Short Research Article

On the Stability Analysis of Commensurate Fractional-Order Systems in the Caputo Sense

Abstract

Dynamics of fractional order systems has presently become a very extensively studied research area. This is due to hereditary properties, more degrees of freedom, and other advantages of fractional modeling involving differential equations. Devising methods for determining the stability of fractional order systems has become necessary. In this paper, we study some techniques for stability analysis of commensurate fractional order systems in the Caputo sense. Also, we demonstrate these methods on fractional Lorenz and reverse butterfly-shaped systems exploring their performance on these systems.

Keywords: Fractional Caputo Derivative, Lorenz System, Reverse Butterfly-Shaped System, Schur Complement, Sylvester Criterion, Routh-Hurwitz Criterion, Boundary Locus Technique

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1 Introduction

A model's response to significantly small perturbations in initial conditions is at the heart of modeling with differential equations. This qualitative behavior is generally called stability. In the field of dynamical systems, stability analysis aims to establish necessary and sufficient conditions for which trajectories close to initial conditions remain so at all future times or tend to stationary solutions [(1; 2)]. Lyapunov laid the foundation of stability in his Ph.D. dissertation titled: 'The General Problem of Motion Stability' in 1892 which proposed two techniques based on the structure of the differential equations or Hurwitz-type polynomials [(3; 4)]. The famous Routh-Hurwitz criterion provides necessary and sufficient conditions which guarantee the stability of integer order systems from their characteristic polynomial [(5; 6; 7)].

With the advent of fractional derivatives, which are known to perform better than integer derivatives in modeling the behavior of physical systems with differential equations, developing techniques for the stability analysis of fractional order differential equations or systems is essential. In 1996, Matignon [(8)] proposed a theorem, popularly called the Matignon stability theorem, which provides necessary and sufficient conditions for ensuring the stability of systems of fractional differential equations. This great leap has sparked researchers' interest in studying fractional systems [(9; 10; 11; 12; 13; 14)]. According to the Matignon stability theorem, the necessary and sufficient condition for the stability of a fractional order system is that the spectra of the associated linearized system must lie inside the Magtinon stability sector to be stable. The standeer or classica Routh-Hurwitz stability criterion is only a sufficient condition to this criteria [(15)].

Several methods have been employed in research to satisfy the Matignon stability criteria for fractional order systems with some resulting in conditions that can not be considered explicit. Some results could only allow for the determination of stability at particular points and do not allow for bifurcation analysis. This paper studies some techniques for the stability analysis of fractional systems and their demonstration of the Lorenz and reverse butterfly-shaped systems. The rest of the paper is structured as follows, section two provides a preliminary discussion on the subject, section three details some fractional techniques and their demonstration on the Lorenz and reverse butterfly-shaped system, and section four states conclusions and recommendations.

2 Preliminaries

In [(16)], Caputo reformulated the definition of the Riemann-Liouville fractional derivatives by switching the order of the ordinary derivative with the fractional integral operator. By doing so, the Laplace transform of this new derivative depends on integer order initial conditions which are different from the initial conditions when we use the Riemann–Liouville fractional derivative, which involves fractional order conditions.

Definition 2.1. (Caputo – Type Derivative) Let $f:\mathbb{R}_+\to\mathbb{R}$ and $\alpha\in(n-l,n), n\in\mathbb{N}$, $n=\lceil\alpha\rceil$ then

$$^{C}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau \tag{2.1}$$

where Γ is the gamma function, is called the Caputo fractional derivative of order α, if it exists [(16; 17; 18)].

From the above definition (2.1), systems of fractional differential equations can be formulated as;

$$\begin{cases} ^{C}D^{\alpha}\mathbf{x}(t)=f(x)\\ \mathbf{x}(t_0)=\mathbf{x}_0.\end{cases} \tag{2.2}$$

where $f(x)$ is continuous and $\mathbf{x}(t_0)$ is initial condition.

Definition 2.2. A point $x_0\in\Re^n$ is called an equilibrium point or critical point of (2.2) if $f(x_0)=0$. An equilibrium point x_0 is called a hyperbolic equilibrium point of (2.2) if none of the eigenvalues of the matrix $Df(x_0)$ have zero real part. The linear system

$$^{C}D^{\alpha}\mathbf{x}(t)=A\mathbf{x} \tag{2.3}$$

with the matrix $A=Df(x_0)$ is called the linearization of (2.2) at x_0 [(19)].

Theorem 2.1. *(Hartman-Globman Theorem) Let E be an open subset of $\Re^n$ containing the origin, let $f\in C^1(E)$, and let ϕ_t be the flow of the nonlinear system (2.2). Suppose that $f(0)=0$ and that* *the matrix $A = Df(0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $I_0 \subset \Re$ containing zero such that for all $x_0 \in U$ and $t \in I_0$,*

$$H \circ \phi_t(\mathbf{x}_0) = e^{At} H(\mathbf{x}_0);$$

i.e., H maps trajectories of (2.2) near the origin onto trajectories of (2.3) near the origin and preserves the parameterization by time [(19)].

Theorem 2.2. *(Sylvester Criterion) Let H be an $n \times n$ symmetric matrix. Then, H is negative definite if and only if its n leading minors alternate in sign as follows [(20; 21)]*

$$|H_1| < 0, |H_2| > 0, |H_3| < 0, ...$$

Theorem 2.3. *(Schur complement) Given a symmetric matrix*

$$H = \begin{pmatrix} M & N \\ N^T & L \end{pmatrix}$$

with L negative definite. The matrix H is negative definite if and only if the Schur complement $M - NL^{-1}N^T$ is [(22; 23)].

Theorem 2.4. *(Matignon Stability Theory) Autonomous system*

$${}^{C}D^{\alpha}x(t) = Ax(t)$$

with $x(t_0) = x_0$ and $0 < \alpha < 1$, is asymptotically stable (i.e. all the eigenvalues associated with A have negative real part) if and only if

$$|arg(specA)| > \frac{\alpha\pi}{2}, \tag{2.4}$$

where spec(A) is the set of all eigenvalues of A. Also, state vector x(t) decays towards 0 and meets the following condition $x(t) < Nt^{-\alpha}$, $t > 0$, $\alpha > 0$, $N \in \Re^{+}$ [(8)].

Theorem 2.5. *The fractional system (2.2) is asymptotically stable if and only the Jacobian matrix of the associated linearized system J has k-multiple zero eigenvalues corresponding to the Jordan block diag($J_1, J_2, J_3....J_i$) where J_I is a Jordan canonical form with order n_l, ${}_{l=1}^{i}$ and $n_l\alpha < 1$, $1 \leq i$ [(24)].*

3 Techniques for Stability Analysis of Fractional Order Systems

3.1 Sylvester Criterion and Schur Compliment

This approach to stability analysis of fractional order system uses the Sylvester criterion and Schur complement stated in theorems (2.2) and (2.3). To demonstrate this we use the famous Lorenz system. From the equation (2.2), the fractional order Lorenz system in the Caputo sense is defined as [(25)]

$$\begin{aligned} {}^{C}D^{\alpha}x &= \sigma(y - x) \\ {}^{C}D^{\alpha}y &= rx - xz - y \\ {}^{C}D^{\alpha}z &= xz - bz \end{aligned} \tag{3.1}$$

where σ, b and r are positive paremeter and $0 < \alpha < 1$. The system (3.1) has three set of equilibria namely; $o(0,0,0)$ and $E^{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$. From definition (2.2), linearization of the system (3.1) at the equilibrium point $o(0,0,0)$ yields

$$H_o = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \tag{3.2}$$

H_o is symmetric if and only if $\sigma = r$ and then,

$$H_o = \begin{bmatrix} -\sigma & r & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \tag{3.3}$$

From the matrix equation (3.3), the leading principal minors of H_o alternate in sign if and only if $r < 1$. Thus;

$$|H_o(1)| = -r < 0, |H_o(2)| = r(r-1) > 0, |H_o(3)| = rb(r-1) < 0, \text{iff} r < 1.$$

Therefore, the equilibrium point $o(0,0,0)$ is asymptotically stable according to Sylvester criterion in theorem (2.2), and that

$$|arg(spec(H_o))| > \frac{\alpha\pi}{2}.$$

Alternatively, by the Schur complement theorem (2.3), H_o can be partitioned as

$$H_o = \begin{pmatrix} M & N \\ N^T & L \end{pmatrix} \tag{3.4}$$

where

$$M = \begin{pmatrix} -R & R \\ R & -1 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $L = -b$ is negative definite. The Schur complement

$$M - NL^{-1}N^T = M$$

where M is negative definite if and only if $R < 1$. To determine the stability of the twin equilibria $E^{\pm}$, we linearize the system at these points. The Jacobian matrix associated with the equilibria $E^{\pm}$ are respectively;

$$H_{E^+} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{bmatrix} \tag{3.5}$$

and

$$H_{E^-} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{b(r-1)} \\ -\sqrt{b(r-1)} & -\sqrt{b(r-1)} & -b \end{bmatrix} \tag{3.6}$$

H_{E^+} and H_{E^-} will be symmetric if and only if $\sigma = r = 1$, and then

$$H_{E^+} = H_{E^-} \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} = H_E \tag{3.7}$$

with eigenvalues $\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = -b$. Since all the eigenvalues of H_E are distinct, H_E is similar to the following Jordan canonical form;

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -b \end{bmatrix} \tag{3.8}$$

For $b \neq 2$, H_E has one zero eigenvalue corresponding to one Jordan block J, therefore $E^{\pm}$ are asymptotically stable by theorem (2.5).
We give a similar analysis for the fractional reverse butterfly-shape system which is defined as;

$$\begin{aligned} {}^{C}D^{\alpha}x &= a(y-x) \\ {}^{C}D^{\alpha}y &= bx + kxz \\ {}^{C}D^{\alpha}z &= -cz - hxy \end{aligned} \tag{3.9}$$

where a, b, c, h, k are positive parameters and $0 < \alpha < 1$. The fractional system (3.9) has three set of equilibria namely $o(0,0,0)$ and $E^{\pm} = (\pm\sqrt{\frac{bc}{kh}}, \pm\sqrt{\frac{bc}{kh}}, -\frac{b}{k})$. The corresponding linearized system of (3.9) at the equilibria E^{+} and E^{-} respectively are

$$H_{E^{+}} = \begin{bmatrix} -a & a & 0 \\ 0 & 0 & k\sqrt{\frac{bc}{hk}} \\ -h\sqrt{\frac{bc}{hk}} & -h\sqrt{\frac{bc}{hk}} & -c \end{bmatrix} \tag{3.10}$$

and

$$H_{E^{-}} = \begin{bmatrix} -a & a & 0 \\ 0 & 0 & -k\sqrt{\frac{bc}{hk}} \\ h\sqrt{\frac{bc}{hk}} & h\sqrt{\frac{bc}{hk}} & -c \end{bmatrix} \tag{3.11}$$

However, $H_{E^{+}}$ and $H_{E^{-}}$ can not be symmetric since the system parameter a must be positive. Hence the stability of the fractional system (3.9) can not be determined using the theorems (2.2) and (2.3). For stability analysis of the classical reverse butterfly-shaped system, see [(26)]. Unfortunately, the stability results from this approach give no information about the fractional order $alpha$ and its values for which the fractional system is stable and unstable.

3.2 Routh-Hurwitz Criterion

According to the Routh-Hurwitz stability criterion, to determine the nature of stability of the nonlinear system expressed in the form (2.3), it is relevant to observe the nature of the roots of the characteristic polynomial associated with A of the linearized system.

Definition 3.1. A polynomial with real coefficients is said to be Hurwitz if all its roots have a negative real part, that is if all its roots lie in $\mathbf{C}$, the left plane of the complex plane

$$\mathbf{C}^{-} = \{a + bi : a < 0\} \tag{3.12}$$

Theorem 3.1. *(Routh-Hurwitz Theorem) The polynomial*

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-2}\lambda^2 + a_{n-1}\lambda^n + a_n \tag{3.13}$$

with its positive leading coefficient ($a_n > 0$) is a Hurwitz polynomial if and only if all the diagonal principal minors of the Hurwitz matrix are positive [(27)].

We demonstrate this approach on the stability of the equilibria $E^{\pm}$ of the fractional order reverse butterfly-shaped dynamic system (3.9). The linearization of the fractional system (3.9), H_{E^+} and H_{E^-} yield the same characteristic polynomial;

$$P^{E^{\pm}}_{a,b,c}(\lambda) = \lambda^3 + (a+c)\lambda^2 + (ac+bc)\lambda + 2abc \tag{3.14}$$

A MATLAB simulation reveals the characteristic polynomial (3.14) one negative root and a pair of complex conjugate roots with negative real parts. For detailed discussions on the classical reverse butterfly-shaped system (3.14), see [(26)]. Assuming the following eigenvalues $\lambda_1 < 0, \lambda_2 = \gamma + \delta i$ and $\lambda_2 = \gamma - \delta i$, the characteristic polynomial (3.14) can be written as

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0 \tag{3.15}$$

or

$$\lambda^3 - (2\gamma + \lambda_1)\lambda^2 + \left(2\gamma\lambda_1 + \gamma^2 + \delta^2\right)\lambda - \lambda_1\left(\gamma^2 + \delta^2\right) = 0 \tag{3.16}$$

Comparing equation (3.14) and (3.16), we have that

$$\begin{aligned} &a + c = -2\gamma - \lambda_1 \\ &c(a+b) = 2\gamma\lambda_1 + \gamma^2 + \delta^2 \\ &2abc = -\lambda_1\left(\gamma^2 + \delta^2\right) \end{aligned} \tag{3.17}$$

From equation $(3.17)_1$, we obtain

$$\gamma = -\frac{a+c+\lambda_1}{2} \tag{3.18}$$

Dividing both sides of equation $(3.17)_3$ by γ^2 yields

$$\frac{2abc}{\gamma^2} = -\lambda_1\left(1 + \left(\frac{\delta}{\gamma}\right)^2\right). \tag{3.19}$$

Thus;

$$\left(\frac{\delta}{\gamma}\right)^2 = \frac{2abc}{-\lambda\gamma^2} - 1 \tag{3.20}$$

Substituting the value of γ into the right side of the equation (3.20) gives

$$\left(\frac{\delta}{\gamma}\right)^2 = \frac{8abc}{-\lambda_1\left(a+c+\lambda_1\right)^2} - 1 \tag{3.21}$$

and as such,

$$\begin{aligned} |\arg(\lambda_i)| &= \arctan\left(\frac{\delta}{\gamma}\right) \\ &= \arctan\left(\sqrt{\frac{8abc}{-\lambda_1\left(a+c+\lambda_1\right)^2} - 1}\right) \end{aligned} \tag{3.22}$$

$i = 2, 3.$
From the above discussion, the zeros of the characteristic polynomial (3.14) will lie inside the Matignon stability sector (i.e. the left half plane) if the condition (3.23) is satisfied. Hence,

$$|\arg(\lambda_i)| = \arctan\left(\sqrt{\frac{8abc}{-\lambda_1\left(a+c+\lambda_1\right)^2} - 1}\right) > \frac{\alpha\pi}{2}. \tag{3.23}$$

A similar approach has been used for the stability analysis of a new fractional-order system with one saddle and two node foci in [(28)]. This result can not be considered explicit as the condition in equation (3.23) depends on an unspecified parameter λ_1. Also, this approach is silent on the effect of α on the stability of the fractional order system and does not allow for bifurcation analysis (3.9).

3.3 Boundary Locus Technnique

In the classical 3-dimensional system (the case $\alpha = 1$) of characteristic polynomial;

$$P(\lambda; p, q, r) = \lambda^3 + p\lambda^2 + q\lambda + r \tag{3.24}$$

where p, q, r are real coefficients, λ_i, $i = 1, 2, 3$ has negative real parts if and only if $p > 0, q > 0$, and $0 < r < pq$ according to standard Routh-Hurwitz conditions. This is only a sufficient condition for all the zeros of λ_i, $i = 1, 2, 3$ of (3.24) to be located inside the Matignon stability

$$|arg(\lambda)| > \frac{\alpha\pi}{2} \tag{3.25}$$

for fractional order cases. To guarantee (3.25) of (3.24), the characteristic polynomial of a fractional system, Čermák and Nechvátal [(29)] proposed the boundary locus technique. According to [(29)], we define the boundary locus $BL(\alpha)$ as follows;

$$\begin{aligned} BL(\alpha) = \{(p, q, r) \in \mathbb{R}\} : \exists\lambda \in \mathbb{C}, |arg(\lambda)| > \frac{\alpha\pi}{2} \\ \text{and} \\ P(\lambda; p, q, r) = 0, 0 < \alpha < 1. \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} \lambda =&\omega e^{i\alpha\pi/2} \\ =&\omega \left(\cos\left(\frac{\alpha\pi}{2}\right) + i\sin\left(\frac{\alpha\pi}{2}\right)\right) \\ =&\left(\omega\left(\cos\left(\frac{\alpha\pi}{2}\right) + i\sin\left(\frac{\alpha\pi}{2}\right)\right)\right)^3 \\ &+p\left(\omega\left(\cos\left(\frac{\alpha\pi}{2}\right) + i\sin\left(\frac{\alpha\pi}{2}\right)\right)\right)^2 \\ &+\text{q}\left(\omega\left(\cos\left(\frac{\alpha\pi}{2}\right) + i\sin\left(\frac{\alpha\pi}{2}\right)\right)\right) + r \end{aligned} \tag{3.27}$$

for a suitable $\omega \geq 0$.
Separating the real and imaginary parts, we obtain

$$\begin{aligned} \omega^3\cos\left(\frac{3\alpha\pi}{2}\right) + p\omega^2\cos(\alpha\pi) + r\omega\cos\left(\frac{\alpha\pi}{2}\right) + l = 0 \\ \omega\left(\omega^2\sin\left(\frac{3\alpha\pi}{2}\right) + p\omega\sin(\alpha\pi) + r\sin\left(\frac{\alpha\pi}{2}\right)\right) = 0 \end{aligned} \tag{3.28}$$

with the two solutions:
Solution 1: $l = 0, \omega = 0$
Solution 2:

$$\begin{aligned} &q = -\omega(4\cos(\frac{\alpha\pi}{2})^2\omega + 2\cos(\frac{\alpha\pi}{2})p - \omega), \\ &r = 2\cos(\frac{\alpha\pi}{2})\omega^3 + \omega^2 p \end{aligned} \tag{3.29}$$

From $(3.29)_1$,

$$\omega^{\pm} = \frac{-\cos\left(\frac{\alpha\pi}{2}\right)p \pm \sqrt{\cos\left(\frac{\alpha\pi}{2}\right)^2 p^2 - 4\cos\left(\frac{\alpha\pi}{2}\right)^2 q + q}}{4\cos\left(\frac{\alpha\pi}{2}\right)^2 - 1} \tag{3.30}$$

To determine the dependence of the constant r on p and q, we substitute equation (3.30) into $(3.29)_2$;

$$r^{\pm} = \frac{-pq \pm 2k_\alpha\left(p^2 - 4qk_\alpha^2 + q\right)\sqrt{k_\alpha^2 p^2 - 4k_\alpha^2 q + q} + \Gamma}{(4k_\alpha^2 - 1)^3} \tag{3.31}$$

where

$$k_\alpha = cos(\frac{\alpha\pi}{2}),$$

and

$$\Gamma = 2pk_\alpha^2\left(-p^2 + 4qk_\alpha^2 + q\right)$$

Theorem 3.2. *Let $2/3 < \alpha < 1$. All the zeros (3.24) satisfy (3.25) if and only if the following conditions holds:*

(i) $p > 0, q > 0, 0 < r < r^-(p, q; \alpha)$;

(ii) $p \leq 0, \bar{q} > p^2/4cos^2(\alpha\pi/2), 0 < r < r^-(p, q; \alpha)$

(iii) $p > 0, \hat{q}(p; \alpha) \leq q \leq 0, r^+(p, q; \alpha) < r < r^-(p, q; \alpha)$.

Proof. See [(29)]. □

Theorem 3.3. *Let $1/2 < \alpha < 2/3$, All the zeros λ_i of (3.24) satisfy (3.25) if and only if any of the following conditions holds:*

(i) $p < 0, q \leq \bar{q}(p; \alpha), r > r^+(p, q; \alpha)$;

(ii) $p < 0, \bar{q} < q < \hat{q}(p; \alpha), 0 < r < r^-(p, q; \alpha)$ *or* $, r > r^+(p, q; \alpha)$;

(iii) $p < 0, q > \hat{q}(p; \alpha), r > 0$;

(iv) $p \geq 0, q < 0, r > r^+(p, q; \alpha)$;

(v) $p \geq 0, q \geq 0, r > 0$.

Proof. See [(29)]. □

Theorem 3.4. *Let $0 < \alpha \leq 1/2$, All the zeros λ_i of (3.24) satisfy (3.25) if and only if any of the following conditions holds:*

(i) $p < 0, q \leq \bar{q}(p; \alpha), r > r^+(p, q; \alpha)$;

(ii) $p < 0, q > \bar{q}(q; \alpha), r > 0$;

(iii) $p \geq 0, q \geq 0, r > 0$.

Proof. See [(29)]. □

Next, we demonstrate the application of theorems (3.2), (3.3), and (3.4) to determine the stability of the equilibria $E^{\pm}$ of the fractional system (3.9). From the characteristic polynomial (3.14), $p = a + c > 0$, $q = c(a + b) > 0$, and $r = 2abc > 0$. Applying theorem (3.4), it is easy to see that equilibria $E^{\pm}$ are asymptotically stable for $0 < \alpha < 1/2$. Also, according to theorem (3.3), the system is asymptotically stable for $0 < \alpha < 2/3$. We therefore conclude that $E^{\pm}$ has all zeros located inside the Matignon sector (3.25), which guarantees that the equilibria $E^{\pm}$ of the fractional system (3.9) are locally asymptotically stable for $0 < \alpha < 2/3$.
For $2/3 < \alpha < 1$, we apply theorem (3.2). Then, it is relevant that the inequality $r < r^-(p, q, r)$ is satisfied. And so, substituting $p = a+c$, $q = c(a+b)$, and $r = 2abc$ into $r < r^-(p, q, r)$ we obtain that;

$$\begin{aligned} 2abc < \{&-c(a+c)(a+b) - 2k_\alpha\left((a+c)^2\right.\\ &- 4c(a+b)k_\alpha^2 + c(a+b)\\ &\times\sqrt{k_\alpha^2(a+c)^2 - 4c(a+b)k_\alpha^2 + c(a+b)}\\ &+2(a+c)\left(-(a+c)^2 + 4c(a+b)k_\alpha^2\right.\\ &+ c(a+b)k_\alpha^2\}/\left(4k_\alpha^2 - 1\right)^3 \end{aligned} \tag{3.32}$$

where $k_\alpha = \cos\left(\frac{\alpha\pi}{2}\right)$.
Further simplification of (3.32) yields

$$Ac^3 + Bc^2 + Dc + E > (Fc^2 + Gc + H)\sqrt{Ic^2 + Jc + K} \tag{3.33}$$

where
$A = 2\cos^2\left(\frac{\alpha\pi}{2}\right)$
$B = -2(a+b)\cos^2\left(\frac{\alpha\pi}{2}\right) - 8(a+b)\cos^4\left(\frac{\alpha\pi}{2}\right) + 6a\cos^2\left(\frac{\alpha\pi}{2}\right) + (a+b)$
$D = -8a(a+b)\cos^4\left(\frac{\alpha\pi}{2}\right) + \left(2ab - 4a^2\right)\cos^2\left(\frac{\alpha\pi}{2}\right) + a^2 + ab\left(1 + 2\left(4\cos^4\left(\frac{\alpha\pi}{2}\right) - 1\right)^3\right)$
$E = 2a^3\cos^2\left(\frac{\alpha\pi}{2}\right)$
$F = -2\cos\left(\frac{\alpha\pi}{2}\right)$
$G = -4a\cos\left(\frac{\alpha\pi}{2}\right) + 8\cos^3\left(\frac{\alpha\pi}{2}\right) - 2\cos\left(\frac{\alpha\pi}{2}\right)(a+b)$
$H = -2a^2\cos\left(\frac{\alpha\pi}{2}\right)$
$I = \cos^2\left(\frac{\alpha\pi}{2}\right)$
$J = (-2a - 4b)\cos^2\left(\frac{\alpha\pi}{2}\right) + (a+b)$
$K = a^2\cos^2\left(\frac{\alpha\pi}{2}\right)$.
We note that, taking the limit as $\alpha \to 1$ gives $F = G = H = 0$ which forces the right side of inequality (3.33) to zero. However, on the left side, we get that;

$$A = 0, B = (a+b), D = a(a-b), E = 0. \tag{3.34}$$

Substituting these into inequality (3.33) gives

$$c > \frac{a(b-a)}{a+b}. \tag{3.35}$$

The inequality (3.35) is exactly the stability condition in the classical case $\alpha = 1$ using the standard Routh-Hurwitz stability criterion, see [(26)]. Therefore, the criterion in the inequality (3.33) is the corresponding fractional extension of the Routh-Hurwitz criterion. This explicit result allows for bifurcation analysis of the fractional systems using the criterion in the equation (3.33). With this, one can easily determine the values of the order α for which the fractional-order system is stable and unstable. With c as an adjustable control parameter, one can determine the topological nature of the fractional-order system for various values of α.

4 Conclusions & Recommendations

In conclusion, the various techniques for the stability analysis of fractional order systems have been studied and applied to the fractional Lorenz and reverse butterfly-shaped dynamic systems. It has been demonstrated that the Schur Complement and Sylvester criterion provide stability conditions for the fractional Lorenz system, however, this approach can not be used to perform stability analysis of the fractional reverse butterfly-shaped since the matrix associated with the linearized system can not be symmetric. Also, results from this approach give no information on the effect of the fractional order α on the stability of the fractional system. This challenge is similar to the Routh-Hurwitz criterion which we have demonstrated with a fractional reverse butterfly-shaped system. Both approaches do not give explicit results and allow for bifurcation analysis. However, the Boundary Locus overcomes these challenges. The Boundary Locus technique gives explicit results that can determine the values of the fractional order α for which the fractional system is stable and unstable. Also, it allows for the variation of the control parameter c and α for bifurcation analysis of the fractional-order system as in the classical case. The stability criterion for fractional systems of more than 4-dimension using the boundary locus technique is open and recommended for future studies. We recommend future stability analysis of the fractional reverse butterfly-shaped system for the specific parameter values $a = 10, b = 40, k = 16$ and $h = 1$ and the control adjustable parameter c using the derived criterion (3.33).

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