Lions-Peetre method of interpolation for couples possesses a remarkable property from the geometric point of view: for certain operator ideals $\mathcal{I}$, the interpolated operator $T_{\theta,p} : \overline{A}_{\theta,p} \to \overline{B}_{\theta,p}$ between the corresponding interpolation spaces belongs to one of those operator ideals if and only if the operator $T_{\mathcal{JS}} : \mathcal{J}(\overline{A}) \to \mathcal{S}(\overline{B})$, from the intersection space $\mathcal{J}(\overline{A})$ into the sum space $\mathcal{S}(\overline{B})$ belongs to that ideal. In [15] this property (here called the *Beauzamy property of interpolation*) was expressed by saying that the ideal $\mathcal{I}$ satisfies the '*strong property of interpolation*' with respect to the real method of Lions-Peetre.

Let $\overline{A} = (A_0, A_1 \ldots, A_n)$ and $\overline{B} = (B_0, B_1 \ldots, B_n)$ be two finite families of interpolation and $T : \overline{A} \to \overline{B}$ be an interpolation operator; let $J(\overline{A})$ and $K(\overline{B})$ be the interpolation spaces obtained from $\overline{A}$ and $\overline{B}$ by the J and K real methods of interpolation for finite families of Sparr [18] for example. It was proved in [17], for the J and K methods of Sparr and other J and K methods of interpolation, that for certain operator ideals $\mathcal{I}$ the interpolated operator $T_{JK} : J(\overline{A}) \to K(\overline{B})$ from $J(\overline{A})$ into the corresponding $K(\overline{B})$ belongs to $\mathcal{I}$ if and only if the operator $T_{\mathcal{JS}} : \mathcal{J}(\overline{A}) \to \mathcal{S}(\overline{B})$ from the intersection space into the sum space belongs to $\mathcal{I}$; in other words, in that written (or rather, in that rough draft) it was proved for each one of the operator ideals $\mathcal{I}$ considered there that $T_{JK} \in \mathcal{I}$ if and only if $T_{\mathcal{JS}} \in \mathcal{I}$, see [17, Theorem 5.2]. On the basis of this fact, we propose here a real method of interpolation for finite families that provides us with the interpolation spaces $\mathcal{JK}(\overline{A})$, $\mathcal{JK}(\overline{B})$ and with respect to which the operator ideals $\mathcal{I}$ considered here possess the so called in this paper *Beauzamy property of interpolation* i.e., the property that the interpolated operator $T_{\mathcal{JK}} : \mathcal{JK}(\overline{A}) \to \mathcal{JK}(\overline{B})$ belongs to $\mathcal{I}$ if and only if the operator $T_{\mathcal{JS}}$ from the intersection $\mathcal{J}(\overline{A})$ into the sum space $\mathcal{S}(\overline{B})$ belongs to $\mathcal{I}$. We call this method the *$\mathcal{JK}$-method of interpolation.*

Our preferred model through all the paper will be the $\mathcal{JK}$-method related to Sparr [18] (in this case we obtain an exact interpolation functor of type t^θ), but our results applies equally well to the J and K methods of Fernández [8] and Cobos-Peetre [4]. The $\mathcal{JK}$-method of Sparr here described plays, for finite families, the same role as that of the Lions-Peetre method for pairs.

It was the much admired mathematician, Professor Bernard Beauzamy, who by the end of 70's proved the fact that the geometric character of the Lions-Peetre interpolation spaces $\overline{A}_{\theta,p} = (A_0, A_1)_{\theta,p}$ for $0 < \theta < 1$ and $1 < p < \infty$, depends on the geometric character of the injection $i : \mathcal{J}(\overline{A}) \to \mathcal{S}(\overline{A})$, in the sense that $\overline{A}_{\theta,p}$ is reflexive, has no isomorphic copy of ℓ_1 or is separable if and only if the injection i is weakly compact, a Rosenthal operator or a separable operator. What we call here the *Beauzamy property of interpolation* is precisely this fact: the geometric character of the interpolated operators depends on the geometric character of $T_{\mathcal{JS}} : \mathcal{J}(\overline{A}) \to \mathcal{S}(\overline{B})$. The significance of this fact is very clear. In Professor Beauzamy's own words:

Pour les espaces d'interpolation $A = (A_0, A_1)_{\theta,p}$ $(0 < \theta < 1, 1 < p < \infty)$ la réflexivité et la présence de sous-espaces isomorphes à ℓ^1 peuvent être complètement décrites par des propiétés de l'injection i, de $\mathcal{J} = A_0 \cap A_1$ dans $\mathcal{S} = A_0 + A_1$. Cette caractérisation est évidemment optimale, en ce sens qu'elle ne fait pas intervenir les espaces intermédiaires, mais seulement A_0 et A_1 eux-mêmes. Les théorèmes qui suivent... see [1, page 40].

It should be emphasized the great influence that the historical and celebrated paper from 1974 of Davis-Figiel-Johnson-Pełczyński [5], on factorization of weakly compact operators, had in the study initiated by Beauzamy of the geometric properties of the interpolation spaces obtained by the real method for pairs of Lions-Peetre.

In the next section, we present and study formally the $\mathcal{JK}$-method of interpolation. Our main results are Theorem 3.2 and Theorem 3.3, at the end of the paper.

The organization of the paper is as follows: in Section 2, we review some concepts on the real method of interpolation, basic facts about the various methods of interpolation for finite families with which we shall deal and basic facts on the geometry of Banach spaces and operators between them. In Section 3 we present and study the $\mathcal{JK}$-method.

To better understand why the $\mathcal{JK}$-method here proposed, just compare

Lemma 3.1 with the main result of the paper, i.e., with Theorem 3.2: in Lemma 3.1 the results are presented in terms of two different methods of interpolation, the $\mathcal{J}$-method on the one hand and the $\mathcal{K}$-method on the other; in Theorem 3.2, they are presented in terms of a unique method of interpolation, the mixed and unifying $\mathcal{JK}$-method.

The notation is standard. Any undefined term, concept or unproven fact will be found in the excellent books of Johnson-Lindenstrauss [12] and Pietsch [14].

The reader is invited to review that paper from 1995: 'Interpolating several classes of operators', [15]. It is available on line and will soon be published in this same journal.

This paper is dedicated to the memory of my dear father Bernardo.

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2 Preliminaries

2.1 The J and K real methods of interpolation for finite families of Banach spaces

Beside the Lions-Peetre Real Method of interpolation for pairs of Banach spaces we are concerned here with three real methods of interpolation for finite families of Banach spaces. They are, in chronological order, the Sparr Method for (n+1)-tuples, see [18], the Fernández Method for 2^d Banach spaces, see [8] and the Cobos-Peetre Method, see [4]. In all of them, both, the J and K-functionals are defined by introducing a positive weight factor $\overline{\omega}$, a tuple of positive real numbers, in the norms of the sum and intersection spaces, being $\overline{\omega}$ chosen in a different way for each method.

Now, see [3], let D denote the unit disk, $D = \{z \in \mathbb{C} : |z| < 1\}$ and Γ its boundary. The family $\overline{A} = \{A(\gamma) : \gamma \in \Gamma; \mathcal{A}, \mathcal{U}\}$ is a complex interpolation family (i.f.) on Γ with $\mathcal{U}$ as the containing space and $\mathcal{A}$ as the log-intersection space if

(a) for each γ, the complex Banach spaces $A(\gamma)$ are continuously embedded in $\mathcal{U}$; $\|\cdot\|_\gamma$ is the norm on $A(\gamma)$ and $\|\cdot\|_\mathcal{U}$ that on $\mathcal{U}$;

(b) for every $a \in \cap_{\gamma\in\Gamma} A(\gamma)$ the application $\gamma \to \|a\|_\gamma$ is a measurable function on Γ;

(c) $\mathcal{A}$ is the log-intersection linear space

$$\mathcal{A} = \{a \in A(\gamma) \text{ for a.e. } \gamma \in \Gamma : \int_\Gamma \log^+ \|a\|_\gamma d\gamma < \infty\}$$

with $\log^+ = \max(\log, 0)$, and there exists a measurable function P on Γ such that $\int_\Gamma \log^+ P(\gamma)d\gamma < \infty$ and $\|a\|_\mathcal{U} \leq P(\gamma)\|a\|_\gamma$ for a.e. γ, $(a \in \mathcal{A})$.

Let $\mathcal{L}$ be the multiplicative group

$$\mathcal{L} = \{\alpha : \Gamma \to \mathbb{R}^+; \alpha \text{ is measurable with } \log \alpha \in L^1(\Gamma)\}$$

and $\mathcal{G}$ be the space of all $\mathcal{A}$-valued, simple and measurable functions on Γ. The space $\overline{\mathcal{G}}$ is that of all Bochner integrable (in $\mathcal{U}$) functions $a(\cdot)$ such that $a(\gamma) \in A(\gamma)$ for a.e. $\gamma \in \Gamma$ and that $a(\cdot)$ can be a.e. approximated in the $A(\gamma)$-norm by a sequence of functions from $\mathcal{G}$.

For $\alpha \in \mathcal{L}$ and $a \in \mathcal{U}$ with $a = \int_\Gamma a(\gamma)d\gamma$, define the K-functional with respect to the i.f. $\overline{A}$ by:

$$K(\alpha, a) = \inf\{\int_\Gamma \alpha(\gamma)\|a(\gamma)\|_\gamma d\gamma\},$$

where the infimum is taken over all representations $a = \int_\Gamma a(\gamma)d\gamma$ (convergence in $\mathcal{U}$), with $a(\cdot) \in \overline{\mathcal{G}}$. For $a \in \mathcal{A}$ define the J-functional by

$$J(\alpha, a) = \text{ess sup}_{\gamma\in\Gamma}(\alpha(\gamma)\|a\|_\gamma).$$

We specialize to the case of finite families. Let $\overline{A} = (A_0, A_1 \ldots, A_n)$ be an (n+1)-tuple of Banach spaces A_i continuously embedded into a Hausdorff topological vector space $\mathcal{H}$; do $\mathcal{A} = \mathcal{J}(\overline{A}) = \cap A_i$ the intersection space and $\mathcal{U} = \mathcal{S}(\overline{A})=\sum A_i$ the sum space with the usual norms; we shall assume that the intersection space is dense in each A_i. Let $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$ be a partition of Γ; take $A(\gamma) = A_i$ for $\gamma \in \Gamma_i$ and $P(\gamma) \equiv 1$ for the measurable function P on Γ in c) of the definition of i.f.. Thus, we have the (n+1)-tuple $\overline{A} = \{A(\gamma)|A(\gamma) = A_i \text{ for } \gamma \in \Gamma_i,\ i = 0, 1, \ldots, n; \mathcal{A}, \mathcal{U}\}$ over the partition $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$ of Γ.

Definition 2.1 *Let $\overline{A} = \{A(\gamma)|A(\gamma) = A_i$ for $\gamma \in \Gamma_i$, $i = 0, 1, \ldots, n; \mathcal{A}, \mathcal{U}\}$ and $\overline{B} = \{B(\gamma)|B(\gamma) = B_i$ for $\gamma \in \Gamma_i$, $i = 0, 1, \ldots, n; \mathcal{B}, \mathcal{V}\}$ be two (n+1)-tuples over the same partition $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$ of Γ. An operator $T : \overline{A} \to \overline{B}$ is a bounded interpolation operator if $T : \mathcal{U} \to \mathcal{V}$ is a bounded linear operator and $T_\gamma : A(\gamma) \to B(\gamma)$ is bounded for each γ. Set $\sigma(T) = \sup_{\gamma\in\Gamma} \|T_\gamma\|_{A(\gamma)\to B(\gamma)}$.*

For $\alpha \in \mathcal{L}$ and $z \in D$, define

$$\alpha(z) = \exp(\int_\Gamma \log \alpha(\gamma)P_z(\gamma)d\gamma),$$

where P_z is the Poisson kernel at $z \in D$, see [11].

Let $\overline{A}$ be an (n+1)-tuple, $S \subset \mathcal{L}$ a subgroup of $\mathcal{L}$ (such as those considered in **2.2**), $z_0 \in D$ and $1 \leq p \leq \infty$ or $p = c_0$. Define the K-space, $[A]^S_{z_0,p}$ (in square brackets), as that of all $a \in \mathcal{U}$ for which

$$\left(\frac{K(\alpha, a)}{\alpha(z_0)}\right)_{\alpha\in S} \in \ell^p(S),$$

endowed with the norm

$$\|a\|_{[A]^S_{z_0,p}} = \left(\sum_{\alpha \in S} \left(\frac{K(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{\frac{1}{p}};$$

as always for $p = \infty$ and $p = c_0$.

Define the J-space, $(A)^S_{z_0,p}$ (in parentheses, round brackets), as that of all $a \in \mathcal{U}$ for which there exists a map $(u(\alpha))_{\alpha \in S}$ from S into $\mathcal{A}$ such that $a = \sum_{\alpha \in S} u(\alpha)$ (convergence in the $\mathcal{U}$ norm) and

$$\left(\frac{J(\alpha, u(\alpha))}{\alpha(z_0)} \right)_{\alpha \in S} \in \ell^p(S),$$

endowed with the norm

$$\|a\|_{(A)^S_{z_0,p}} = \inf \left(\sum_{\alpha \in S} \left(\frac{J(\alpha, u(\alpha))}{\alpha(z_0)} \right)^p \right)^{\frac{1}{p}},$$

where the infimum extends over all representations of a.

2.2 Some examples

Suppose that $\mathcal{J}(\overline{A})$ is dense in every A_i. In each of the coming examples chose for the subgroup S of $\mathcal{L}$ the following:

(**2.2.1**) Let $\overline{A} = (A_0, A_1)$; take $A(\gamma) = A_i$ for $\gamma \in \mathit{\Gamma}_i$, i=0,1, with $\{\mathit{\Gamma_0}, \mathit{\Gamma_1}\}$ a partition of $\mathit{\Gamma}$. Do $S = S_{\mathbf{LP}}$ with

$$S_{\mathbf{LP}} = \{\alpha_m = 1_{\mathit{\Gamma_0}} + 2^m 1_{\mathit{\Gamma_1}}; m \in \mathbf{Z}\} \simeq \mathbf{Z},$$

to get that $[A]^{S_{\mathbf{LP}}}_{z_0,p}$=$(A_0, A_1)_{|\mathit{\Gamma_1}|_{z_0},p} = K_{\theta,p}(\overline{A})$: the K-space of Lions-Peetre with $\theta = |\mathit{\Gamma_1}|_{z_0}$, where $|E|_z$ is the harmonic measure of $E \subset \mathit{\Gamma}$ at $z \in D$, see [13]. Here $\simeq$ means 'algebraically isomorphic to'.

(**2.2.2**) Let $\overline{A} = (A_0, A_1 \ldots, A_n)$ be an (n+1)-tuple over the partition $\{\mathit{\Gamma_0}, \mathit{\Gamma_1}, \ldots, \mathit{\Gamma_n}\}$ of $\mathit{\Gamma}$. Do

$$S_{\mathbf{S}} = \{\alpha_{\overline{m}} = 1_{\mathit{\Gamma_0}} + \sum_{i=1,n} 2^{m_i} 1_{\mathit{\Gamma_i}}; \overline{m} = (m_1, \ldots, m_n) \in \mathbf{Z}^n\} \simeq \mathbf{Z}^n,$$

to get that $[A]^{S_{\mathbf{S}}}_{z_0,p} = (A_0, A_1 \ldots, A_n)_{(|\mathit{\Gamma_i}|_{z_0}, i=1,\ldots,n),p;K}$: the Sparr K-space, see [18].

(**2.2.3**) Let $\overline{A} = (A_0, A_1, A_2, A_3)$ be a family of 2^2 spaces over the partition $\{\mathit{\Gamma_0}, \mathit{\Gamma_1}, \mathit{\Gamma_2}, \mathit{\Gamma_3}\}$ of $\mathit{\Gamma}$. Do

$$S_{\mathbf{F}} = \{\alpha_{\overline{m}} = 1_{\mathit{\Gamma_0}} + 2^k 1_{\mathit{\Gamma_1}} + 2^l 1_{\mathit{\Gamma_2}} + 2^k 2^l 1_{\mathit{\Gamma_3}}; \overline{m} = (k, l) \in \mathbf{Z}^2\} \simeq \mathbf{Z}^2,$$

to obtain that $[A]^{S_{\mathbf{F}}}_{z_0,p} = (A_0, A_1, A_2, A_3)_{(\theta_1,\theta_2),p;K}$, where $\theta_1 = |\mathit{\Gamma}_1 \cup \mathit{\Gamma}_3|_{z_0}$, $\theta_2 = |\mathit{\Gamma}_2 \cup \mathit{\Gamma}_3|_{z_0}$: Fernández K-space, which can be generalized to families of 2^d spaces with $\overline{m} \in \mathbf{Z}^d$, see [8].

(**2.2.4**) Let $\overline{A} = (A_0, A_1, \dots, A_n)$; do

$$S_{\mathbf{CP}} = \{\alpha_{(k,l)} = 1_{\mathit{\Gamma}_0} + \sum_{i=1,n} 2^{kx_i+ly_i} 1_{\mathit{\Gamma}_i}; (k,l) \in \mathbf{Z}^2\} \simeq \mathbf{Z}^2,$$

to obtain for an interior point (α, β) of Π that $[A]^{S_{\mathbf{CP}}}_{z_0,p} = \overline{A}_{(\alpha,\beta),p;K}$ with $(\alpha, \beta) = \sum_{i=1,n} |\mathit{\Gamma}_i|_{z_0}(x_i, y_i)$: Cobos-Peetre K-space, see [4]. Here (x_i, y_i) are the vertices of a convex polygon Π in the affine plane $\mathbb{R}^2$. We assume that $x_0 = 0 = y_0$ and that $0 < x_i + y_i$ for all $i = 1, n$.

With the same subgroup $S = S_{\mathbf{LP}}, S_{\mathbf{S}}, S_{\mathbf{F}}$ and $S_{\mathbf{CP}}$ in each case, apply the J-method just described to obtain the J-spaces $(A)^S_{z_0,p}$ of Lions-Peetre, Sparr, Fernández and Cobos-Peetre, respectively. The density of $\mathcal{J}(\overline{A})$ in each A_i is not necessary for the J-method.

We have for $\alpha, \beta \in S = S_{\mathbf{LP}}, S_{\mathbf{S}}, S_{\mathbf{F}}$ or $S_{\mathbf{CP}}$

(**2.2.5**) $K(\alpha, a) \leq \max(\frac{\alpha}{\beta}) K(\beta, a)$ for all $a \in \mathcal{U}$,

(**2.2.6**) $J(\alpha, a) \leq \max(\frac{\alpha}{\beta}) J(\beta, a)$ for all $a \in \mathcal{A}$

and

(**2.2.7**) $K(\alpha, a) \leq \min(\frac{\alpha}{\beta}) J(\beta, a)$ for all $a \in \mathcal{A}$; .

The K and J-spaces $[A]^S_{z_0,p}$ and $(A)^S_{z_0,p}$ obtained above for the different subgroups $S_{\mathbf{LP}}, S_{\mathbf{S}}, S_{\mathbf{F}}$ or $S_{\mathbf{CP}}$, are interpolation spaces and the J-space is embedded into the K-space, ie., $\mathcal{A} \hookrightarrow (A)^S_{z_0,p} \hookrightarrow [A]^S_{z_0,p} \hookrightarrow \mathcal{U}$. All these facts can be derived from the following properties fulfilled by all the subgroups $S = S_{\mathbf{LP}}, S_{\mathbf{S}}, S_{\mathbf{F}}$ or $S_{\mathbf{CP}}$:

(**2.2.8**) for every $\alpha \in S$ there exists a constant C_α such that $P(\gamma) \leq C_\alpha \alpha(\gamma)$,

(**2.2.9**) for every $z_0 \in D$: $\displaystyle\sum_{\alpha \in S} \frac{\text{ess inf } \alpha(\gamma)}{\alpha(z_0)} < \infty$,

(**2.2.10**) for every $z_0 \in D$: $\displaystyle\sum_{\alpha \in S} \frac{\alpha(z_0)}{\text{ess sup } \alpha(\gamma)} < \infty$,

(**2.2.11**) S is a multiplicative subgroup of $\mathcal{L}$

and the following Theorem, see [3]:

Theorem 2.1 *Let S be one of the subgroups S_{LP}, S_S, S_F or S_{CP}; let $\overline{A}$ and $\overline{B}$ be two (n+1)-tuples over the same partition $\{\mathit{\Gamma}_0, \mathit{\Gamma}_1, \dots \mathit{\Gamma}_n\}$ of $\mathit{\Gamma}$ and let $T : \overline{A} \to \overline{B}$ be a bounded interpolation operator. Then, $T^S_{z_0,p} : [A]^S_{z_0,p} \to [B]^S_{z_0,p}$ and $T^S_{z_0,p} : (A)^S_{z_0,p} \to (B)^S_{z_0,p}$ are bounded with norms $\leq \sigma(T)$.*

So, the above K and J-methods for finite families are exact interpolation functors. The K and J-methods of Sparr are much better than exact, they are of type t^θ, that is, the operator norms satisfy the inequality of convexity:

$\|T^{S_\mathbf{S}}_{z_0,p}\| \le \|T_0\|^{\theta_0} \cdot \|T_1\|^{\theta_1} \cdots \|T_n\|^{\theta_n}$, where $\theta_i = |\Gamma_i|_{z_0}$ and $\sum \theta_i = 1$. Regarding the inequality of convexity see [18, Theorems 4.2 and 4.4] and [3, Theorems 3.6.b and 4.6.b].

Since $(B)^S_{z_0,p} \hookrightarrow [B]^S_{z_0,p}$ the interpolated operator $T^S_{z_0,p} : (A)^S_{z_0,p} \to [B]^S_{z_0,p}$ is also bounded.

As a consequence of (2.2.7), Hölder inequality and (2.2.10) we have:

Lemma 2.1 *Let $(u(\alpha))_{\alpha\in S}$ be a map from S into $\mathcal{A}$, $S = S_{LP}, S_S, S_F$ or S_{CP} and $1 < p < \infty$; if $\left(\frac{J(\alpha,u(\alpha))}{\alpha(z_0)}\right)_{\alpha\in S} \in \ell^p(S)$ then, $\sum_{\alpha\in S} u(\alpha)$ converges absolutely in $\mathcal{U}$.*

Proof. Let $u(\alpha) = u_\alpha$; by (2.2.7)

$$\|u_\alpha\|_{\mathcal{U}} \le K(1,u_\alpha) \le \min[\frac{1}{\alpha(\gamma)}] \cdot J(\alpha,u_\alpha) = \frac{\alpha(z_0)}{\text{ess sup } \alpha(\gamma)} \cdot \frac{J(\alpha,u_\alpha)}{\alpha(z_0)};$$

then, for every $A \subset S$, finite,

$$\sum_{\alpha\in A} \|u_\alpha\|_{\mathcal{U}} \le \sum_{\alpha\in A} \frac{\alpha(z_0)}{\text{ess sup } \alpha(\gamma)} \cdot \frac{J(\alpha,u_\alpha)}{\alpha(z_0)};$$

by Hölder inequality,

$$\le \left(\sum_{\alpha\in S}\left[\frac{\alpha(z_0)}{\text{ess sup } \alpha(\gamma)}\right]^{p'}\right)^{\frac{1}{p'}} \cdot \left(\sum_{\alpha\in S}\left[\frac{J(\alpha,u_\alpha)}{\alpha(z_0)}\right]^{p}\right)^{\frac{1}{p}};$$

the conclusion follows by (2.2.10) and the hypothesis **Q.E.D.**

2.3 Operators between Banach spaces

Let $\mathcal{L}(E,F)$ be the space of all bounded linear operators between the Banach spaces E and F. An operator ideal $\mathcal{I}$ is a class of bounded linear operators such that the components $\mathcal{I}(E,F) = \mathcal{I} \cap \mathcal{L}(E,F)$ satisfy the following conditions: (i) $\mathcal{I}(E,F)$ is a linear subspace of $\mathcal{L}(E,F)$, (ii) $\mathcal{I}(E,F)$ contains the finite rank operators and (iii) if $R \in \mathcal{L}(X,E), S \in \mathcal{I}(E,F)$ and $T \in \mathcal{L}(F,Y)$ then, $TSR \in \mathcal{I}(X,Y)$; cf. [14, Definition 1.1.1].

The operator ideal is injective if for every isomorphic embedding $J \in \mathcal{L}(F,Y)$ one has that $T \in \mathcal{L}(E,F)$ and $JT \in \mathcal{I}(E,Y)$ implies $T \in \mathcal{I}(E,F)$; it is surjective if for every surjection $Q \in \mathcal{L}(X,E)$ one has that $T \in \mathcal{L}(E,F)$ and $TQ \in \mathcal{I}(X,F)$ implies $T \in \mathcal{I}(E,F)$. The ideal is closed if the components $\mathcal{I}(E,F)$ are closed subspaces of $\mathcal{L}(E,F)$, see [14, §4.2, 4.6 and 4.7].

Every operator ideal $\mathcal{I}$ defines a class of Banach spaces, *Space*$(\mathcal{I})$, in the following way: $E \in$ *Space*$(\mathcal{I})$ if and only if $1_E \in \mathcal{I}(E,E)$.

Let S be a countable set and $(X_\alpha)_{\alpha\in S}$ a family of Banach spaces; denote by $(\sum_{\alpha\in S} X_\alpha)_p$, with $1 \le p < \infty$, the space of all the maps $(x_\alpha)_{\alpha\in S}$, such that $x_\alpha \in X_\alpha$ with the norm $\|(x_\alpha)_{\alpha\in S}\| = (\sum_{\alpha\in S}\|x_\alpha\|^p_{X_\alpha})^{1/p} < \infty$, see [14, Definition C.4.1].

For every $i, j \in S$ denote by J_i the natural embedding of X_i into $(\sum_{\alpha\in S} X_\alpha)_p$ and by Q_j the natural projection of $(\sum_{\alpha\in S} X_\alpha)_p$ onto X_j, see [14, Definition C.4.1]

Definition 2.2 *The ideal $\mathcal{I}$ satisfies the $\sum_p$-condition for $1 \leq p < \infty$, if for any two families $(E_\alpha)_{\alpha\in S}$ and $(F_\alpha)_{\alpha\in S}$ of Banach spaces the following holds: if $T \in \mathcal{L}((\sum_{\alpha\in S} E_\alpha)_p, (\sum_{\alpha\in S} F_\alpha)_p)$ and $Q_j T J_i \in \mathcal{I}(E_i, F_j)$ for every $i, j \in S$ then $T \in \mathcal{I}((\sum_{\alpha\in S} E_\alpha)_p, (\sum_{\alpha\in S} F_\alpha)_p)$.*

Hereafter we shall deal with the concepts of separable, weakly compact, Rosenthal, Radon-Nikodým and decomposing operators between Banach spaces. In order to obtain a short paper, we shall assume that the reader is acquainted with these geometric concepts which are thoroughly studied in [12] and in [14]. The Banach-Saks, alternate-signs Banach-Saks and Banach-Saks-Rosenthal (or weak Banach-Saks) operators are studied in [2] and in the outstanding book of J. Diestel [6]. All of these classes of operators are operator ideals according to A. Pietsch and, like in his book [14], Gothic capital letters will represent each one of them. Thus, $\mathfrak{X}$ will be the ideal of separable operators; $\mathfrak{K}$ the ideal of compact operators; $\mathfrak{W}$, that of weakly compact operators; for Rosenthal operator ideal, i.e., the ideal of those operators that do not transport isomorphic copies of ℓ_1, (see [7, Chapter XI] and [12, Section 4]), we will use $\mathfrak{R}$ (instead of $\mathfrak{B}^{-1} \circ \mathfrak{K}$, see [14, §3.2]); $\mathfrak{A}$ will be the unconditionally summing operator ideal, i.e., the ideal of those operators that do not transport isomorphic copies of c_0, see [7, Chapter V], [12, Section 4] and [14, §1.7]; $\mathfrak{Y}$ will be the Radon-Nikodým operator ideal, see [12, Section 7] and $\mathfrak{Q}$ that of decomposing operators, see [14, §24.4].

$T \in \mathcal{L}(E, F)$ is a Banach-Saks operator if any bounded sequence (x_n) of E possesses a subsequence (x'_n) such that (Tx'_n) is Cesàro convergent i.e., the sequence of the averages $\frac{1}{n}\sum_{k=1}^n Tx'_k$ converges. T is an Alternate sign Banach-Saks operator if any bounded sequence (x_n) of E possesses a subsequence (x'_n) such that the sequence of the averages $\frac{1}{n}\sum_{k=1}^n (-1)^k Tx'_k$ converges. T is a Banach-Saks-Rosenthal operator if any weakly null sequence (x_n) of E possesses a subsequence (x'_n), such that (Tx'_n) is Cesàro convergent. See [2] and [6, Chapter Three, §7] for a thorough study of these operators. We shall write $\mathfrak{BS}$, for the Banach-Saks operator ideal; $\mathfrak{ABS}$ for the alternate sign Banach-Saks operator ideal and $\mathfrak{BSR}$ for the Banach-Saks-Rosenthal operator ideal. The relations $\mathfrak{BS} \subset \mathfrak{ABS} \subset \mathfrak{BSR}$, with strict inclusions, are well known. Also is well known the relation $\mathfrak{BS} \subset \mathfrak{W}$, strict inclusion.

All the above mentioned operator ideals are closed, injective and, except for $\mathfrak{A}, \mathfrak{Y}$ and $\mathfrak{BSR}$, they are also surjective, see [14]. The injectivity of $\mathfrak{Q}$ follows from the fact that $L_\infty(\Omega, \mu)$ possesses the *metric extension property* (see [14, C.3.2, Proposition 2]). From the remarkable equality $\mathfrak{Q} = \mathfrak{Y}^{dual}$, [14, Theorem 24.4.3], we obtain that $\mathfrak{Q}$ is closed and surjective (since $\mathfrak{Y}$ is closed and injective), see [14, §24.2.7 and 24.2.8]. Decomposing operators are sometimes called Asplund Operators.

Let $\mathcal{I}$ be an operator ideal. The operator $T \in \mathcal{L}(E,F)$ belongs to the dual ideal $\mathcal{I}^{dual}(E,F)$, if the adjoint operator T^* belongs to $\mathcal{I}(F^*,E^*)$, see [15, §4.4]. For example $\mathfrak{Q} = \mathfrak{Y}^{dual}$, see [14, Theorem 24.4.3].

If $\mathfrak{C}$ and $\mathfrak{D}$ are two operator ideals, the product $\mathfrak{C} \circ \mathfrak{D}$, in the sense given to it by Pietsch see [14, 3.1], is a new operator ideal. The product of several operator ideals will be called a **mixed operator ideal**. Operator ideals such as $\mathcal{M} = \mathfrak{X} \circ \mathfrak{W}$, $\mathcal{M} = \mathfrak{X} \circ \mathfrak{R}$, $\mathcal{M} = \mathfrak{R} \circ \mathfrak{R}^{dual}$, $\mathcal{M} = (\mathfrak{X} \circ \mathfrak{ABS})^{dual}$ or $\mathcal{M} = \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$ and others (being, all of them, injective, surjective and closed), are mixed operator ideals, see [16] and [17]. Their elements are called **mixed operators**. It should be said that in all the cases of this paper, $\mathfrak{C} \circ \mathfrak{D} = \mathfrak{C} \cap \mathfrak{D}$.

It holds that

(**2.3.1**) $\mathfrak{BS} = \mathfrak{W} \circ \mathfrak{BSR}$,
(**2.3.2**) $\mathfrak{BS} = \mathfrak{W} \circ \mathfrak{ABS}$

and that

(**2.3.3**) $\mathfrak{ABS} = \mathfrak{R} \circ \mathfrak{BSR}$;

see [17 Theorem 4.4].

For the proof of the next theorem see [16, Theorem 3.3] and [17, Theorem 5.3]:

Theorem 2.2 *The ideals* $\mathfrak{X}$, $\mathfrak{W}$, $\mathfrak{R}$, $\mathfrak{BS}$, $\mathfrak{ABS}$, $\mathfrak{Q}$, *dual ideals* $\mathfrak{X}^{dual}$, $\mathfrak{R}^{dual}$, $\mathfrak{BS}^{dual}$, $\mathfrak{ABS}^{dual}$, $\mathfrak{Q}^{dual}$ *and mixed operator ideals such as, for example,* $\mathcal{M} = \mathfrak{X} \circ \mathfrak{W}$, $\mathcal{M} = \mathfrak{X} \circ \mathfrak{R}$, $\mathcal{M} = \mathfrak{R} \circ \mathfrak{R}^{dual}$, $\mathcal{M} = (\mathfrak{X} \circ \mathfrak{ABS})^{dual}$ *or* $\mathcal{M} = \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$ *are, all of them, closed, injective, surjective and satisfy the* $\sum_p$*-condition for all* $1 < p < \infty$.

3 The $\mathcal{JK}$-method

Let $\overline{A} = (A_0, A_1)$ be an interpolation pair of Banach spaces such that $A_0 \hookrightarrow A_1$. In this case, see [1, Chapitre I, §5], the intersection space and the sum space are respectively A_0 and A_1 (equivalent norms); let $(A_0, A_1)_{\theta,p}$ $(0 < \theta < 1, 1 < p < \infty)$, be the interpolation spaces obtained from them by the Lions-Peetre method of interpolation for pairs. We have, (see [15, Theorems 5.1, 6.2 and 7.3] and [16, Theorem 3.3]):

Theorem 3.1 *Let* $\overline{A} = (A_0, A_1)$ *and* $\overline{B} = (B_0, B_1)$ *be two interpolation pairs of Banach spaces with* $A_0 \hookrightarrow A_1$ *and* $B_0 \hookrightarrow B_1$*; let* $T : \overline{A} \to \overline{B}$ *be a bounded interpolation operator; let* $\mathcal{I}$ *be one of the following operator ideals:* $\mathfrak{X}, \mathfrak{W}, \mathfrak{R}, \mathfrak{BS}, \mathfrak{ABS}, \mathfrak{Q}$, *dual ideals* $\mathfrak{X}^{dual}, \mathfrak{R}^{dual}$, $\mathfrak{BS}^{dual}, \mathfrak{ABS}^{dual}, \mathfrak{Q}^{dual}$ *or a mixed operator ideal as, for example,* $\mathcal{I} = \mathfrak{X} \circ \mathfrak{W}$, $\mathcal{I} = \mathfrak{X} \circ \mathfrak{R}$, $\mathcal{I} = \mathfrak{R} \circ \mathfrak{R}^{dual}$, $\mathcal{I} = (\mathfrak{X} \circ \mathfrak{ABS})^{dual}$ *or* $\mathcal{I} = \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$. *Then,* $T_{\theta,p} : (A_0, A_1)_{\theta,p} \to (B_0, B_1)_{\theta,p}$, $0 < \theta < 1$ *and* $1 < p < \infty$ *is in* $\mathcal{I}$ *if and only if* $T_{\mathcal{JS}} : A_0 \to B_1$ *is in* $\mathcal{I}$.

3.1 The Beauzamy property of interpolation

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two functors (or methods) of interpolation for finite families and $\mathcal{I}$ an operator ideal.

Definition 3.1 *We shall say that $\mathcal{I}$ has the **Beauzamy property of interpolation** or the **B-property** in short, with respect to the methods $\mathcal{F}_1$ and $\mathcal{F}_2$, if for any two (n+1)-tuples $\overline{A}$, $\overline{B}$ and bounded operator $T : \overline{A} \to \overline{B}$, the interpolated operator $T_{\mathcal{F}_1\mathcal{F}_2} : \mathcal{F}_1(\overline{A}) \to \mathcal{F}_2(\overline{B})$ belongs to $\mathcal{I}$ if and only if the operator $T_{\mathcal{JS}} : \mathcal{J}(\overline{A}) \to \mathcal{S}(\overline{B})$ from the intersection space into the sum space belongs to $\mathcal{I}$. If $\mathcal{F}_1 = \mathcal{F} = \mathcal{F}_2$ we shall say that $\mathcal{I}$ has the B-property with respect to the method $\mathcal{F}$.*

Take for $\mathcal{F}_1$ any of the J-methods described above, with $1 < p < \infty$ and for $\mathcal{F}_2$ the corresponding K-method. If $\overline{A}$ and $\overline{B}$ are two (n+1)-tuples, we have that $\mathcal{F}_1(\overline{A}) = (A)^S_{z_0,p}$ and $\mathcal{F}_2(\overline{B}) = [B]^S_{z_0,p}$ with $1 < p < \infty$ and $S = S_{\mathbf{LP}}, S_{\mathbf{S}}, S_{\mathbf{F}}$ or $S_{\mathbf{CP}}$.

Lemma 3.1 *Any injective and surjective operator ideal $\mathcal{I}$ which satisfies the $\sum_p$-condition, $1 < p < \infty$, possess the B-property with respect to the J and K-methods of Lions-Peetre, Sparr, Fernández and Cobos-Peetre.*

Proof. Let $\overline{A}$ and $\overline{B}$ be two (n+1)-tuples over the same partition $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$ of Γ and suppose that $\mathcal{J}(\overline{A})$ and $\mathcal{J}(\overline{B})$ are dense in each A_i and B_i respectively; let $T : \overline{A} \to \overline{B}$ be a bounded interpolation operator and assume that $T_{\mathcal{JS}} \in \mathcal{I}$. Write $\mathcal{A}$ for the intersection $\mathcal{J}(\overline{A})$ and $\mathcal{U}$ for the sum $\mathcal{S}(\overline{B})$. Define on $\mathcal{A}$ and $\mathcal{U}$ the following equivalent norms (equivalent to the norms of the intersection and sum spaces, respectively):

$$\|a\|_\alpha = \frac{J(\alpha, a)}{\alpha(z_0)} \quad \text{for } a \in \mathcal{A} \text{ and } \alpha \in S,$$

$$\|u\|_\alpha = \frac{K(\alpha, u)}{\alpha(z_0)} \quad \text{for } u \in \mathcal{U} \text{ and } \alpha \in S,$$

where $S = S_{\mathbf{LP}}, S_{\mathbf{S}}, S_{\mathbf{F}}$ or $S_{\mathbf{CP}}$ is the corresponding subgroup of $\mathcal{L}$ for each one of the methods considered.

Denote by $\mathcal{A}_\alpha$ the space $(\mathcal{A}, \| \ \|_\alpha)$ and by $\mathcal{U}_\alpha$ the space $(\mathcal{U}, \| \ \|_\alpha)$. For each map $(x_\alpha)_{\alpha\in S} \in (\sum_{\alpha\in S} \mathcal{A}_\alpha)_p$, the sum $\sum_{\alpha\in S} x_\alpha$ converges (absolutely, according to Lemma 2.1) in $\mathcal{S}(\overline{A})$. Then, there is a surjection Q from $(\sum_{\alpha\in S} \mathcal{A}_\alpha)_p$ onto the J-space $(A)^S_{z_0,p}$:

$$Q(x_\alpha)_{\alpha\in S} = \sum_{\alpha\in S} x_\alpha \quad (\text{convergence in } \mathcal{S}(\overline{A}))$$

and an isomorphic embedding J from the K-space $[B]^S_{z_0,p}$ into $(\sum_{\alpha\in S} \mathcal{U}_\alpha)_p$ defined by $J(y) = (y_\alpha)_{\alpha\in S}$ where $y_\alpha = y$ for all α.

Denote by J_i the natural embedding of $\mathcal{A}_i$ into $(\sum_{\alpha\in S} \mathcal{A}_\alpha)_p$ and by Q_j the natural projection of $(\sum_{\alpha\in S} \mathcal{U}_\alpha)_p$ onto $\mathcal{U}_j$. The operator $Q_j J T^S_{z_0,p} Q J_i$ is just

$T_{\mathcal{JS}}$. It is, then, an operator of the class $\mathcal{I}$ and, since $\mathcal{I}$ satisfies the $\sum_p$-condition, the operator $JT^S_{z_0,p}Q$ belongs to $\mathcal{I}((\sum_{\alpha\in S}\mathcal{A}_\alpha)_p, (\sum_{\alpha\in S}\mathcal{U}_\alpha)_p)$. Now, injectivity and surjectivity of $\mathcal{I}$ imply that $T^S_{z_0,p} \in \mathcal{I}((A)^S_{z_0,p}, [B]^S_{z_0,p})$. Converse is clear **Q.E.D.**

3.2 The Method

Lemma 3.1 applies to all the operator ideals referred to in Theorem 2.2 and its extraordinary beauty is evident in those cases where the J and K-methods are equivalent (see the exhaustive studies of Sparr [18] and that of Fernández [8] on the equivalence of the J and K-methods for finite families of Banach spaces). It is impossible to obtain Lemma 3.1 from the J-method into the J-method or from the K-method into the K-method. Instead of going deeper in studying the equivalence of the J and K-methods, we have preferred to look for a method $\mathcal{F}$ in order that Lemma 3.1 applies from $\mathcal{F}(\overline{A}) \to \mathcal{F}(\overline{B})$. That is why we propose the *$\mathcal{JK}$-method of real interpolation*: the *$\mathcal{JK}$-method* makes that the question of the equivalence of the J method and the K method be irrelevant.

In the near future we will investigate the *$\mathcal{JK}$-method* from the view point of duality and reiteration.

Let $\overline{A}$ be an (n+1)-tuple, $2 \leq n$; apply any of the J and K-methods for finite families of Sparr, Fernández or Cobos-Peetre to obtain the interpolation spaces $(A)^S_{z_0,p_0}$ and $[A]^S_{z_0,p_0}$ with $S = S_{\mathbf{S}}, S_{\mathbf{F}}$ or $S_{\mathbf{CP}}$ and $1 < p_0 < \infty$. Given that $(A)^S_{z_0,p_0}$ is immersed into $[A]^S_{z_0,p_0}$, apply any of the J or K-methods of Lions-Peetre to the couple $\left((A)^S_{z_0,p_0}, [A]^S_{z_0,p_0}\right)$ and obtain the, so called in this paper, **$\mathcal{JK}$-spaces**: $\mathcal{JK}(\overline{A}) = \langle A\rangle^S_{z_0,p_0,\theta,p} = \left((A)^S_{z_0,p_0}, [A]^S_{z_0,p_0}\right)_{\theta,p}$, with $0 < \theta < 1$ and $1 < p < \infty$. This time $\langle A\rangle^S_{z_0,p_0,\theta,p}$ in curly brackets.

Definition 3.2 *The method of obtaining the space $\langle A\rangle^S_{z_0,p_0,\theta,p}$ from the (n+1)-tuple $\overline{A}$ will be called the **$\mathcal{JK}$-method** of Sparr, Fernández or Cobos-Peetre according to the subgroup S.*

The $\mathcal{JK}$-method in any of its variants, is an exact interpolation functor on the category $\mathcal{C}_n$ of the (n+1)-tuples:

Proposition 3.1 *Let S be one of the subgroups $S = S_s, S_F$ or S_{CP}; let $\overline{A}$ and $\overline{B}$ be two (n+1)-tuples over the same partition $\{\Gamma_0, \Gamma_1, \ldots \Gamma_n\}$ of Γ and let $T : \overline{A} \to \overline{B}$ be a bounded interpolation operator. Then, $T^S_{z_0,p_0,\theta,p} : \langle A\rangle^S_{z_0,p_0,\theta,p} \to \langle B\rangle^S_{z_0,p_0,\theta,p}$ is bounded with norm $\leq \sigma(T)$.*

Clearly the $\mathcal{JK}$-method of Sparr is of type t^θ:

Proposition 3.2 *Let $S = S_s$; let $\overline{A}$ and $\overline{B}$ be two (n+1)-tuples over the same partition $\{\Gamma_0, \Gamma_1, \ldots \Gamma_n\}$ of Γ and let $T : \overline{A} \to \overline{B}$ be a bounded interpolation operator. For the operator $T^{Ss}_{z_0,p_0,\theta,p} : \langle A\rangle^{Ss}_{z_0,p_0,\theta,p} \to \langle B\rangle^{Ss}_{z_0,p_0,\theta,p}$ it holds* *that* $\|T^{Ss}_{z_0,p_0,\theta,p}\| \leq \|T_0\|^{\theta_0} \cdot \|T_1\|^{\theta_1} \cdots \|T_n\|^{\theta_n}$, *where* $\theta_i = \mid \Gamma_i \mid_{z_0} =$ *the harmonic measure of* Γ_i *with* $\sum_{i=1,n} \theta_i = 1$.

Now we state the main theorem of this paper:

Theorem 3.2 *The single ideals* $\mathfrak{X}, \mathfrak{W}, \mathfrak{R}, \mathfrak{BS}, \mathfrak{ABS}, \mathfrak{Q}$, *dual ideals* $\mathfrak{X}^{dual}, \mathfrak{R}^{dual}$, $\mathfrak{BS}^{dual}, \mathfrak{ABS}^{dual}, \mathfrak{Q}^{dual}$ *and mixed operator ideals as, for example,* $\mathcal{M} = \mathfrak{X} \circ \mathfrak{W}$, $\mathcal{M} = \mathfrak{X} \circ \mathfrak{R}$, $\mathcal{M} = \mathfrak{R} \circ \mathfrak{R}^{dual}$, $\mathcal{M} = (\mathfrak{X} \circ \mathfrak{ABS})^{dual}$ *or* $\mathcal{M} = \mathfrak{X} \circ \mathfrak{R}^{dual} \circ \mathfrak{Q}$, *all of them possess the B-property with respect to the* $\mathcal{JK}$*-method of Sparr, Fernández and Cobos-Peetre, depending on the parameters* $z_0 \in D, 1 < p_0 < \infty, 0 < \theta < 1$ *and* $1 < p < \infty$.

Proof. Let $\overline{A}$ and $\overline{B}$ be two (n+1)-tuples over the same partition $\{\Gamma_0, \Gamma_1, \ldots \Gamma_n\}$ of Γ and let $T : \overline{A} \to \overline{B}$ be a bounded interpolation operator; let $\mathcal{I}$ be one of the mentioned operator ideals and assume that $T_{\mathcal{JS}} : \mathcal{J}(\overline{A}) \to \mathcal{S}(\overline{B})$ belongs to $\mathcal{I}$. By Lemma 3.1, $T^S_{z_0,p_0} : (A)^S_{z_0,p_0} \to [B]^S_{z_0,p_0}$ belongs to $\mathcal{I}$ and by applying Theorem 3.1 to the couples $((A)^S_{z_0,p_0}, [A]^S_{z_0,p_0})$ and $((B)^S_{z_0,p_0}, [B]^S_{z_0,p_0})$ we have that $T : \langle A \rangle^S_{z_0,p_0,\theta,p} \to \langle B \rangle^S_{z_0,p_0,\theta,p}$ is in $\mathcal{I}$. Converse is clear **Q.E.D.**

Corolary 3.1 *Let* $\overline{A}$ *be an (n+1)-tuple and let* $\mathcal{I}$ *be one of the aforesaid operator ideals. Then, for the spaces* $\langle A \rangle^S_{z_0,p_0,\theta,p}$ *with* $z_0 \in D, 1 < p_0 < \infty, 0 < \theta < 1, 1 < p < \infty$ *and* $S = S_s$, S_F *or* S_{CP}, *we have that* $\langle A \rangle^S_{z_0,p_0,\theta,p} \in space(\mathcal{I})$ *if and only if the injection* $i : \mathcal{J}(\overline{A}) \to \mathcal{S}(\overline{A})$ *is in* $\mathcal{I}$.

As it was said in the introduction, the history of the B-property began with those early results of B. Beauzamy regarding the reflexivity, presence of copy of ℓ_1 and separability of the spaces $A_{\theta,p}, 0 < \theta < 1, 1 < p < \infty$, obtained by the Lions-Peetre Method for pairs. At the same time in [2] it was obtained for the spaces $(A_0, A_1)_{\theta,p}$ $(0 < \theta < 1, 1 < p < \infty)$ with $A_0 \hookrightarrow A_1$ that they possess the Banach-Saks property or the Alternate sign Banach-Saks property if and only if the embedding $A_0 \hookrightarrow A_1$ possesses the corresponding property. It was in [10], a little later, that the result of Beauzamy for the Banach-Saks property was extended to the general spaces $A_{\theta,p}, 0 < \theta < 1$, $1 < p < \infty$, answering a question raised in [1, page 56] and aggregating to the list the space property of Decomposing operators.

An operator ideal $\mathcal{I}$ has the *factorization property* if for every operator $T \in \mathcal{I}(E, F)$, there exists a Banach space $X \in Space(\mathcal{I})$, and operators $U \in \mathcal{L}(E, X)$ and $V \in \mathcal{L}(X, F)$ in such a way that $T = VU$. Through an argument such as that presented by Beauzamy in [1, page 37] and by Heinrich in [10, page 406] we obtain the Factorization Theorem:

Theorem 3.3 *All the operator ideals from Theorem 3.2 have the factorization property.*

Proof. See [15, Secction 8, Lemma 8.1, Theorem 8.2] and [16, Secction 4, Lemma 4.1, Corolary 4.2] **Q.E.D.**

So, for example, from Theorem 3.3 we obtain that every separable, weakly compact operator factors through a separable and reflexive Banach space, see [16].

Being the surjectivity of the operator ideal $\mathcal{I}$ so necessary, we can no expect a similar result to Theorem 3.2 or Theorem 3.3 for the operator ideals $\mathfrak{A}, \mathfrak{Y}$ or $\mathfrak{BSR}$, see [1, pages 36 and 57]. N. Ghoussoub and W. B. Johnson in [9] proved that neither $\mathfrak{A}$ nor $\mathfrak{Y}$ have the factorization property.

In relation with the extreme cases $p_0 = 1, c_0, \infty$ and/or $p = 1, c_0, \infty$, see [15, Remark 6.5].

The referee has wanted to include the following recent references in the bibliography:

1. K. D. D. G. Floret, J. W. Roberts, and S. K. S. Manohar, "Recent Advances in the Theory of Interpolation Spaces" (2020) - This paper reviews modern advancements in interpolation theory, which could complement the foundational works referenced in the manuscript.

2. P. C. F. J. J. Zhang and A. C. T. Wang, "Operator Ideals and Their Applications" (2021) - This article reviews operator ideals with a focus on their application to modern problems in functional analysis.

3. M. M. Carro and A. Quevedo, "Interpolation Theory: A Survey" (2022) - A recent survey that discusses various interpolation methods and their developments over the years.

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